Stability of Cauchy's Problem in Space for Analytic Systems of Arbitrary Type

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1. Introduction. For analytic partial differential equations of the second order in two independent variables there is a well developed and effective theory of the extension of solutions into the complex domain. Methods based on such an extension have proved to be useful for the discussion of questions that can be classified in essentially three different categories. The first of these is concerned with the CAUCHY problem for linear or nonlinear equations of the elliptic type, which has usually been treated by the CAUCHY-KOWALEWSKI power series procedure, but which can be solved numerically in a systematic way through analytic continuation into the domain of complex values of the independent variables, provided that the CAUCHY data permit such a continuation in the large [3]. Secondly, there is the theorem that all solutions of an analytic elliptic equation must themselves be analytic, which LEWY [5] has proved in the nonlinear case by using a hyperbolic system to define complex extensions that satisfy the CAUCHY-RIEMANN equations. Finally, LEWY [6] has generalized the SCHWARZ reflection principle to obtain rules for continuing the solution of an elliptic equation of the second order across a boundary where it obeys an analytic boundary condition.

The original treatment of the three results we have described depended on the introduction of characteristic coordinates and therefore was confined to partial differential equations in two independent variables. It is of interest to inquire about the status of the corresponding questions for an equation in more independent variables, a situation where characteristic coordinates, reduction to a simple canonical form, and the techniques of conformal mapping are not available. It turns out that all three problems can be handled successfully, but by methods which are quite different in each case. In this paper we shall be primarily concerned with the CAUCHY problem, and we shall present a new approach to it that is valid for any analytic system of partial differential equations in any number of independent variables with analytically prescribed initial data. These are precisely the hypotheses of the classical CAUCHY-KOWALEWSKI theorem, and our investigation will provide an alternate explanation of that result.

The principal tool of our analysis is a reduction of the general analytic system of arbitrary type to one that is symmetric and hyperbolic, but involves more

independent variables and more unknowns. This symmetric hyperbolic normal form is achieved for analytic solutions in a quite direct and elementary fashion by continuing them into the complex domain and then adding to the original system there a suitable combination of the CAUCHY-RIEMANN equations, which are, of course, identically fulfilled. After an analytic continuation of the CAUCHY data has been performed, the CAUCHY problem becomes properly set for the new hyperbolic system. The instabilities associated with the elliptic case, for example, appear exclusively here in the step of extending the data into the complex domain. When this has been done, the remaining problem can even be solved numerically by any of the standard stable procedures based on finite differences.

Useful applications of a technique of the above kind can be made to the inverse solution of free boundary problems arising in plasma physics and in fluid dynamics. The construction of flows with a detached shock wave is a case in point. A significant advantage of the new approach to be presented here over an earlier one [3] involving characteristic coordinates is that it is adequate for the treatment of genuinely three-dimensional flows not exhibiting axial symmetry, since the partial differential equations of fluid dynamics form an analytic system, albeit one not always of a distinguishable type. Moreover, for two-dimensional shock waves the new and simpler method permits calculation of the transonic flow regime with complete deafness to the issue of type and thus even overcomes difficulties encountered previously near the sonic line.

In order to put the main body of our analysis in proper perspective, we take this opportunity to review briefly the solution in higher dimensional space of the other two problems we mentioned at the beginning. A proof of the analyticity of solutions of a nonlinear analytic elliptic equation of the second order in more than two independent variables was given by HOPF [4]. To do this he constructed the extension of solutions into the complex domain by means of an integral equation which he obtained by application of GREEN's theorem to a divergence expression built around a parametrix associated with the equation in question. He selected as the domain of integration a section of a very flat cone with its vertex located in the complex domain, but with its perimeter lying in the real domain. This permitted him to avoid a detailed investigation of the geometry of the complex characteristics of the elliptic equation. However, the integral involved here can be shown with the aid of CAUCHY's theorem to be independent of the manifold over which it is evaluated, an observation which enables us to conclude that HOPF's analyticity proof is the most natural generalization to higher dimensional space of the proof developed by LEWY in the plane.

