A Variational Principle for Convection of Heat

Part I

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Part II

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1. Introduction. The equations of heat transfer can be expressed in the variational language in a variety of ways. A variational principle for the equation of heat conduction in which only the time derivative of the temperature is varied, the temperature itself remaining unchanged, has been given by CHAMBERS [1]. ROSEN [2] has formulated a variational principle for diffusional processes which is a modification of ONSAGER's principle of minimum dissipation. In his formulation the time derivative of the temperature is held fixed while the temperature is varied. Recently BIOT [3] has developed a variational principle for the equation of heat conduction using the concepts of thermal potential, dissipation function and generalized thermal force. It appears to be most general and capable of dealing with non-linear and surface effects. It fits into a unified thermodynamic theory, embracing a large category of physical phenomena, leading to the equations of the Lagrangian type for the thermal flow field.

In Part I of this paper the variational principle of BIOT is extended to *transient* heat transfer in fluids under forced flow including the heat generated by viscous dissipation. In Part II a similar variational principle is developed for the problems of *steady* heat transfer in channel flows for large PÉCLET numbers (axial heat conduction neglected). The method is applied to heat transfer between two semi-infinite parallel plates for (i) slug flow and (ii) parabolic flow. It is found that the temperature distribution is in good agreement with that obtained by "orthogonal functions expansion method" (SCHNEIDER [4] and V. D. DOES & SCHENK [5]).

2. The variational principle. Consider an isotropic medium with thermal conductivity $k(x,y,z)$ and heat capacity $c(x,y,z)$ per unit volume. H being the

heat flow vector field, the conservation of energy is expressed by the relation

$$(2.1) \quad c\theta = -\operatorname{div} \mathbf{H}.$$

Let a thermal potential V and a differential invariant δD be defined respectively as

$$(2.2) \quad V = \frac{1}{2} \int_{\tau} c\theta^2 d\tau,$$

$$(2.3) \quad \delta D = \int_{\tau} \frac{1}{k} \left\{ \frac{\partial \mathbf{H}}{\partial t} + \mathbf{U} \operatorname{div} \mathbf{H} + \mathbf{M} \right\} \cdot \delta \mathbf{H} d\tau, \quad \operatorname{div} \mathbf{M} = \theta \mathbf{U} \cdot \operatorname{grad} c + \Phi,$$

where Φ is the viscous dissipation function and $d\tau$ an element of volume occupied by the fluid. With these definitions the variational principle is stated as

$$(2.4) \quad \delta V + \delta D = \int_S \theta \mathbf{n} \cdot \delta \mathbf{H} dS,$$

the surface integral being extended over the boundary S of the domain τ with the unit normal $\mathbf{n}$ to the boundary taken positive inward. Equation (2.2) gives

$$(2.5) \quad \delta V = - \int_{\tau} \theta \operatorname{div} (\delta \mathbf{H}) d\tau = \int_{\tau} \delta \mathbf{H} \cdot \operatorname{grad} \theta d\tau + \int_S \theta \mathbf{n} \cdot \delta \mathbf{H} dS.$$

Using (2.3) and (2.5) the equation (2.4) yields

$$(2.6) \quad \int_{\tau} \left[\operatorname{grad} \theta + \frac{1}{k} \left\{ \frac{\partial \mathbf{H}}{\partial t} + \mathbf{U} \operatorname{div} \mathbf{H} + \mathbf{M} \right\} \right] \cdot \delta \mathbf{H} d\tau = 0.$$

Equating the coefficient of $\delta \mathbf{H}$ to zero, one finds

$$(2.7) \quad k \operatorname{grad} \theta + \frac{\partial \mathbf{H}}{\partial t} + \mathbf{U} \operatorname{div} \mathbf{H} + \mathbf{M} = 0.$$

On taking divergence, (2.7) gives

$$(2.8) \quad \operatorname{div} (k \operatorname{grad} \theta) + \Phi = c \frac{\partial \theta}{\partial t} + c \mathbf{U} \cdot \operatorname{grad} \theta,$$

which is identical with the heat convection equation. The variational principle is thus valid.

