Unilateral Stability and Almost Periodicity

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1. Let x and $f(x)$ denote n -component vectors, where f is of class C^1 on some open invariant set Ω of the x -space, and consider the system

$$(1) \quad x' = f(x) \quad (x' = dx/dt).$$

As in [3] a solution $x = x(t)$ will be called future (past) stable in the sense of LIAPOUNOFF if, for every $\epsilon > 0$, there exists a $\delta = \delta_\epsilon > 0$ such that

$$|y(t) - x(t)| < \epsilon \quad \text{for } t \geq 0 \quad (t \leq 0)$$

whenever $y(0)$, hence $y(t)$, is in Ω and $|y(0) - x(0)| < \delta$. It was shown in [3] that if (1) is incompressible, so that

$$(2) \quad \operatorname{div} f = 0,$$

and if $x(t)$ is an almost periodic solution (in the sense of BOHR) of (1) which is either future or past stable, then it must be both future and past stable. Moreover, it was shown in [2] that, if (1) is incompressible and if $x(t)$ is any bounded stable (both future and past) solution, then $x(t)$ must be almost periodic. The purpose of the present note is to point out the connection between these results in the following:

Theorem. *If (1) satisfies (2) and if $x(t)$ is bounded on $0 \leq t < \infty$ ($-\infty < t \leq 0$) and is a future (past) stable solution then necessarily (i) $x(t)$ is both future and past stable and (ii) $x(t)$ is Bohr almost periodic.*

That (i) (and (ii)) may fail to hold if the boundedness assumption on $x(t)$ is dropped can be seen from the following example. Suppose $n = 2$, so that $x = (x_1, x_2)$ and $f = (f_1, f_2)$. Then put $f_1(x_1, x_2) \equiv 1$ and $f_2(x_1, x_2) = 0$ or x_1^2 according as $x_1 \geq 0$ or $x_1 < 0$, so that f is of class C^1 and (1) satisfies (2). Hence if $x_1 \geq 0$, then $x'_1 = 1$, $x_1 = t + a$ and $x_2 = b$, where a and b are constants. If a and b are chosen so that the solution path goes through the point $(0, c)$ when $t = 0$, then $a = 0$, $b = c$ and one gets $x_1 = t$, $x_2 = c$ whenever $x_1 \geq 0$, that is, whenever $t \geq 0$. Similarly, $x_1 = t$ and $x_2 = \frac{1}{3}t^3 + c$ if $x_1 < 0$, that is, if $t < 0$. Thus, not only do the solutions exist for $-\infty < t < \infty$, but also any

open invariant set has an infinite measure, and every solution is future, but not past, stable. That any Ω *fails* to satisfy

$$(3) \quad \text{meas } \Omega < \infty$$

is not accidental since, as will be clear from the proof of the theorem, the boundedness assumption on $x(t)$ assures (3), which, in turn, makes available the POINCARÉ recurrence theorem, an essential ingredient of the proof. Condition (3) was vital in [2] and [3] also.

2. Proof of the Theorem. It is clearly sufficient to consider the case of future stability only. Let A be an open sphere about $x(0)$ and let $B = \bigcup_0^\infty A_t$, where A_t denotes the image of the set A after time t under the flow defined by (1). Then if A is sufficiently small, the boundedness and stability of $x(t)$ on $0 \leq t < \infty$ imply that B is bounded and, in particular, that $\text{meas } B < \infty$; cf. [3]. Since $\text{meas } B_t = \text{meas } B$ ($B = B_0$) for any t and since $B_t \subset B_s$ for $t > s$, it is clear that

$$\Omega = \bigcup_{-\infty < t \leq 0} B_t = \bigcup_{-\infty < t < \infty} A_t$$

is an open invariant set containing $x(t)$ and satisfying (3). Moreover, B and Ω are open, B is contained in Ω , and both sets have the same measure. Since B is bounded so also is Ω ; as a consequence, $x(t)$ is bounded on $-\infty < t < \infty$.

Next, choose $\epsilon > 0$. Then, according to the hypothesis of future stability,

$$(4) \quad |y(t) - x(t)| < \epsilon \quad \text{for } t \geq 0 \quad \text{if } y(0) \text{ is in } C,$$

where C is a sufficiently small sphere with center $x(0)$. It is supposed that the radius of C is so small that C is contained in A and hence also in the invariant set Ω . In view of the continuity properties of the solutions of (1), for every $\epsilon > 0$ and every positive integer n , there exists a number $\delta = \delta(\epsilon, n)$ such that

$$(5) \quad |y(t) - x(t)| < \epsilon \quad \text{for } -n \leq t \leq n \quad \text{if } |y(0) - x(0)| < \delta.$$

Since $x(t)$ is future stable, it follows from POINCARÉ's theorem that there exists a number $r = r_n \geq n$ (in fact, arbitrarily large numbers r) satisfying

$$(6) \quad |x(r_n) - x(0)| < \delta.$$

Otherwise there would exist a sphere about $x(0)$ having no point in common with its image, after time r , whenever r is sufficiently large.

Let $x_n(t) = x(t + r_n)$. Then, by (4),

$$(7) \quad |z(t) - x_n(t)| < \epsilon \quad \text{for } t \geq -r_n \quad \text{if } z(0) \text{ is in } C^{(n)},$$

where $C^{(n)}$ is the image of the set C after time r_n ; and, by (5) and (6), $|x_n(t) - x(t)| < \epsilon$ on $-n \leq t \leq n$. It now follows from relation (7) and the inequality $r_n \geq n$ that

$$(8) \quad |z(t) - x(t)| < 2\epsilon \quad \text{for } -n \leq t \leq n \quad \text{if } z(0) \text{ is in } C^{(n)}.$$

Let

$$\Gamma = \bigcap_m \left(\bigcup_{n>m} C^{(n)} \right),$$

the set of points belonging to $C^{(n)}$ for an infinity of positive integers n . Clearly $\text{meas } \Gamma \geq \text{meas } C > 0$; while $\text{meas } \Gamma < \infty$, since C , and hence Γ , belongs to the invariant set Ω . Relation (8) implies that

$$(9) \quad |z(t) - x(t)| < 2\epsilon \quad \text{for} \quad -\infty < t < \infty \quad \text{if} \quad z(0) \text{ is in } \Gamma.$$

Next, it will be shown that $x(t)$ is almost periodic; the argument will be similar to that used in [2], p. 902. Since Γ has a positive measure, it follows from a theorem of KHINTCHINE, namely

$$\lim (v - u)^{-1} \int_u^v \text{meas } (\Gamma \Gamma_t) dt \geq (\text{meas } \Gamma)^2 / \text{meas } \Omega, \quad \text{as} \quad v - u \rightarrow \infty,$$

that there cannot exist arbitrarily large intervals $u \leq t \leq v$ for which $\Gamma \Gamma_t$ is empty. Thus there exists a number L (depending on Γ , hence on ϵ) with the property that every t -interval of length L contains at least one number t^* for which $\Gamma \Gamma_{t^*}$ is not empty. If then $z(0)$ belongs both to Γ and to Γ_{t^*} , condition (9) holds, as well as the implied condition $|z(t) - x(t + t^*)| < 2\epsilon$ for $-\infty < t < \infty$. Consequently $|x(t + t^*) - x(t)| < 4\epsilon$ for all t and so t^* is, in BOHR's terminology (cf. [1]), a translation number of $x(t)$ belonging to 4ϵ . This shows that $x(t)$ must be almost periodic and part (ii) of the theorem is proved. Assertion (i) follows from [3] and the proof is now complete.

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