Indeed, LEWY's method actually involves the classical PICARD successive approximation scheme for an integro-differential equation taken over a triangle of dependence bounded by an initial line in the real domain, together with two complex characteristics. Now in the two-dimensional case HOPF's cone of integration can be deformed until it collapses onto the two sides of LEWY's triangle up in the complex domain, since the integrals involved are independent of path.

Furthermore, the parametrix that HOPF uses is a familiar logarithm of geodesic distance in the plane problem, and when his cone degenerates to a triangle, as described above, the logarithm changes its value from one side to the other only by the amount πi . Thus the parametrix cancels out except for this increment, which only appears because the two characteristics bordering the triangle are branch lines of the logarithm. More careful examination of this contour integration procedure serves to establish that HOPF's integral equation really reduces here, as a limiting case, to the one required for LEWY's construction, and therefore both methods of proving the analyticity of solutions of analytic elliptic equations essentially coincide for two independent variables. We emphasize that HOPF's device for avoiding the characteristics when more variables occur can be interpreted as an appropriate rearrangement of the underlying integral equation, combined with a suitable choice of the manifold of integration, which he visualized, however, as a generalized form of the disc that is used in PICARD's scheme for an elliptic equation in the real domain.

The final problem of reflection is the most difficult of the three we have considered. Apparently it can only be settled satisfactorily in space of more than two dimensions by a detailed analysis of the complex geometry of the actual characteristics of the elliptic equation being studied. This is feasible for a linear equation, and a reflection rule can be found which depends on the structure of a reflected characteristic and of a characteristic conoid emanating from some centrally located boundary point [2]. However, determination of the precise region into which it will always be possible to continue the reflected solution is a more intricate question. Presumably the answer depends on how far one can proceed and still retain the ability to find a family of bicharacteristics within the reflected characteristic whose envelope, or focal curve, remains beyond a judiciously chosen section of the central characteristic conoid.

2. Reduction of arbitrary analytic systems to a symmetric hyperbolic normal form. We are interested in CAUCHY's problem for an analytic system of partial differential equations with data assigned analytically on a surface that is not characteristic. Standard substitutions [1] enable us to reduce the general system to a quasi-linear one

$$(1) \quad \frac{\partial u}{\partial t} = \sum_{i=1}^n A_i \frac{\partial u}{\partial z_i}$$

of the first order, with the data prescribed at $t = 0$. Here u is an m -dimensional vector of unknown functions of the $n + 1$ independent variables $t, z_1, \dots, z_n$, and the coefficients $A_i = A_i(t, z, u)$ are m by m square matrices whose elements $a_{\mu\nu}^{(i)}$ are real analytic functions of those variables and of u . The canonical form (1) is convenient for application of the CAUCHY-KOWALEWSKI power series method to determine the solution of analytic initial value problems for an arbitrary system. To start with we shall adopt the point of view that such a series solution is known to us in, say, a neighborhood of the origin, and we shall

consider its analytic continuation into the complex domain obtained by direct substitution of complex values $z_i = x_i + iy_i$ of the independent variables $z_1, \dots, z_n$ into the series.

The partial differential equations (1) remain satisfied as identities in the complex domain, and the solution $u = u(t, z)$ reduces there on the initial hyperplane $t = 0$ to the analytic continuation of the original CAUCHY data

$$(2) \quad u(0, z) = f(z).$$

Since the components of u are analytic functions of the complex variables z_i , they also fulfill the CAUCHY-RIEMANN equations

$$(3) \quad \frac{\partial u}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial u}{\partial x_j} + i \frac{\partial u}{\partial y_j} \right) = 0, \quad j = 1, \dots, n.$$

Furthermore, we have

$$(4) \quad \frac{\partial u}{\partial z_j} = \frac{1}{2} \left(\frac{\partial u}{\partial x_j} - i \frac{\partial u}{\partial y_j} \right) = \frac{\partial u}{\partial x_j}, \quad j = 1, \dots, n.$$