3. Lagrangian formulation. Express the thermal flow field $\mathbf{H}$ as

$$(3.1) \quad \mathbf{H} = \mathbf{H}(q_1, q_2, \dots, q_n, x, y, z),$$

where $q_1, q_2, \dots, q_n$ are the generalized coordinates, a set of n parameters, functions of time only defining the configuration of the thermal field. The variations $\delta \mathbf{H}$ in this case are entirely due to the variations of q_i , namely,

$$(3.2) \quad \delta \mathbf{H} = \sum \left(\frac{\partial \mathbf{H}}{\partial q_i} \right) \delta q_i.$$

Also we may write

$$(3.3) \quad \frac{\partial \mathbf{H}}{\partial t} = \sum^i \left(\frac{\partial \mathbf{H}}{\partial q_i} \right) \dot{q}_i .$$

Hence

$$(3.4) \quad \frac{\partial}{\partial \dot{q}_i} \left(\frac{\partial \mathbf{H}}{\partial t} \right) = \frac{\partial \mathbf{H}}{\partial q_i} .$$

The invariant

$$\left[\frac{\partial \mathbf{H}}{\partial t} + \mathbf{U} \operatorname{div} \mathbf{H} + \mathbf{M} \right] \cdot \delta \mathbf{H}$$

in (2.3) may be rewritten as

$$(3.5) \quad \sum^i \left[\frac{\partial \mathbf{H}}{\partial t} + \mathbf{U} \operatorname{div} \mathbf{H} + \mathbf{M} \right] \cdot \frac{\partial}{\partial \dot{q}_i} \left(\frac{\partial \mathbf{H}}{\partial t} \right) \delta q_i .$$

The variation equation (2.4) then becomes

$$(3.6) \quad \delta V + \sum^i \delta q_i \frac{\partial D}{\partial q_i} = \int_s \theta \mathbf{n} \cdot \delta \mathbf{H} dS ,$$

with

$$(3.7) \quad \frac{\partial D}{\partial \dot{q}_i} = \int_\tau \frac{1}{k} \left[\frac{\partial \mathbf{H}}{\partial t} + \mathbf{U} \operatorname{div} \mathbf{H} + \mathbf{M} \right] \cdot \frac{\partial}{\partial \dot{q}_i} \left(\frac{\partial \mathbf{H}}{\partial t} \right) d\tau .$$

The dissipation function D is

$$(3.8) \quad D = \frac{1}{2} \int_\tau \frac{1}{k} \left[\frac{\partial \mathbf{H}}{\partial t} + \mathbf{U} \operatorname{div} \mathbf{H} + \mathbf{M} \right]^2 d\tau .$$

In virtue of (3.2)

$$(3.9) \quad \int_s \theta \mathbf{n} \cdot \delta \mathbf{H} dS = \sum^i Q_i \delta q_i ,$$

where Q_i is the thermal force defined in exactly the same way as a mechanical force, namely, as the work done by a temperature θ on a virtual displacement $\delta \mathbf{H}$. The variation in V can be expressed as

$$(3.10) \quad \delta V = \sum^i \left(\frac{\partial V}{\partial q_i} \right) \delta q_i .$$

The variational principle thus gives n Lagrangian equations for the parameters q_i ,

$$(3.11) \quad \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i , \quad i = 1, 2, \dots n .$$

The effect of surface heat transfer can be easily included in the above formul-

ation. The local rate of heat flow across the surface is

$$(3.12) \quad \frac{\partial H_n}{\partial t} = h(\theta_\alpha - \theta),$$

where h is the heat transfer coefficient of the surface, θ_α is the temperature of the surroundings and H_n is the normal component of the heat flow vector $\mathbf{H}$ at the surface. The thermal force at the surface is

$$(3.13) \quad \begin{aligned} Q_i &= \int_s \theta \left(\frac{\partial H_n}{\partial q_i} \right) dS \\ &= - \int_s \frac{1}{h} \left(\frac{\partial H_n}{\partial t} \right) \left(\frac{\partial H_n}{\partial q_i} \right) dS + \int_s \theta_\alpha \left(\frac{\partial H_n}{\partial q_i} \right) dS. \end{aligned}$$

