Remarks on Stefan-Type Free Boundary Problems for Parabolic Equations

AVNER FRIEDMAN

Communicated by H. LEWY

Introduction. Free boundary problems for parabolic equations started to receive rigorous treatment in the last decade. Various methods have been devised to obtain existence and uniqueness theorems. We refer in particular to works (on Stefan problems) of J. DOUGLAS [1], I. I. KOLODNER [5], W. T. KYNER [6], W. I. MIRANKER [7], L. I. RUBINSTEIN [9] and G. SESTINI [10], [11].

Recently we have investigated various free boundary problems from a unified point of view. The first step in solving each problem consists in reducing it (by a unified technique) to a nonlinear integral equation of Volterra type $v = Tv$ (T nonlinear). The existence of a unique solution for *all* future times has been proved for a variety of Stefan problems [2], for a drop in gas [3] and for a bubble in liquid [4]. The method of proving existence and uniqueness in these papers consists of two steps: (i) a proof that T is locally a contraction, (ii) construction step-by-step of a solution in the large, deriving first appropriate *a priori* inequalities. There are however Stefan problems for which it is not known to us whether T is locally a contraction or even bounded. Such a problem arises physically if there is initially no water (only ice). In the first part of this paper we prove an existence theorem for such a problem.

In [3], [4] we have also derived the asymptotic behavior of the solution as the time t tends to infinity. In the second part of this paper we derive asymptotic formulas ($t \rightarrow \infty$) for two Stefan problems. The method has a general feature in common with the methods of [3], [4], namely, it reduces the problem of evaluating the free boundary $s(t)$ to that of estimating a certain integral. Physically, this integral measures how much heat is wasted in heating the water instead of melting the ice. The crucial step is to obtain a sharp estimate of this integral. The method used in this paper for estimating it is more effective (for the present problems) than the method of [3], [4].

We give a brief summary of the paper. In §1 we prove an existence theorem for the Stefan problem (1.1)–(1.4). The proof is given in 1.2 and various generali-

Prepared under Contract Nonr 710(16) (NR 044 004) between the Office of Naval Research and the University of Minnesota.

zations are given in 1.3; in particular, we prove an extension to nonhomogeneous equations. J. DOUGLAS [1] and W. T. KYNER [6] proved (by a different method) existence and uniqueness for quasi-linear (homogeneous) parabolic equations, with boundary conditions analogous to (1.2)–(1.4).

In §2 we derive asymptotic bounds on $s(t)$: for the system (1.1)–(1.4) in 2.1, and for the system (2.15)–(2.21) (a two-phase problem) in 2.2. Various open questions are mentioned at the end of 1.3 and 2.3.

1. AN EXISTENCE THEOREM

1.1. Statement of the main result. In this chapter we shall be interested in the following problem: Find $s(t) > 0$ and $u(x, t)$ such that

$$(1.1) \quad u_{xx} = u_t \quad \text{for } 0 < x < s(t), \quad 0 < t < \infty$$

$$(1.2) \quad u_x(0, t) = f(t) \quad \text{for } f(t) \leq 0, \quad 0 < t < \infty$$

$$(1.3) \quad u(s(t), t) = 0 \quad \text{for } 0 < t < \infty, \quad \text{and } s(0) = 0$$

$$(1.4) \quad \alpha u_x(s(t), t) = -\dot{s}(t) \equiv -\frac{ds(t)}{dt} \quad \text{for } 0 < t < \infty.$$

$x = s(t)$ is the free boundary and α is a positive constant. In the case $s(0) > 0$ we have proved existence and uniqueness in [2] for $\alpha = 1$ and for (1.2) replaced by

$$(1.2') \quad u(x, 0) = \varphi(x) \quad \text{for } 0 < x < s(0); \quad u(0, t) = f(t) \quad \text{for } 0 < t < \infty$$

where $\varphi \geq 0, f \geq 0$. As was mentioned there, the proof can easily be modified to establish existence and uniqueness also in case $u_x(0, t)$ is given instead of $u(0, t)$, and also for any $\alpha > 0$. In case $s(0) = 0$ however, the method of [2] does not seem to work for the following reason. The existence and uniqueness problem for the differential system are reduced to solving uniquely a nonlinear integral equation $v = Tv$, but in case $s(0) = 0$ we are unable to prove directly that T has a fixed point, by establishing that T is completely continuous or that T is a contraction.

We shall prove the existence of a solution u, s of (1.1)–(1.4) in the following way. We first replace the condition (1.3) by

$$(1.3') \quad u(s(t), t) = 0 \quad \text{for } 0 < t < \infty, \quad \text{and } s(0) = b > 0$$

$$(1.3'') \quad u(x, 0) = \varphi(x) \quad \text{for } 0 < x < b, \quad \text{where } \varphi(x) \equiv 0.$$

The new system has a (unique) solution $u^b(x, t), s^b(t)$. Next we take $b \rightarrow 0$ through an appropriate sequence $b = b_n$ ($n = 1, 2, \dots$) and prove that $u^b \rightarrow u, s^b \rightarrow s$ where u, s is a solution of (1.1)–(1.4).

Clearly there is no restriction in assuming that $f(t) \equiv 0$ in any interval $0 < t < \epsilon$ ($\epsilon > 0$). Indeed, in the contrary case we let t_0 be the least number for which $f(t) \equiv 0$ in any interval $t_0 < t < t_0 + \epsilon$ ($\epsilon > 0$) and set $u \equiv 0, s \equiv 0$ for $0 < t < t_0$.

We now state the main result of this chapter. Various extensions will be given later on.

Theorem 1. *Assume that $f(t)$ ($0 \leq t < \infty$) is continuous, that $f(t) \leq 0$, and that $f(t) \equiv 0$ in any interval $0 < t < \epsilon$ ($\epsilon > 0$). Then there exists a solution $u(x, t)$, $s(t)$ of the system (1.1)–(1.4) for all $t < \infty$, and the function $s(t)$ is strictly monotone increasing.*

1.2. Proof of Theorem 1. It is enough to prove the existence of a solution for $0 \leq t \leq \sigma$, for some $\sigma > 0$, since then we can apply the results of [2] and thus conclude the existence of a solution for $0 \leq t < \infty$. Furthermore, the proof of the monotonicity of $s(t)$ is similar to that given in [2] and will therefore be omitted.

To prove existence for $0 \leq t \leq \sigma$ (σ will be determined in the course of the proof) we consider the system (1.1), (1.2), (1.3'), (1.3''), (1.4) and denote the solution by u^b, s^b . Introducing the Neumann function

$$(1.5) \quad N(x, t; \xi, \tau) = K(x, t; \xi, \tau) + K(-x, t; \xi, \tau)$$

where

$$(1.6) \quad K(x, t; \xi, \tau) = \frac{1}{2\pi^{\frac{1}{2}}(t - \tau)^{\frac{1}{2}}} \exp \left\{ -\frac{(x - \xi)^2}{4(t - \tau)} \right\}$$

and using Green's formula with $u^b(\xi, \tau)$, $N(x, t; \xi, \tau)$ in the domain $0 < \xi < s^b(\tau)$, $0 < \tau < t$, we obtain

$$(1.7) \quad u^b(x, t) = - \int_0^t f(\tau) N(x, t; 0, \tau) d\tau + \int_0^t v^b(\tau) N(x, t; s^b(\tau), \tau) d\tau$$

where

$$(1.8) \quad v^b(\tau) = \frac{\partial}{\partial x} u^b(s^b(\tau), \tau).$$

Differentiating (1.7) with respect to x and letting $x \rightarrow s^b(t) = 0$ we obtain, as in [2],

$$(1.9) \quad v^b(t) = -2 \int_0^t f(\tau) N_x(s^b(t), t; 0, \tau) d\tau + 2 \int_0^t v^b(\tau) N_x(s^b(t), t; s^b(\tau), \tau) d\tau.$$

Also, by (1.4),

$$(1.10) \quad s^b(t) = b - \alpha \int_0^t v^b(\tau) d\tau.$$

It can be shown, as in [2], that if we solve the integral equation (1.9) with s^b defined by (1.10), then the right side of (1.7) is a solution of the system (1.1), (1.2), (1.3'), (1.3''), (1.4). As was already mentioned above, the existence and uniqueness of a solution v^b of (1.9) follows by the methods of [2]. For simplicity we shall restrict b from now on to be smaller than 1.