It follows that we can add to (1) any linear combination of the CAUCHY-RIEMANN equations (3) in order to derive for u a new system of partial differential equations

$$(5) \quad \frac{\partial u}{\partial t} = \sum_{i=1}^n A_i \frac{\partial u}{\partial x_i} + \sum_{i=1}^n B_i \left(\frac{\partial u}{\partial x_i} + i \frac{\partial u}{\partial y_i} \right),$$

where $B_1, \dots, B_n$ are m by m square matrices of arbitrary elements $b_{\mu\nu}^{(i)}$.

Our aim is to choose the matrices B_i so that (5) becomes a symmetric hyperbolic system for the real and imaginary parts of u in the $(2n + 1)$ -dimensional space with real coordinates $t, x_1, \dots, x_n, y_1, \dots, y_n$. We rewrite (5) in the form

$$(6) \quad \frac{\partial u}{\partial t} = \sum_{i=1}^n (A_i + B_i) \frac{\partial u}{\partial x_i} + \sum_{i=1}^n iB_i \frac{\partial u}{\partial y_i}$$

and recall that it is hyperbolic when the matrix

$$(7) \quad C = \sum_{i=1}^n \alpha_i (A_i + B_i) + \sum_{i=1}^n i\beta_i B_i$$

has exclusively real characteristic roots and has a full set of m linearly independent characteristic vectors for every real choice of the parameters $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$. In particular, we say that the system is symmetric hyperbolic when the matrices $A_i + B_i$ and iB_i are Hermitian for $j = 1, \dots, n$, so that C is always Hermitian too. Note that if a symmetric hyperbolic system (6) is broken down into $2m$ partial differential equations for the real and imaginary parts of the components of the unknown vector u , the resulting real matrices of coefficients are symmetric.

It is apparent that to achieve in (6) a symmetric hyperbolic normal form for the original system (1) we have only to require that

$$(8) \quad a_{\mu\nu}^{(i)} + b_{\mu\nu}^{(i)} = \bar{a}_{\nu\mu}^{(i)} + \bar{b}_{\nu\mu}^{(i)}, \quad b_{\mu\nu}^{(i)} = -\bar{b}_{\nu\mu}^{(i)}$$

for $\mu, \nu = 1, \dots, m$ and $j = 1, \dots, n$. For this purpose it suffices to set

$$(9) \quad b_{\mu\nu}^{(i)} = \frac{1}{2} \left(\bar{a}_{\nu\mu}^{(i)} - a_{\mu\nu}^{(i)} \right),$$

a formula which defines $b_{\mu\nu}^{(i)}$ analytically in its dependence on the variables $t, x_1, \dots, x_n, y_1, \dots, y_n$ and on the real and imaginary parts of u . In such a way we are led to replace the arbitrary analytic system (1) in $(n+1)$ -dimensional space by the symmetric hyperbolic one

$$(10) \quad \frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i=1}^n (A_i + \bar{A}_i) \frac{\partial u}{\partial x_i} + \frac{1}{2i} \sum_{i=1}^n (A_i - \bar{A}_i) \frac{\partial u}{\partial y_i}$$

for twice as many unknowns in a space of dimension $2n+1$, where a customary notation for transposed and for complex conjugate matrices has been assumed. For the hyperbolic system CAUCHY'S problem is well posed and can be solved by a variety of methods, including finite differences. It is illuminating to exploit this remark in order to obtain a novel approach to the CAUCHY-KOWALEWSKI theorem.