The first term on the right hand side may be derived from a surface dissipation function D_s , and may be coupled with the volume dissipation function D . The function D_s is

$$(3.14) \quad D_s = \frac{1}{2} \int_s \frac{1}{h} \left(\frac{\partial H_n}{\partial t} \right)^2 dS.$$

The total dissipation function thus becomes

$$(3.15) \quad D = \frac{1}{2} \int_\tau \frac{1}{k} \left[\frac{\partial \mathbf{H}}{\partial t} + \mathbf{U} \operatorname{div} \mathbf{H} + \mathbf{M} \right]^2 d\tau + \frac{1}{2} \int_s \frac{1}{h} \left(\frac{\partial H_n}{\partial t} \right)^2 dS,$$

and the new thermal force

$$(3.16) \quad Q_i = \int_s \theta_\alpha \left(\frac{\partial H_n}{\partial q_i} \right) dS.$$

The equations (3.11) are valid with D and Q_i given by (3.15) and (3.16) respectively.

The extension to the case when heat is generated within the fluid and c and k are functions of temperature is purely formal and follows the same line of treatment as in heat conduction [3].

Part II

4. The equations of heat transfer in channel flow and the analogy. In theoretical investigations the channel is assumed to be uniform and semi-infinite, $0 \leq x \leq \infty$, $\Phi(y, z) = C$, the x -axis being along its length. For the sake of simplicity the entry conditions are neglected and the flow is assumed to be fully developed laminar flow, $U = U(y, z)$. It has been shown by many workers (SCHNEIDER [4], MILLSAPS & POHLHAUSEN [6], SINGH [7]) that for Pe (PÉCLET number) > 100 the effect of axial conduction is almost negligible. The equation for steady heat convection with axial conduction and viscous dissipation neglected

is given by

$$(4.1) \quad \frac{\partial}{\partial y} \left(k' \frac{\partial \theta}{\partial y} \right) + \frac{\partial}{\partial z} \left(k' \frac{\partial \theta}{\partial z} \right) = c' U \frac{\partial \theta}{\partial x},$$

where $k'(y, z)$ is the thermal conductivity, $c'(y, z)$ the heat capacity per unit volume of the fluid and $U(y, z)$ is the fully developed laminar velocity in the channel.

The equation of heat conduction in two dimensions for transient heat flow in an isotropic medium with thermal conductivity $k(y, z)$ and heat capacity per unit volume $c(y, z)$ is

$$(4.2) \quad \frac{\partial}{\partial y} \left(k \frac{\partial \theta}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial \theta}{\partial z} \right) = c \frac{\partial \theta}{\partial t}, \quad 0 \leq t \leq \infty.$$

If $U = U_0 U'$, where U_0 is a typical velocity and U' is a dimensionless function of y, z , the equation (4.1) may be transformed by introducing a time-like variable $\zeta = x / U_0$ into

$$(4.3) \quad \frac{\partial}{\partial y} \left(k' \frac{\partial \theta}{\partial y} \right) + \frac{\partial}{\partial z} \left(k' \frac{\partial \theta}{\partial z} \right) = c' U' \frac{\partial \theta}{\partial \zeta}, \quad 0 \leq \zeta \leq \infty.$$

In the case of slug flow, U' is constant and the analogy between (4.2) and (4.3) is perfect. Thus the transient heat flow with appropriate boundary conditions in a slab is analogous to the steady heat transfer in a channel of the same cross-section. The problem can be solved using Bror's variational principle after taking k' as the conductivity and $c'U'$ as the heat capacity of the slab. The analogy breaks down when U' is a variable quantity vanishing on the channel walls, since it leads to an inadmissible zero heat capacity on the surface of the slab. The variational principle therefore needs a reformulation which is given in the next section.