Lemma 1. For every $\sigma > 0$ there exists a positive constant M such that

$$(1.11) \quad -M < v^b(t) \leq 0 \quad \text{for } 0 \leq t \leq \sigma.$$

Proof. Using the maximum principle [8] it easily follows that $u^b \geq 0$ and hence $v^b(t) \leq 0$. To prove the left inequality of (1.11) we introduce W as the set of points t such that $v^b(\tau) > -M$ for $0 \leq \tau \leq t$. W is nonempty since $\alpha v^b(0) = -d\varphi(b)/dx = 0 > -M$. Clearly W is an open set in the open interval $(0, \sigma)$. Hence if we show that W is a closed set in the closed interval $[0, \sigma]$, it would follow that W coincides with this interval and the proof is completed. To prove that W is closed it is sufficient to show that if $v(\tau) \geq -M$ for $0 \leq \tau \leq t$, then $v(t) > -M$.

Since

$$(1.12) \quad N_z(x, t; \xi, \tau) = -\frac{x - \xi}{2(t - \tau)} K(x, t; \xi, \tau) - \frac{x + \xi}{2(t - \tau)} K(x, t; \xi, \tau)$$

and since $s^b(t) \geq s^b(\tau)$ if $t > \tau$, we see that the integrand in the second integral on the right side of (1.9) is non-negative. Hence,

$$(1.13) \quad v^b(t) \geq -2 \int_0^t |f(\tau)| \frac{s^b(t)}{t - \tau} K(s^b(t), t; 0, \tau) d\tau.$$

Using the inequality

$$(1.14) \quad \int_0^t \frac{x}{t - \tau} K(x, t; 0, \tau) d\tau \leq \frac{1}{\pi^{\frac{1}{3}}} \int_{x^{\frac{1}{3}/4t}}^{\infty} \frac{e^{-z}}{z^{\frac{1}{3}}} dz \leq 1 \quad (x \geq 0),$$

we obtain from (1.13),

$$(1.15) \quad v^b(t) \geq -2 \mathop{\text{l.u.b.}}_{0 < \tau < t} |f(\tau)|.$$

Hence, $v^b(t) > -M$ provided we take

$$(1.16) \quad M > 2 \mathop{\text{l.u.b.}}_{0 < \tau < \sigma} |f(\tau)|.$$

Lemma 2. The family of functions $\{s^b(t)\}$ is equicontinuous in the closed interval $[0, \sigma]$.

Proof. Combining Lemma 1 and (1.10) we get

$$(1.17) \quad |s^b(t) - s^b(\tau)| \leq \alpha M |t - \tau|$$

and the proof is completed.

Lemma 3. The family of functions $\{u^b(x, t)\}$ (each function in its respective domain $0 \leq x \leq s^b(t)$, $0 \leq t \leq \sigma$) is equicontinuous.

Proof. It is clearly sufficient to prove the equicontinuity of the right side of (1.7) in the rectangular domain $0 \leq x \leq A$, $0 \leq t \leq \sigma$, for any fixed $A > 0$. We start with the second integral on the right side of (1.7) and denote it by $I(x, t)$. We first prove equicontinuity with respect to x .

For any two points x, x' we introduce $\lambda = |x - x'|^{2\epsilon}$ where $\epsilon > 0$ is any fixed number. If $\lambda < t$ then we write

$$(1.18) \quad I(x, t) - I(x', t) = I_1 + I_2$$

where

$$I_1 = \int_{t-\lambda}^t v^b(\tau) N(x, t; s^b(\tau), \tau) d\tau - \int_{t-\lambda}^t v^b(\tau) N(x', t; s^b(\tau), \tau) d\tau$$

$$I_2 = \int_0^{t-\lambda} v^b(\tau) [N(x, t; s^b(\tau), \tau) - N(x', t; s^b(\tau), \tau)] d\tau.$$

Estimating each term in I_1 separately and using Lemma 1, we get

$$(1.19) \quad |I_1| \leq 4M\lambda^{\frac{1}{2}} = 4M |x - x'|^{\epsilon}.$$

To estimate I_2 we use the mean value theorem and the inequality $ye^{-y} \leq \text{constant}$. We obtain

$$(1.20) \quad |I_2| \leq A_1 M \int_0^{t-\lambda} \frac{|x - x'|}{t - \tau} d\tau \leq A_2 M |x - x'| \log \frac{1}{\lambda}$$

where A_i are used to denote appropriate universal constants. Inserting (1.19), (1.20) into (1.18) and taking, for instance, $\epsilon = 1$ we conclude that

$$(1.21) \quad |I(x, t) - I(x', t)| \leq A_3 M |x - x'|^{\theta} \quad \text{for any } 0 < \theta < 1$$

and A_3 depends on θ .

If $\lambda = |x - x'|^{2\epsilon} \geq t$ then we take $I_2 \equiv 0$ and in the I_1 above replace $t - \lambda$ by 0. Instead of (1.19) we then obtain

$$|I_1| \leq 4Mt^{\frac{1}{2}} \leq 4M |x - x'|^{\epsilon}$$

and, hence, (1.21) holds with $\theta = 1$.

(1.21) establishes the equicontinuity of $I(x, t)$ with respect to x . The proof with respect to t is analogous and is therefore omitted. The proof of the equicontinuity of the first integral on the right side of (1.7) is the same as the above proof for I is we replace $v^b(\tau)$ by $f(\tau)$ and $s^b(\tau)$ by 0.

Combining Lemmas 2, 3 and using a theorem of ARZELÁ we conclude that *there exists a sequence $b = b_n \rightarrow 0$ such that*

$$(1.22) \quad u^b(x, t) \rightarrow u(x, t), \quad s^b(t) \rightarrow s(t)$$

uniformly for $0 \leq x \leq s^b(t)$, $0 \leq t \leq \sigma$. The function $s(t)$ is continuous and monotone nondecreasing, the function $u(x, t)$ is continuous for $0 \leq x \leq s(t)$, $0 \leq t \leq \sigma$, and

$$(1.23) \quad u(s(t), t) = 0 \quad \text{for } 0 \leq t \leq \sigma.$$

We proceed to show that u, s form a solution of (1.1)–(1.4).