To such an end we shall suppose that the CAUCHY data (2) have been continued analytically into the complex domain and have then been used in conjunction with the symmetric hyperbolic system (10) to define u as a function of $t, x_1, \dots, x_n, y_1, \dots, y_n$. This stable CAUCHY problem for u might be treated by any of the standard procedures based on successive approximations and a priori estimates. In order to conclude that the solution satisfies the original system (1) in the real domain it is necessary, however, to establish that it fulfills the CAUCHY-RIEMANN equations (3), or, in other words, that it is analytic in its dependence on $z_1, \dots, z_n$. We shall be able to prove (3) on the basis of the uniqueness theorem for a linear homogeneous CAUCHY problem to be derived for the set of vectors

$$(11) \quad v^{(l)} = \frac{\partial u}{\partial \bar{z}_l}, \quad l = 1, \dots, n,$$

and thus we shall see that the symmetric hyperbolic system (10) serves to generate analytic solutions of (1). In this connection it is remarkable that the analyticity of the coefficients A_i in their dependence on t will not actually be required for our analysis.

We have only to apply the differential operators

$$(12) \quad \frac{\partial}{\partial \bar{z}_l} = \frac{1}{2} \left(\frac{\partial}{\partial x_l} + i \frac{\partial}{\partial y_l} \right), \quad l = 1, \dots, n,$$

to the equations (10) to obtain the system

$$(13) \quad \begin{aligned} \frac{\partial v^{(l)}}{\partial t} = & \frac{1}{2} \sum_{i=1}^n (A_i + \bar{A}_i) \frac{\partial v^{(l)}}{\partial x_i} + \frac{1}{2i} \sum_{i=1}^n (A_i - \bar{A}_i) \frac{\partial v^{(l)}}{\partial y_i} \\ & + \sum_{i=1}^n \frac{\partial \bar{A}_i'}{\partial \bar{z}_l} v^{(i)} + \sum_{i=1}^n \sum_{k=1}^m \frac{\partial A_i}{\partial u_k} \frac{\partial u}{\partial z_i} v_k^{(i)}, \quad l = 1, \dots, n, \end{aligned}$$

for all the vectors $v^{(1)}, \dots, v^{(n)}$ simultaneously, since the coefficients A_l depend analytically on $z_1, \dots, z_n$ and u . Here u_k and $v_k^{(l)}$ stand for the k th components of the vectors u and $v^{(l)}$. In addition, we conclude from (2) that

$$(14) \quad v^{(l)} = \frac{\partial f}{\partial \bar{z}_l} = 0, \quad l = 1, \dots, n,$$

at $t = 0$. Since the equations (13) are symmetric hyperbolic and homogeneous, the only solution with vanishing initial data (14) is the obvious one

$$(15) \quad v_1 = v_2 = \dots = v_n = 0.$$

This observation is equivalent to our assertion that u fulfills the CAUCHY-RIEMANN equations (3).

Our proof that the solution u of the canonical system (10) determined by analytic CAUCHY data (2) must itself be complex analytic has interesting consequences in the case where the system (1) is linear and symmetric hyperbolic to begin with. In such a situation the solutions of (1) and of (10) must coincide in the real domain, which shows that every solution of (1) with real analytic initial data has to be an analytic function of its real arguments $z_1, \dots, z_n$. The HOLMGREN uniqueness theorem and the CAUCHY-KOWALEWSKI theorem can be applied to establish that u is also analytic in its dependence on t , and since (1) has been assumed to be linear, these conclusions are valid in the large. Furthermore, the definition (9) shows that the coefficients of the partial derivatives of u with respect to $y_1, \dots, y_n$ in (10) are small when these imaginary parts of the variables $z_1, \dots, z_n$ are small. It follows that in order to calculate u from (10) in a neighborhood of the real domain we only need the analytic continuation of the data (2) in some neighborhood of the real initial hyperplane. Therefore real analyticity of the CAUCHY data is enough to ensure global real analyticity of the solution of an analytic symmetric hyperbolic system of linear partial differential equations (1). Incidentally, it is evident that the addition to (1) of inhomogeneous linear terms in u does not affect the validity of this result.