Lemma 4. *The function $s(t)$ is positive for $t > 0$.*

Proof. Let

$$(1.24) \quad \alpha \text{ l.u.b. } u^b(x, t) = H, \quad \text{where } 0 \leq x \leq s^b(t), \quad 0 \leq t \leq \sigma.$$

Integrating the equation $\partial^2 u^b / \partial \xi^2 = \partial u^b / \partial \tau$ in the domain $0 < \xi < s^b(\tau)$, $0 < \tau < t$, and using the initial and boundary conditions which u^b satisfies, we obtain

$$(1.25) \quad \frac{s^b(t)}{\alpha} - \frac{b}{\alpha} + \int_0^t f(\tau) d\tau = - \int_0^{s^b(t)} u^b(x, t) dx.$$

Using (1.24) we have

$$(1.26) \quad \int_0^{s^b(t)} u^b(x, t) dx \leq \frac{H}{\alpha} s^b(t).$$

Substituting (1.26) into (1.25) we get, upon taking $b = b_n \rightarrow 0$,

$$(1.29) \quad s(t) \geq \frac{\alpha}{H+1} \int_0^t |f(\tau)| d\tau > 0$$

and the proof is completed.

Consider now the family $\{v^b(t)\}$ for $b = b_n$. Since it is uniformly bounded, the L^2 -norms are of course also uniformly bounded and hence we can find a subsequence which converges weakly in $L^2(0, \sigma)$ to a function $v(t)$. For simplicity we denote this subsequence again by b_n . We thus have:

$$(1.30) \quad \lim_{b=b_n \rightarrow 0} \int_0^\sigma v^b(t) \psi(t) dt = \int_0^\sigma v(t) \psi(t) dt$$

for every function ψ in $L^2(0, \sigma)$. Taking in particular ψ to be the characteristic function of $(0, t)$, we get

$$(1.31) \quad \lim_{b=b_n \rightarrow 0} \int_0^t v^b(\tau) d\tau = \int_0^t v(\tau) d\tau.$$

By (1.30) it follows that for every fixed t and ϵ , where $0 < \epsilon < t < \sigma$,

$$(1.32) \quad \lim_{b=b_n \rightarrow 0} \int_0^{t-\epsilon} v^b(\tau) N_x(s(t), t; s(\tau), \tau) d\tau = \int_0^{t-\epsilon} v(\tau) N_x(s(t), t; s(\tau), \tau) d\tau.$$

Also, since $|v^b(\tau)| \leq M$,

$$(1.33) \quad \lim_{b=b_n \rightarrow 0} \int_0^{t-\epsilon} v^b(\tau) [N_x(s^b(t), t; s^b(\tau), \tau) - N_x(s(t), t; s(\tau), \tau)] d\tau = 0.$$

Here we made use of (1.22).

Next, using (1.14), (1.17) to estimate the two integrals corresponding to the two terms of N_x , we get

$$(1.34) \quad \left| \int_{t-\epsilon}^t v^b(\tau) N_x(s^b(t), t; s^b(\tau), \tau) d\tau \right| \leq \alpha M^2 \epsilon^{\frac{1}{2}} + A_0 M \int_\gamma^\infty e^{-y} \frac{dy}{y^{\frac{1}{2}}}$$

where $\gamma = [s^b(t)]^2 / 4\epsilon$, and A_0 is a universal constant.

Finally, for any $0 < \delta < \epsilon$ we have, by (1.30),

$$\begin{aligned}
 & \int_{t-\epsilon}^{t-\delta} |v(\tau)N_x(s(t), t; s(\tau), \tau)| d\tau \\
 & \leq \limsup_{b=b_n \rightarrow 0} \int_{t-\epsilon}^{t-\delta} |v^b(\tau)N_x(s(t), t; s(\tau), \tau)| d\tau \\
 (1.35) \quad & \leq M \int_{t-\epsilon}^{t-\delta} |N_x(s(t), t; s(\tau), \tau)| d\tau \\
 & \leq \alpha M^2 \epsilon^{\frac{1}{2}} + A_0 M \int_{\beta}^{\infty} e^{-y} \frac{dy}{y^{\frac{1}{2}}}
 \end{aligned}$$

where $\beta = [s(t)]^2/4\epsilon$. The last inequality follows (as (1.34)) by using (1.14) and (1.17) with s^b replaced by s . Note also that the first inequality in (1.35) follows by applying (1.30) with $\psi = [\text{sgn}(vN_x)]N_x$. Since the right side of (1.35) does not depend on δ and tends to zero as $\epsilon \rightarrow 0$, the integral on the left side of (1.35) is convergent for $\delta = 0$ (by Cauchy's criterion) and is bounded by the right side of (1.35).

We now use (1.22), (1.29) and conclude that if $t \geq \eta > 0$ (for any $\eta > 0$, $\eta < \sigma$) then the right sides of (1.34) and (1.35) tend to zero as $\epsilon \rightarrow 0$, uniformly with respect to t and $b = b_n$. Combining this remark with (1.32), (1.33) we have thus proved that, as $b = b_n \rightarrow 0$, the second integral on the right side of (1.9) converges to

$$\int_0^t v(\tau)N_x(s(t), t; s(\tau), \tau) d\tau,$$

in any interval $\eta \leq t \leq \sigma$ ($\eta > 0$).

A similar assertion can now be stated and easily proved for the first integral on the right side of (1.9). Using (1.9) we conclude that as $b = b_n \rightarrow 0$, $v^b(t)$ converges in any interval $\eta \leq t \leq \sigma$ ($\eta > 0$) to a function which we denote by $v^0(t)$. Since $|v^b(t)| \leq M$ for $0 < t < \sigma$, we can apply Lebesgue's Theorem. It follows that for every function $\psi(\tau)$ in $L^2(0, \sigma)$:

$$\lim_{b=b_n \rightarrow 0} \int_0^\sigma v^b(\tau)\psi(\tau) d\tau = \int_0^\sigma v^0(\tau)\psi(\tau) d\tau.$$

Taking ψ to be the characteristic function of the interval $(0, t)$ and comparing with (1.31) we find that

$$\int_0^t [v(\tau) - v^0(\tau)] d\tau = 0$$

for all $0 < t \leq \sigma$. Hence $v(\tau) = v^0(\tau)$ almost everywhere.

Taking $b = b_n \rightarrow 0$ in (1.9) we conclude that, for almost all t , $0 < t < \sigma$,

$$(1.36) \quad v(t) = -2 \int_0^t f(\tau)N_x(s(t), t; 0, \tau) d\tau + 2 \int_0^t v(\tau)N_x(s(t), t; s(\tau), \tau) d\tau.$$

We now change the definition of $v(t)$ on a set of measure zero and define it there to be equal to the right side of (1.36) (that is, to $v^0(t)$). Denoting the "corrected" function again by $v(t)$, we conclude that $v(t)$ is a continuous solution of (1.36).

Finally, letting $b = b_n \rightarrow 0$ in (1.10) and using (1.31) we find that the $s(t)$ in (1.36) satisfies

$$(1.37) \quad s(t) = -\alpha \int_0^t v(\tau) d\tau.$$

This completes the proof of Theorem 1.

1.3. Generalizations. (a) General equations. Theorem 1 can be generalized to the parabolic equation

$$Lu \equiv a(x, t)u_{xx} + b(x, t)u_x + c(x, t)u - u_t = 0.$$

Indeed, the only essential difference is that the Neumann function takes a more complicated form. In proving Lemma 4 we have to make the following modification: An analogue of (1.25) is obtained by writing Lu in the form

$$(au_x)_x + (\tilde{b}u)_x + \tilde{c}u.$$

(b) Nonhomogeneous equations. The results of [2] ($b > 0$) can be generalized to nonhomogeneous equations

$$(1.38) \quad u_{xx} - u_t = h(x, t)$$

(h Hölder continuous in x)

provided $h \leq 0$. Indeed, noting that in this case the free boundary $x = s(t)$ is still an increasing function of t , all one needs to verify is that the integral

$$V(t; s) = \int_0^t \int_0^{s(\tau)} N_x(s(t), t; \xi, \tau) h(\xi, \tau) d\xi d\tau$$

satisfies the two properties:

(i) Define

$$||g|| = \text{l.u.b.}_{0 < t < \sigma} |g(t)|.$$

Let $||v|| \leq M$ ($M > 0$) and let $s(t)$ be defined by $s(t) = b - \alpha \int v(\tau) d\tau$. Then

$$(1.39) \quad |V(t; s)| \leq A |h|_0 \sigma^{\frac{1}{2}}$$

(A depends only on b, α, M) where $|h|_0 = \text{l.u.b. } |h|$.

(ii) If $||v - v^0|| = \epsilon$ and s^0 depends on v^0 in the same manner in which s depends on v , then

$$(1.40) \quad |V(t; s) - V(t; s^0)| \leq A' |h|_1 \sigma^{\frac{1}{2}} \epsilon$$

(A' depends only on b, α, M and $\sigma \leq \sigma_0$, σ_0 depending on M) where

$$|h|_1 = |h|_0 + |\partial h / \partial x|_0.$$

The proof of (i) is very simple. The proof of (ii) is given in the Appendix to the present chapter, in the case that h and $\partial h/\partial x$ are continuous functions.

We can now extend the proof of Theorem 1 to the nonhomogeneous equation (1.38) provided $h \leq 0$. We conclude that *Theorem 1 remains true if we replace (1.1) by (1.38), provided $h \leq 0$ and provided h and $\partial h/\partial x$ are continuous for $0 \leq x < \infty, 0 < t < \infty$.*

(c) **Small α .** In physical experience α is a small number. Thus, for water-ice α is approximately $1/80$. As in [3], [4] we can prove existence and uniqueness theorems also for Stefan problems ($b > 0$) with nonmonotone free boundary $s(t)$, provided α is sufficiently small. In particular we can solve nonhomogeneous equations (1.38) without any restriction on $\text{sgn } [h(x, t)]$; the range of the permissible α 's essentially covers the range arising in physics. We shall not derive here estimates on the permissible α 's. We also mention that the methods of [3], [4] can be extended to prove that the solution u, s is analytic in α . Thus, in particular,

$$s(t) \equiv s(t, \alpha) = \sum_{n=0}^{\infty} s_n(t) \alpha^n$$

$$\dot{s}(t) \equiv \dot{s}(t, \alpha) = \sum_{n=0}^{\infty} \dot{s}_n(t) \alpha^n$$

$$u(x, t) = U(y, t, \alpha) = \sum_{n=0}^{\infty} U_n(y, t) \alpha^n, \quad \text{where } y = x - s(t, \alpha),$$

provided α is sufficiently small.

(d) **Open questions.** (1) Under the assumptions of Theorem 1, to prove uniqueness by considerations which involve our integral equation $v = Tv$.

(2) Prove existence and uniqueness when (1.2) is replaced by

$$(1.41) \quad u(0, t) = f(t) \quad \text{for } f(t) \geq 0, \quad 0 < t < \infty.$$

(3) Prove that the solution depends continuously on the data $f(t)$. (This is obvious in the case $b > 0$).

The above questions can be answered in part if one could prove that the free boundary depends monotonically (in some broad sense) on the data, since then one could compare the given system with a simpler system that can be solved explicitly.

Added in proof. W. T. KYNER has recently (*Quarterly of Applied Math.*, **17** (1959), 307–310) modified the method of [1], [6] and derived existence and uniqueness of solutions of (1.1)–(1.4) by successive approximations. His proof can be used to establish (3).

APPENDIX: PROOF OF (1.40).

$N_x(x, t; \xi, \tau)$ consists of two terms, $K_x(x, t; \xi, \tau)$ and $K_x(-x, t; \xi, \tau)$. Denote by V_1 the integral V when N_x is replaced by the first term and by V_2 the integral

V when N_x is replaced by the second term. Since $s(t) \geq \frac{1}{2}b > 0$, $s^0(t) \geq \frac{1}{2}b > 0$, if $\sigma \leq \sigma_0$, where $\sigma_0 = \sigma_0(M)$ is sufficiently small, it is easily seen that

$$(1.42) \quad |V_2(t; s) - V_2(t; s^0)| \leq H_0 |h|_0 M \sigma \epsilon$$

where the H_i are used to denote constants depending only on b, α, M

We next introduce the function

$$\lambda(t) = \min \{s(t), s^0(t)\}$$

and the sets

$$B = \{(\xi, \tau); \lambda(\tau) < \xi < s(\tau), 0 < \tau < t\}$$

$$C = \{(\xi, \tau); 0 < \xi < \lambda(\tau), 0 < \tau < t\},$$

and write

$$(1.43) \quad V_1(t; s) = \int_0^t \int_0^{s(\tau)} = \iint_B + \iint_C = V_{11}(s) + V_{12}(s).$$

We have

$$(1.44) \quad \begin{aligned} |V_{11}(s)| &\leq \left| \int_0^t \int_{\lambda(\tau)}^{s(\tau)} \frac{\partial}{\partial \xi} \{K(s(t), t; \xi, \tau) h(\xi, \tau)\} d\xi d\tau \right| \\ &\quad + \left| \int_0^t \int_{\lambda(\tau)}^{s(\tau)} \frac{\partial h(\xi, \tau)}{\partial \xi} K(s(t), t; \xi, \tau) d\xi d\tau \right| \\ &= \int_0^t |K(s(t), t; s(\tau), \tau) h(s(\tau), \tau) - K(s(t), t; \lambda(\tau), \tau) h(\lambda(\tau), \tau)| d\tau \\ &\quad + |h|_1 \int_0^t \int_{\lambda(\tau)}^{s(\tau)} K(s(t), t; \xi, \tau) d\xi d\tau \\ &\leq H_1 |h|_1 \epsilon \sigma^{\frac{1}{2}} + |h|_0 \int_0^t |K(s(t), t; s(\tau), \tau) - K(s(t), t; \lambda(\tau), \tau)| d\tau. \end{aligned}$$

We split the integral on the right side of (1.44) into

$$(1.45) \quad \int_0^{t-\delta} + \int_{t-\delta}^t \text{ where } \delta = \epsilon^2 \sigma.$$

The second integral is bounded by

$$\int_{t-\delta}^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} = 2\delta^{\frac{1}{2}} = 2\epsilon\sigma^{\frac{1}{2}}.$$

To estimate the first integral we use the inequalities

$$(1.46) \quad |s(t) - s^0(t)| \leq \alpha \epsilon t, \quad |s(t) - s(\tau)| \leq \alpha M(t - \tau).$$

The integrand can be written as

$$(1.47) \quad K(s(t), t; s(\tau), \tau) \left[1 - \exp \left\{ \frac{(s(t) - s(\tau))^2 - (s(t) - \lambda(\tau))^2}{4(t - \tau)} \right\} \right].$$

Using (1.46) and the elementary inequality $|1 - e^x| \leq \text{const. } |x|$ for $|x|$ small, we find that the expression (1.47) is bounded by

$$H_2 \frac{1}{(t - \tau)^{\frac{3}{2}}} \left[\frac{\alpha^2 \epsilon^2 t^2}{t - \tau} + \alpha^2 \epsilon M t \right]$$