The simple rule given by (9) is not the only one that might be used to transform an arbitrary analytic system to some hyperbolic normal form in a higher dimensional space of the complex domain. Such an absence of uniqueness in our procedure can best be illustrated in the case of a single analytic elliptic partial differential equation of the second order. Suppose that coordinates have been chosen so that the equation in question can be written in the form

$$(16) \quad \frac{\partial^2 u}{\partial t^2} + \sum_{0 \leq i \leq k \leq n} \alpha_{ik} \frac{\partial^2 u}{\partial z_i \partial z_k} + \dots = 0$$

with $t = z_0$ and $\alpha_{00} \equiv 0$, where the dots indicate terms involving derivatives of at most the first order. Assuming that analytic initial data have been assigned at $t = 0$, we perform an extension to complex values $z_i = x_i + iy_i$ of the variables $z_1, \dots, z_n$. Then we express $\alpha_{ik} = \beta_{ik} + i\gamma_{ik}$ in terms of its real and imaginary

parts β_{ik} and γ_{ik} and use the CAUCHY-RIEMANN equations to rewrite (16) as an equation

$$(17) \quad \frac{\partial^2 u}{\partial t^2} = - \sum_{0 \leq i \leq k \leq n} \beta_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} - \sum_{0 \leq i \leq k \leq n} \gamma_{ik} \frac{\partial^2 u}{\partial x_i \partial y_k} + \dots$$

in the real independent variables $t, x_1, \dots, x_n, y_1, \dots, y_n$. Since u is a harmonic function of each of the pairs of variables x_i and y_i separately, we can add an arbitrary linear combination of Laplacians to (17) to derive

$$(18) \quad \frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^n \lambda_i \left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial y_i^2} \right) - \sum_{0 \leq i \leq k \leq n} \beta_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} - \sum_{0 \leq i \leq k \leq n} \gamma_{ik} \frac{\partial^2 u}{\partial x_i \partial y_k} + \dots$$

This final equation will be of the hyperbolic type when the coefficients λ_i are selected to be sufficiently large and positive. The fact that there is no unique rule specifying the parameters λ_i here makes explicit the freedom we are allowed in our choice of a hyperbolic equation equivalent to the original analytic one in the complex domain.

A special example of the phenomenon we have been describing is provided by LAPLACE's equation

$$(19) \quad \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

in the two independent variables t and z . The hyperbolic equation (18) replacing (19) in the three-dimensional space generated by real values of t and complex values of $z = x + iy$ takes the form

$$(20) \quad \frac{\partial^2 u}{\partial t^2} = (\lambda - 1) \frac{\partial^2 u}{\partial x^2} + \lambda \frac{\partial^2 u}{\partial y^2},$$

with $\lambda > 1$. The limiting case $\lambda = 1$ is of particular interest because it involves only the two independent variables t and y . Such a degenerate situation corresponds to the introduction of characteristic coordinates. In this connection observe that it is suggestive to interpret our present scheme of complex substitutions as a method of ascent to be contrasted with the classical method of descent.

3. A stable finite difference scheme applicable through the sonic line. Any of the hyperbolic equations (10), (18) or (20) can be used to set up convergent difference procedures for the numerical solution of the analytic CAUCHY problems from which they originated. For example, the finite difference approximation

$$(21) \quad \begin{aligned} u(t + \delta t, x, y) &= 2u(t, x, y) - u(t - \delta t, x, y) \\ &+ (\lambda - 1) \frac{\delta t^2}{\delta x^2} [u(t, x + \delta x, y) - 2u(t, x, y) + u(t, x - \delta x, y)] \\ &+ \lambda \frac{\delta t^2}{\delta y^2} [u(t, x, y + \delta y) - 2u(t, x, y) + u(t, x, y - \delta y)] \end{aligned}$$

to the partial differential equation (20) is stable for CAUCHY's problem when

$$(22) \quad (\lambda - 1) \frac{\delta t^2}{\delta x^2} + \lambda \frac{\delta t^2}{\delta y^2} \leq 1,$$

provided that $\lambda > 1$. On the other hand, observe that if $\lambda = 0$, so that no extension into the complex domain is involved, (21) degenerates into an unstable difference method for the initial value problem associated with LAPLACE's equation (19) in the real plane. The case $\lambda = 1$ has already been identified with the use of characteristic coordinates and is stable for $0 < \delta t \leq \delta y$. When $\lambda > 1$ it is possible to interpret the three-dimensional rule (21) as a linear combination of these two simpler procedures which were based on simultaneous consideration of only two independent variables.