(note that $t - \tau \geq \epsilon^2 \sigma$, $t \leq \sigma$ and σ is small). We easily find that the first integral in (1.45) is bounded by

$$H_3 (\alpha^2 \epsilon \sigma^{\frac{3}{2}} + \alpha^2 \epsilon M \sigma^{\frac{3}{2}}).$$

Combining the estimates of the two integrals of (1.45), we get

$$(1.48) \quad |V_{11}(s)| \leq H_4 |h|_1 \epsilon \sigma^{\frac{3}{2}}.$$

Similarly, decomposing $V_1(t, s^0)$ analogously to (1.43), we have

$$(1.49) \quad |V_{11}(s^0)| \leq H_4 |h|_1 \epsilon \sigma^{\frac{3}{2}}.$$

It remains to estimate $V_{12}(s) - V_{12}(s^0)$. Clearly we may assume that $s^0(t) \geq s(t)$.

$$(1.50) \quad \begin{aligned} & |V_{12}(s) - V_{12}(s^0)| \\ & \leq \left| \int_0^t \int_0^{\lambda(\tau)} \frac{\partial}{\partial \xi} \{ [K(s(t), t; \xi, \tau) - K(s^0(t), t; \xi, \tau)] h(\xi, \tau) \} d\xi d\tau \right| \\ & \quad + \left| \int_0^t \int_0^{\lambda(\tau)} \frac{\partial h(\xi, \tau)}{\partial \xi} [K(s(t), t; \xi, \tau) - K(s^0(t), t; \xi, \tau)] d\xi d\tau \right| \\ & \leq \left| \int_0^t [K(s(t), t; \lambda(\tau), \tau) - K(s^0(t), t; \lambda(\tau), \tau)] h(\lambda(\tau), \tau) d\tau \right| \\ & \quad + \left| \int_0^t [K(s(t), t; 0, \tau) - K(s^0(t), t; 0, \tau)] h(0, \tau) d\tau \right| \\ & \quad + |h|_1 \int_{s(t)}^{s^0(t)} \int_0^t \int_0^{\lambda(\tau)} |K_t(\zeta, t; \xi, \tau)| d\xi d\tau d\zeta. \end{aligned}$$

The first integral on the right side is estimated by the method used above to estimate the last integral on the right side of (1.44). The second integral is easily estimated noting that $s(t) \geq \frac{1}{2} b$, $s^0(t) \geq \frac{1}{2} b$. The last integral on the right side of (1.50) is bounded by

$$\begin{aligned} & \int_{s(t)}^{s^0(t)} \int_0^t \int_0^{\lambda(\tau)} \frac{|\zeta - \xi|}{2(t - \tau)} K(\zeta, t; \xi, \tau) d\xi d\tau d\zeta \\ & \leq 2 \int_{s(t)}^{s^0(t)} \int_0^t \frac{d\tau}{2(t - \tau)^{\frac{3}{2}}} d\zeta = 2 |s^0(t) - s(t)| \sigma^{\frac{1}{2}}. \end{aligned}$$

We then obtain

$$(1.51) \quad |V_{12}(s) - V_{12}(s^0)| \leq H_5 |h|_1 \epsilon \sigma^{\frac{3}{2}}.$$

Combining the estimates (1.42), (1.48), (1.49), (1.51), the proof of (1.40) is completed.

2. ASYMPTOTIC BEHAVIOR OF SOLUTIONS

In this chapter we shall derive asymptotic formulas for the free boundary $x = s(t)$ as $t \rightarrow \infty$. Such formulas have been derived by the author [3], [4] for increasing drops and bubbles. The method used here has the same basic feature as the method of [3], [4], namely, we first integrate the differential equation and find an expression for $s(t)$. Each term in this expression is then trivially estimated except for the term $I \equiv \int_0^{s(t)} u(x, t) dx$. The crucial step in this method is in estimating I . In this chapter we use a new method to estimate I .

We shall consider two Stefan systems. The first one is the system (1.1)–(1.4) (the same results are then easily obtained also for the system (1.1), (1.2), (1.3'), (1.3''), (1.4) where $\varphi(\xi)$ is any continuous function). The second one is a system in which the temperature of the ice is not identically zero and we thus have two unknown temperature functions (in two different domains) and one free boundary. We call the first problem a *one-phase Stefan problem* and the second problem a *two-phase Stefan problem*.

2.1. One-phase Stefan problem. We integrate equation (1.1) in the domain $0 < \xi < s(\tau)$, $0 < \tau < t$ and obtain

$$(2.1) \quad \frac{s(t)}{\alpha} = \int_0^t |f(\tau)| d\tau - \int_0^{s(t)} u(x, t) dx.$$

We proceed to estimate

$$(2.2) \quad I = \int_0^{s(t)} u(x, t) dx.$$

In the first place, $u \geq 0$ by the maximum principle. Hence

$$(2.3) \quad I \geq 0.$$

To find an upper bound on I we first find an upper bound on u . Let $w(x, t)$ be a solution of the heat equation in the domain $0 < x < \infty$, $0 < t < \infty$, satisfying the conditions:

$$(a) \quad w_x(0, t) = f(t) \quad \text{for } 0 < t < \infty,$$

$$(b) \quad w(x, 0) = 0 \quad \text{for } 0 < x < \infty,$$

$$(c) \quad \text{as } x \rightarrow \infty, w(x, t) \text{ and } w_x(x, t) \text{ remain bounded uniformly in } t \text{ in finite intervals.}$$

By the maximum principle w is nonnegative and, in particular, $w \geq 0$ for $x = s(t)$. Using again the maximum principle we conclude that $w \geq u$. Hence,

$$(2.4) \quad I = \int_0^{s(t)} u(x, t) dx \leq \int_0^{s(t)} w(x, t) dx.$$

Using Green's formula with $N(x, t; \xi, \tau)$ and $w(\xi, \tau)$ in the domain $0 < \xi < \infty$, $0 < \tau < t$, we obtain for w the integral representation:

$$(2.5) \quad w(x, t) = \int_0^t |f(\tau)| N(x, t; \xi, \tau) d\tau.$$

Substituting (2.5) into (2.4), we get

$$(2.6) \quad I \leq s(t) \int_0^t |f(\tau)| \frac{d\tau}{(t-\tau)^{\frac{1}{2}}}.$$

We are now ready to derive bounds on $s(t)$ for $t \rightarrow \infty$, under appropriate assumptions on $f(t)$.

Case (i): As $t \rightarrow \infty$, $t^\delta f(t) \rightarrow -\gamma$ where $\gamma > 0$, $\frac{1}{2} < \delta < 1$. We have

$$(2.7) \quad 0 \leq I \leq \text{const. } t^{1-\delta} s(t) = o(1)s(t) \quad (o(1) \rightarrow 0 \text{ as } t \rightarrow \infty).$$

Hence, by (2.1),

$$(2.8) \quad \lim_{t \rightarrow \infty} \frac{s(t)}{t^{1-\delta}} = \frac{\alpha\gamma}{1-\delta}.$$

If $\delta = \frac{1}{2}$ then

$$(2.9) \quad \frac{s(t)}{t^{\frac{1}{2}}} = 2\alpha\gamma(1 + o(1)) \quad \text{where } o(1) \rightarrow 0 \text{ as } \alpha \rightarrow 0, \quad t \rightarrow \infty.$$

Case (ii): As $t \rightarrow \infty$, $t^\delta f(t) \rightarrow -\gamma$ where $\gamma > 0$, $0 \leq \delta < \frac{1}{2}$. We have, for any $\epsilon > 0$,

$$(2.10) \quad I \leq \frac{1}{4\epsilon} s^2(t) + \epsilon \left[\int_0^t |f(\tau)| \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} \right]^2.$$

Substituting in (2.1) we get the asymptotic inequality ($t \rightarrow \infty$)

$$(2.11) \quad \frac{1}{4\epsilon} s^2(t) + \frac{s(t)}{\alpha} \geq \frac{\gamma}{1-\delta} t^{1-\delta} - \epsilon\beta\gamma^2 t^{1-2\delta}$$

where $\beta > 0$ is a constant depending only on δ . Taking ϵ such that

$$\epsilon\beta\gamma t^{-\delta} = \frac{1}{2}(1-\delta)^{-1},$$

the inequality (2.11) takes the form

$$(2.12) \quad \frac{1}{2}(1-\delta)\beta\gamma t^{-\delta} s^2(t) + \frac{1}{\alpha} s(t) \geq \frac{\gamma}{2(1-\delta)} t^{1-\delta}.$$

It follows that $s(t) \rightarrow \infty$ as $t \rightarrow \infty$. Furthermore, for every t , either $s(t) \geq \text{const. } t^{1-\delta}$, or $s^2(t) \geq \text{const. } t$. Since $\delta < \frac{1}{2}$, the first possibility implies the second one (for $t \rightarrow \infty$). Thus we have:

$$(2.13) \quad \liminf_{t \rightarrow \infty} \frac{s(t)}{t^{\frac{1}{2}}} \geq \rho \quad \text{where } \frac{1}{\rho} = (1-\delta)^2\beta.$$

To get an upper bound, we substitute $I \geq 0$ in (2.1) and obtain:

$$(2.14) \quad \limsup_{t \rightarrow \infty} \frac{s(t)}{t^{1-\delta}} \leq \frac{\alpha\gamma}{1-\delta}.$$

Remark. One can obtain in the same manner various asymptotic bounds on $s(t)$, under various corresponding assumptions on the behavior of $f(t)$ as $t \rightarrow \infty$. The *same* asymptotic formulas can be derived for the more general system (1.1), (1.2), (1.3'), (1.3''), (1.4) with arbitrary φ . We note that our results are incomplete in case (ii) above.

2.2. Two-phase Stefan problem. We shall consider the asymptotic behavior of $s(t)$ for the system:

$$(2.15) \quad \frac{\partial^2 w_1}{\partial x^2} = a_1^2 \frac{\partial w_1}{\partial t} \quad \text{for} \quad -\infty < x < s(t), \quad 0 < t < \infty$$

$$(2.16) \quad \frac{\partial^2 w_2}{\partial x^2} = a_2^2 \frac{\partial w_2}{\partial t} \quad \text{for} \quad s(t) < x < \infty, \quad 0 < t < \infty$$

$$(2.17) \quad w_1(x, 0) = \psi_1(x) \geq 0 \quad \text{for} \quad -\infty < x < 0$$

$$(2.18) \quad w_2(x, 0) = \psi_2(x) \leq 0 \quad \text{for} \quad 0 < x < \infty$$

$$(2.19) \quad w_1(s(t), t) = w_2(s(t), t) = 0 \quad \text{for} \quad 0 < t < \infty, \quad \text{and} \quad s(0) = 0$$

$$(2.20) \quad \dot{s}(t) = -k_1 \frac{\partial w_1(s(t), t)}{\partial x} + k_2 \frac{\partial w_2(s(t), t)}{\partial x} \quad \text{for} \quad 0 < t < \infty,$$

where k_1, k_2, a_1, a_2 are positive constants.

We shall make the assumptions:

$$\begin{aligned} &\text{as } x \rightarrow -\infty, \quad \psi_1(x) \rightarrow \gamma_1 \geq 0, \quad d\psi_1(x)/dx \rightarrow 0, \\ (\Psi) \quad &\text{as } x \rightarrow \infty, \quad \psi_2(x) \rightarrow \gamma_2 \leq 0, \quad d\psi_2(x)/dx \rightarrow 0, \\ &\int_{-\infty}^0 |\psi_1(x) - \gamma_1| dx < \infty, \quad \int_0^{\infty} |\psi_2(x) - \gamma_2| dx < \infty. \end{aligned}$$

We then require that w_1, w_2 satisfy the additional conditions:

$$(2.21) \quad \text{as } x \rightarrow \pm \infty, w_i \text{ and } \partial w_i / \partial x \ (i = 1, 2) \text{ remain bounded uniformly with respect to } t \text{ in finite intervals.}$$

It is important for our method that right at the beginning we perform the transformation

$$(2.22) \quad u_i = w_i - \gamma_i \quad (i = 1, 2).$$

The system (2.15)–(2.21) then takes the form:

$$(2.23) \quad \frac{\partial^2 u_1}{\partial x^2} = a_1^2 \frac{\partial u_1}{\partial t} \quad \text{for} \quad -\infty < x < s(t), \quad 0 < t < \infty$$

$$(2.24) \quad \frac{\partial^2 u_2}{\partial x^2} = a_2^2 \frac{\partial u_2}{\partial t} \quad \text{for } s(t) < x < \infty, \quad 0 < t < \infty$$

$$(2.25) \quad u_1(x, 0) = \varphi_1(x) \equiv \psi_1(x) - \gamma_1 \quad \text{for } -\infty < x < 0$$

$$(2.26) \quad u_2(x, 0) = \varphi_2(x) \equiv \psi_2(x) - \gamma_2 \quad \text{for } 0 < x < \infty$$

$$(2.27) \quad u_i(s(t), t) = -\gamma_i, \quad (i = 1, 2) \quad \text{for } 0 < t < \infty \quad \text{and } s(0) = 0$$

$$(2.28) \quad \dot{s}(t) = -k_1 \frac{\partial u_1(s(t), t)}{\partial x} + k_2 \frac{\partial u_2(s(t), t)}{\partial x} \quad \text{for } 0 < t < \infty$$

$$(2.29) \quad \text{as } x \rightarrow \pm \infty, u_i \text{ and } \partial u_i / \partial x \text{ (} i = 1, 2 \text{) remain bounded uniformly with respect to } t \text{ in finite intervals.}$$

The free boundary $s(t)$ is not necessarily a monotone function of t . However, the methods of [3], [4] can be used to establish existence and uniqueness of a solution u_1, u_2, s (under the assumption (Ψ)) provided k_i and

$$(2.30) \quad \alpha_i = a_i^2 k_i \quad (i = 1, 2)$$

are sufficiently small. In the physical experience k_i and α_i are indeed small enough to guarantee existence and uniqueness (for ice-water, for instance, $k_i < 5/8000$, $\alpha_i < 1/80$, $i = 1, 2$). We now proceed to derive an asymptotic bound on $s(t)$.