The condition (22) for stability implies that if we do not take δt too large relative to δy , we can include a small contribution from the unstable real scheme without doing any damage. Our success here can be attributed to the fact that the five mesh points for (21) which lie in a plane at the level t have a convex hull which contains the intersections with this plane of the characteristics for (19) passing through the foremost mesh point $(t + \delta t, x, y)$. It is equally significant that on the right in the difference equation (21) the terms involving values of u at mesh points located in the above plane must all appear with non-negative coefficients. The fundamental requirement to be expected in a context of this kind is, of course, that the domain of dependence for a stable difference approximation should enclose in an appropriate sense the corresponding true domain of dependence for the original differential equation being studied.

The approach we have developed avoids direct reference to the characteristics at the expense of bringing in a space of high dimension in the complex domain. Thus for the inverse construction of genuinely three-dimensional flows with a detached shock wave we have to consider a symmetric hyperbolic system in no less than five independent variables. The resulting finite difference procedure would be correspondingly lacking in immediate practical importance. However, our present method is advantageous for treating the CAUCHY problem defined when we prescribe analytically the form of a detached shock wave in plane or axially symmetric flow. Indeed, the new scheme is quite independent of the type of the system of partial differential equations to which it is applied. Consequently there is no awkward distinction in its formulation between the subsonic and the supersonic flow regimes. In particular, it permits a convenient calculation of the solution of the inverse detached shock problem at the sonic line, and no difficulties are experienced in marching right through the region of mixed type there.

For an understanding of these remarks it is helpful to consider the example of TRICOMI's equation

$$(23) \quad \frac{\partial^2 u}{\partial r^2} + r \frac{\partial^2 u}{\partial \theta^2} = 0$$

with analytic CAUCHY data given along the line $r = \theta$. By introducing the new unknowns $v = \partial u / \partial r$ and $w = \partial u / \partial \theta$ and the new independent variables $t = \theta - r$ and $z = \theta + r$, we can transform (23) into the analytic system

$$(24) \quad \frac{\partial v}{\partial t} = \frac{2+t-z}{2-t+z} \frac{\partial v}{\partial z} + \frac{2(z-t)}{2-t+z} \frac{\partial w}{\partial z},$$

$$(25) \quad \frac{\partial w}{\partial t} = -\frac{4}{2-t+z} \frac{\partial v}{\partial z} + \frac{2+t-z}{2-t+z} \frac{\partial w}{\partial z}.$$

The symmetric hyperbolic normal form (10) to be associated with (24) and (25) is given by

$$(26) \quad \begin{aligned} \frac{\partial v}{\partial t} = & \frac{1}{2} \cdot \frac{2+t-x-iy}{2-t+x+iy} \left(\frac{\partial v}{\partial x} - i \frac{\partial v}{\partial y} \right) + \frac{x+iy-t}{2-t+x+iy} \left(\frac{\partial w}{\partial x} - i \frac{\partial w}{\partial y} \right) \\ & + \frac{1}{2} \cdot \frac{2+t-x+iy}{2-t+x-iy} \left(\frac{\partial v}{\partial x} + i \frac{\partial v}{\partial y} \right) - \frac{2}{2-t+x-iy} \left(\frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} \right), \end{aligned}$$

$$(27) \quad \begin{aligned} \frac{\partial w}{\partial t} = & -\frac{2}{2-t+x+iy} \left(\frac{\partial v}{\partial x} - i \frac{\partial v}{\partial y} \right) + \frac{1}{2} \cdot \frac{2+t-x-iy}{2-t+x+iy} \left(\frac{\partial w}{\partial x} - i \frac{\partial w}{\partial y} \right) \\ & + \frac{x-iy-t}{2-t+x-iy} \left(\frac{\partial v}{\partial x} + i \frac{\partial v}{\partial y} \right) + \frac{1}{2} \cdot \frac{2+t-x+iy}{2-t+x-iy} \left(\frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} \right). \end{aligned}$$