We first note that by the maximum principle, $\partial w_i(s(t), t) / \partial x \leq 0$. For brevity, we introduce the functions

$$(2.31) \quad \lambda_i(t) = -\int_0^t \frac{\partial u_i(s(\tau), \tau)}{\partial x} d\tau = \int_0^t \left| \frac{\partial u_i(s(\tau), \tau)}{\partial x} \right| d\tau \quad (i = 1, 2).$$

The $\lambda_i(t)$ are increasing functions in t . Integrating (2.23) we get

$$(2.32) \quad \lambda_1(t) = a_1^2 \int_{-\infty}^0 \varphi(x) dx - a_1^2 \gamma_1 s(t) - a_1^2 \int_{-\infty}^{s(t)} u_1(x, t) dx.$$

Here we made use of the fact that $\partial u_1(x, t) / \partial x \rightarrow 0$ uniformly in t in finite intervals as $x \rightarrow -\infty$. This fact can be proved by representing $u_1(x, t)$ in terms of its boundary data (using Green's identity with u_1 and a fundamental solution), then differentiating with respect to x and using the assumption that $d\varphi_1(x)/dx \rightarrow 0$ as $x \rightarrow -\infty$. We proceed to estimate

$$(2.33) \quad I \equiv a_1^2 \int_{-\infty}^{s(t)} u_1(x, t) dx.$$

Bounds on I . We introduce the function

$$(2.34) \quad \sigma(t) = \text{l.u.b.}_{0 < \tau < t} |s(\tau)|$$

and compare u_1 with two functions, z_1 and z_2 . z_1 is a solution of (2.23) in the

domain $-\infty < \xi < -\sigma(t)$, $0 < \tau < t$ which satisfies the condition (2.29) and, in addition,

$$z_1(x, 0) = \varphi_1(x) \quad \text{for } -\infty < x < -\sigma(t)$$

$$z_1(-\sigma(t), \tau) = -\gamma_1 \quad \text{for } 0 < \tau < t.$$

z_2 is a solution of (2.23) in the domain $-\infty < \xi < \sigma(t)$, $0 < \tau < t$ which satisfies the condition (2.29) and, in addition,

$$z_2(x, 0) = \varphi_1(x) \quad \text{for } -\infty < x < 0, \quad z_2(x, 0) = 0 \quad \text{for } 0 < x < \sigma(t)$$

$$z_2(\sigma(t), \tau) = -\gamma_1 \quad \text{for } 0 < \tau < t.$$

By the maximum principle we conclude that

$$(2.35) \quad z_1(x, t) \leq u_1(x, t) \leq z_2(x, t).$$

We next introduce the fundamental solution of (2.23)

$$(2.36) \quad K_1(x, t; \xi, \tau) = \frac{a_1}{2\pi^{\frac{1}{2}}(t-\tau)^{\frac{1}{2}}} \exp \left\{ -\frac{a_1^2(x-\xi)^2}{4(t-\tau)} \right\} = a_1 K(a_1^2 x, t; a_1^2 \xi, \tau).$$

Using Green's identity with u_1 , $G_1(x + \sigma(t), t; \xi + \sigma(t), \tau)$ where

$$(2.37) \quad G_1(x, t; \xi, \tau) = K_1(x, t; \xi, \tau) - K_1(-x, t; \xi, \tau),$$

we obtain

$$(2.38) \quad \begin{aligned} a_1^2 z_1(x, t) &= a_1^2 \int_{-\infty}^{-\sigma(t)} G_1(x + \sigma(t), t; \xi + \sigma(t), \tau) \varphi_1(\xi) d\xi \\ &\quad + \gamma_1 \int_0^t \frac{\partial}{\partial \xi} G_1(x + \sigma(t), t; 0, \tau) d\tau. \end{aligned}$$

Combining (2.33), (2.35), (2.38), we get

$$(2.39) \quad \begin{aligned} -I &\leq -a_1^2 \int_{-\infty}^{-\sigma(t)} \int_{-\infty}^{-\sigma(t)} G_1(x + \sigma(t), t; \xi + \sigma(t), \tau) \varphi_1(\xi) d\xi dx \\ &\quad - \gamma_1 \int_{-\infty}^{-\sigma(t)} \int_0^t \frac{\partial}{\partial \xi} G_1(x + \sigma(t), t; 0, \tau) d\tau dx + 2a_1^2 \sigma(t) \text{l.u.b.}_{0 < \tau < t} |u_1|. \end{aligned}$$

The first integral on the right side of (2.39) is bounded by $a_1^2 \int |\varphi_1(\xi)| d\xi$. The third term is bounded by $2a_1^2 \sigma(t) B_1$ where $B_1 = \max \{ \gamma_1, \text{l.u.b. } |\varphi_1(x)| \}$. Finally, changing the order of integration in the second term on the right side of (2.39) we find that it is equal to

$$\frac{2\gamma_1 a_1}{\pi^{\frac{1}{2}}} t^{\frac{1}{2}}.$$

Combining these remarks we get

$$(2.40) \quad -I \leq a_1^2 \int_{-\infty}^0 |\varphi_1(\xi)| d\xi + 2a_1^2 B_1 \sigma(t) + \frac{2\gamma_1 a_1}{\pi^{\frac{1}{2}}} t^{\frac{1}{2}}.$$

In a similar way, employing z_2 we obtain

$$(2.41) \quad -I \geq -a_1^2 \int_{-\infty}^0 |\varphi_1(\xi)| d\xi - 2a_1^2 B_1 \sigma(t) + \frac{2\gamma_1 a_1}{\pi^{\frac{1}{2}}} t^{\frac{1}{2}}.$$

Combining (2.41), (2.40) and substituting in (2.32), we get

$$(2.42) \quad \lambda_1(t) = \frac{2\gamma_1 a_1}{\pi^{\frac{1}{2}}} t^{\frac{1}{2}} - \gamma_1 a_1^2 s(t) + \vartheta_1 a_1^2 \int_{-\infty}^0 |\varphi_1(x)| dx + \vartheta_2 a_1^2 B_1 \sigma(t)$$

where $|\vartheta_1| \leq 2$, $|\vartheta_2| \leq 2$.

Similarly we obtain

$$(2.43) \quad \lambda_2(t) = -\frac{2\gamma_2 a_2}{\pi^{\frac{1}{2}}} t^{\frac{1}{2}} + \gamma_2 a_2^2 s(t) + \vartheta_3 a_2^2 \int_0^{\infty} |\varphi_2(x)| dx + \vartheta_4 a_2^2 B_2 \sigma(t)$$

where $|\vartheta_3| \leq 2$, $|\vartheta_4| \leq 2$.

Using the definition of λ_i in (2.31) and substituting (2.42), (2.43) into the equation obtained from (2.28) by integration, we get

$$(2.44) \quad \begin{aligned} s(t) = k_1 \lambda_1(t) - k_2 \lambda_2(t) &= \frac{2}{\pi^{\frac{1}{2}}} (k_1 a_1 \gamma_1 + k_2 a_2 \gamma_2) t^{\frac{1}{2}} - (k_1 \gamma_1 a_1^2 + k_2 \gamma_2 a_2^2) s(t) \\ &+ \vartheta_1 k_1 a_1^2 \int_{-\infty}^0 |\varphi_1(x)| dx - \vartheta_3 k_2 a_2^2 \int_0^{\infty} |\varphi_2(x)| dx \\ &+ (\vartheta_2 k_1 a_1^2 B_1 - \vartheta_4 k_2 a_2^2 B_2) \sigma(t). \end{aligned}$$

We have to eliminate $\sigma(t)$ from the right side of (2.44). To do this, we notice that

$$|\dot{s}(\tau)| \leq k_1 \left| \frac{\partial}{\partial x} u_1(s(\tau), \tau) \right| + k_2 \left| \frac{\partial}{\partial x} u_2(s(\tau), \tau) \right|.$$

Hence,

$$(2.45) \quad \sigma(t) = \text{l.u.b.}_{0 < \rho < t} |s(\rho)| \leq k_1 \lambda_1(t) + k_2 \lambda_2(t).$$

Substituting λ_1 , λ_2 from (2.42), (2.43) into (2.45) we get

$$(2.46) \quad \begin{aligned} \sigma(t) &\leq \frac{2}{\pi^{\frac{1}{2}}} (k_1 \gamma_1 a_1 - k_2 \gamma_2 a_2) t^{\frac{1}{2}} \\ &- (k_1 \gamma_1 a_1^2 - k_2 \gamma_2 a_2^2) s(t) + \vartheta_1 k_1 a_1^2 \int_{-\infty}^0 |\varphi_1(x)| dx \\ &+ \vartheta_3 k_2 a_2^2 \int_0^{\infty} |\varphi_2(x)| dx + (\vartheta_2 k_1 a_1^2 B_1 + \vartheta_4 k_2 a_2^2 B_2) \sigma(t). \end{aligned}$$

We now make use of the assumption that

$$(2.47) \quad k_1, k_2, a_1^2 k_1, a_2^2 k_2$$

are sufficiently small. Using (2.47) we obtain from (2.46) an estimate on $\sigma(t)$.

Inserting it into (2.44) and noting that the coefficient of $s(t)$ on the right side of (2.44) is sufficiently small, we obtain the asymptotic formula

$$(2.48) \quad \limsup_{t \rightarrow \infty} \left| \frac{s(t)}{t^{\frac{1}{3}}} - \frac{2}{\pi^{\frac{1}{3}}} (k_1 a_1 \gamma_1 + k_2 a_2 \gamma_2) \right| \leq o(1)$$

where $o(1) \rightarrow 0$ as $k_i, k_i a_i^2$ tend to zero.

We have thus proved:

Theorem 2. *If the functions ψ_1, ψ_2 satisfy the assumption (Ψ) , then $s(t)$ satisfies the asymptotic formula (2.48).*

2.3. Generalizations. (a) **Asymptotic behavior of u_1, u_2 .** Consider the system (2.15)–(2.21) in the special case that

$$\psi_1(x) \equiv \gamma_1 \geq 0, \quad \psi_2(x) \equiv \gamma_2 \leq 0.$$

We try to find an explicit solution of the form

$$s(t) = \mu t^{\frac{1}{3}}, \quad u_i(x, t) = f_i(z) \quad (i = 1, 2) \quad \text{where} \quad z = \frac{x}{t^{\frac{1}{3}}}.$$

We get

$$(2.49) \quad f_1(z) = C_1 \int_{\mu}^z \exp \left\{ -\frac{a_1^2 \xi^2}{4} \right\} d\xi, \quad f_2(z) = C_2 \int_{\mu}^z \exp \left\{ -\frac{a_2^2 \xi^2}{4} \right\} d\xi$$

where C_1, C_2, μ satisfy the equations

$$(2.50) \quad \frac{1}{2}\mu = -k_1 C_1 \exp \left\{ -\frac{a_1^2 \mu^2}{4} \right\} + k_2 C_2 \exp \left\{ -\frac{a_2^2 \mu^2}{4} \right\}$$

$$(2.51) \quad C_i \int_{\mu}^{-\infty} \exp \left\{ -\frac{a_i^2 \xi^2}{4} \right\} d\xi = \gamma_i \quad (i = 1, 2).$$

For μ small we find the approximate solution

$$(2.52) \quad \mu = \frac{2}{\pi^{\frac{1}{3}}} (k_1 a_1 \gamma_1 + k_2 a_2 \gamma_2)$$

$$(2.53) \quad C_1 = -\frac{a_1 \gamma_1}{\pi^{\frac{1}{3}}}, \quad C_2 = \frac{a_2 \gamma_2}{\pi^{\frac{1}{3}}}.$$

(2.52) is indeed in correspondence with the asymptotic formula (2.48). But the special case just considered gives much more. It leads to the conjecture that asymptotically $u_1(x, t), u_2(x, t)$ depend only on $z = x/t^{\frac{1}{3}}$ and possibly tend to $f_1(z), f_2(z)$. A conjecture of this type has been proved for problems of drops and bubbles [3], [4]. It can also be proved for the present problem, by modifying the methods used there. The restriction (2.47) is essential to the success of the method.

(b) **Other Stefan problems.** Combining the methods of 2.1, 2.2 we can obtain asymptotic bounds for the two-phase Stefan problem where w_1 is defined for $0 < x < s(t)$ and w_2 is defined for $s(t) < x < \infty$. We can also get asymptotic bounds for both the systems in 2.1 and in 2.2 in case of nonhomogeneous differential equations. The equation of equilibrium of flow ((1.4) or (2.20)) can also be replaced by a more general equation.

(c) **Open questions.** (1) Improve the asymptotic bounds derived in 2.1, for $\delta < \frac{1}{2}$. (2) Derive the asymptotic behavior of $s(t)$ for the system (1.1), (1.3), (1.4), (1.41). (3) Consider a two-phase Stefan problem with w_1 defined for $0 < x < s(t)$ and w_2 defined for $s(t) < x < c$ ($c > 0$). Conjecture: if $\lim_{t \rightarrow \infty} w_i(0, t)$ ($i = 1, 2$) exist in some strong sense, then $\lim_{t \rightarrow \infty} s(t)$ exists, provided $s(t)$ does not intersect $x = 0$ or $x = c$ at a finite time. A similar conjecture arises in case $\lim_{t \rightarrow \infty} [\partial w_1(0, t)/\partial x - \partial w_2(0, t)/\partial x]$ exists in some strong sense.

REFERENCES

- [1] J. DOUGLAS, A uniqueness theorem for the solution of a Stefan problem, *Proc. Amer. Math. Soc.*, **8** (1957), 402–408.
- [2] A. FRIEDMAN, Free boundary problem for parabolic equations I. Melting of solids, *J. Math. and Mech.*, **8** (1959), 499–517.
- [3] A. FRIEDMAN, Free boundary problems for parabolic equations II. Condensation and evaporation of a liquid drop. *J. Math. and Mech.*, **9** (1960), 19–66.
- [4] A. FRIEDMAN, Free boundary problems for parabolic equations III. Dissolution of a gas bubble in liquid, *J. Math. and Mech.*, **9** (1960), 327–345.
- [5] I. I. KOLODNER, Free boundary problem for the heat equation with applications to problems of change of phase, *Comm. Pure Appl. Math.*, **9** (1956), 1–31.
- [6] W. T. KYNER, An existence and uniqueness theorem for nonlinear Stefan problem, *J. Math. and Mech.*, **8** (1959), 483–498.
- [7] W. I. MIRANKER, A free boundary value problem for the heat equation, *Quart. Appl. Math.*, **16** (1958), 121–130.
- [8] L. NIRENBERG, A strong maximum principle for parabolic equations, *Comm. Pure Appl. Math.*, **6** (1953), 167–177.
- [9] L. I. RUBINSTEIN, On the determination of the portion of the boundary which separates two phases in the one dimensional problem of Stefan, *Doklady Akad. Nauk SSSR (N.S.)*, **58** (1947), 217–220.
- [10] G. SESTINI, Esistenza di una soluzione in problemi analoghi a quello di Stefan, *Rivista Mat. Univ. Parma*, **3** (1952), 3–23.
- [11] G. SESTINI, Esistenza ed unicità nel problema di Stefan relativo a campi dotati di simmetria, *Rivista Mat. Univ. Parma*, **3** (1952), 103–113.

University of Minnesota
Institute of Technology
Minneapolis, Minnesota