The initial value problem with data prescribed at $t = 0$ can be solved for this system provided that we stay away from the singularity along the line $t = x + 2$, $y = 0$, which is generated by a point where the original initial curve $r = \theta$ is tangent to a characteristic of (23). It is to be observed that (26) and (27) furnish in particular a solution of the CAUCHY problem throughout a full neighborhood of the origin. A sketch of the characteristics of the TRICOMI equation in the real plane shows that such a neighborhood includes regions cut off from the initial line by a cusped pair of these characteristics and by the sonic line $r = 0$. It is significant that our method yields an answer in regions of this kind which might at first sight appear to be forbidden to the solution.

Although we have seen that the elementary technique based on (10) for studying CAUCHY'S problem in the complex domain has many advantages, we should emphasize, nevertheless, that for two independent variables the finite difference scheme found by introducing characteristic coordinates can be adapted to apply at the sonic line. The appropriate modification here is, in fact, suggested by what has just been said about formulation of a hyperbolic system in a direct complex extension of the physical plane. The necessary step is to replace a grid of mesh points which follows the characteristics perfectly by one spaced more closely in the forward direction in such a way that the ratio of mesh sizes assumes reasonable values in terms of the physical coordinates.

To be more specific, let ξ and η denote characteristic coordinates for the TRICOMI equation (23) and consider the equivalent canonical system

$$(28) \quad \frac{\partial \theta}{\partial \tau} = -ir \frac{\partial r}{\partial \zeta}, \quad \frac{\partial r}{\partial \tau} = \frac{i}{r^{\frac{1}{2}}} \frac{\partial \theta}{\partial \zeta},$$

$$(29) \quad \frac{\partial v}{\partial \tau} = -ir \frac{\partial w}{\partial \zeta}, \quad \frac{\partial w}{\partial \tau} = \frac{i}{r^{\frac{1}{2}}} \frac{\partial v}{\partial \zeta},$$

where $\tau = \xi + \eta$ and $\zeta = \xi - \eta$. The initial value problem with data assigned at $\tau = 0$ is well posed for the hyperbolic equations (28), (29), and it can be solved by a difference procedure with mesh sizes $\delta\tau$ and $\delta\zeta$ in the range $0 < \delta\tau \leq \delta\zeta$. In order to avoid an accumulation of truncation errors due to the singularity of the characteristic system (28), (29) at the sonic line $r = 0$, the ratio of mesh sizes $\delta\tau/\delta\zeta$ should be chosen to approach zero there like $|r|^{\frac{1}{2}}$. Furthermore, when treating an analytic CAUCHY problem for (23) by means of (28) and (29) we are at liberty to select the distribution of initial values of one of the unknowns, say of θ , in an arbitrary fashion in its dependence on the real parameter ζ . It is possible to calculate the solution at any point in the neighborhood of the sonic line by picking such a path of initial values in a suitable manner in the complex plane. The associated two-dimensional surface in the complex domain over which the partial differential equation (23) is thus integrated can even be arranged so that it terminates in one of the apparently forbidden regions described earlier as being separated from the initial curve in the real plane by the sonic line and a pair of characteristics. The path of initial data to be used here, which need not be analytic, has to twist around so as to avoid the sonic line and still generate a solution coming out beyond it in the real domain. However, such troublesome difficulties are encountered in specifying the correct branch of $r^{\frac{1}{2}}$ to take on the resulting complex surface of integration that it is preferable to employ our more elementary three-dimensional approach, which does not involve any radicals of this kind.

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