5. The variational principle. In two-dimensions

$$\nabla = j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}, \quad \text{grad} = j \frac{\partial}{\partial y}, \quad k \frac{\partial}{\partial z}.$$

$\mathbf{H}$ being the heat flow vector field, the conservation of energy is given by

$$(5.1) \quad c' U' \theta = -\text{div } \mathbf{H}.$$

An auxiliary reduced heat flow vector field $\mathbf{H}'$ related to $\mathbf{H}$ may be defined as

$$(5.2) \quad c' \theta = -\text{div } \mathbf{H}'.$$

The thermal potential is

$$(5.3) \quad V = \frac{1}{2} \int_A c' \theta^2 dy dz.$$

A variational invariant is defined as

$$(5.4) \quad \delta D = \int_A \frac{1}{k'} \left[U' \frac{\partial \mathbf{H}'}{\partial \zeta} + \mathbf{N} \right] \cdot \delta \mathbf{H}' dy dz, \quad \text{div } \mathbf{N} = \Phi - \frac{\partial \mathbf{H}'}{\partial \zeta} \cdot \text{grad } U',$$

where Φ is the viscous dissipation function. The variational principle is stated as

$$(5.5) \quad \delta V + \delta D = \oint \theta \mathbf{n} \cdot \delta \mathbf{H}' dS,$$

where the line integral extends over the curve enclosing the area A and dS is an element of the curve. From the equation (5.3)

$$(5.6) \quad \delta V = \oint \theta \mathbf{n} \cdot \delta \mathbf{H}' dS + \int_A \text{grad } \theta \cdot \delta \mathbf{H}' dy dz.$$

The variational principle (5.5) gives

$$(5.7) \quad \int_A \left\{ \text{grad } \theta + \frac{1}{k'} \left[U' \frac{\partial \mathbf{H}'}{\partial \zeta} + \mathbf{N} \right] \right\} \cdot \delta \mathbf{H}' dy dz = 0.$$

Putting the coefficient of $\delta \mathbf{H}'$ equal to zero yields

$$(5.8) \quad k' \text{grad } \theta + U' \frac{\partial \mathbf{H}'}{\partial \zeta} + \mathbf{N} = 0.$$

Taking the divergence of (5.8) we get

$$(5.9) \quad \text{div} (k' \text{grad } \theta) + \Phi = -U' \frac{\partial}{\partial \zeta} \text{div } \mathbf{H}' = Uc' \frac{\partial \theta}{\partial x},$$

which is identical with the steady heat convection equation including viscous dissipation. When k' , c' are kept constant and viscous dissipation is neglected and U' is a function of y only, the variational invariant (5.4) becomes

$$(5.10) \quad \delta D = \frac{1}{k'} \int_{y_1}^{y_2} \left[\int U' \frac{\partial}{\partial \zeta} \left(\frac{\partial \mathbf{H}'}{\partial y} \right) dy \right] \cdot \delta \mathbf{H}' dy.$$

When U' is a constant equal to unity, the functions $\mathbf{H}'$ and $\mathbf{H}$ are identical and the variational invariant becomes

$$(5.11) \quad \delta D = \frac{1}{k'} \int_A \left(\frac{\partial \mathbf{H}}{\partial \zeta} \right) \cdot \delta \mathbf{H} dy dz,$$

which is identical with the variational invariant defined by BIOT in his formulation for heat conduction.

6. Lagrangian formulation. The vector field $\mathbf{H}'$ may be expressed as

$$(6.1) \quad \mathbf{H}' = \mathbf{H}'(q_1, q_2, \dots, q_n, y, z),$$

where $q_1, q_2, \dots, q_n$ are the generalized coordinates depending on the time-like coordinate ζ only. The thermal problem consists in determining how these

generalized coordinates vary as a function of ζ in order to represent the heat flow phenomenon. Thus

$$(6.2) \quad \delta \mathbf{H}' = \sum^i \left(\frac{\partial \mathbf{H}'}{\partial q_i} \right) \delta q_i ,$$

$$(6.3) \quad \frac{\partial \mathbf{H}'}{\partial \zeta} = \sum^i \left(\frac{\partial \mathbf{H}'}{\partial q_i} \right) \dot{q}_i ,$$

$$(6.4) \quad \frac{\partial}{\partial \dot{q}_i} \left(\frac{\partial \mathbf{H}'}{\partial \zeta} \right) = \frac{\partial \mathbf{H}'}{\partial q_i} .$$

The invariant

$$\left[U' \frac{\partial \mathbf{H}'}{\partial \zeta} + \mathbf{N} \right] \cdot \delta \mathbf{H}'$$

in equation (5.4) may therefore be written as

$$(6.5) \quad \sum^i \delta q_i \left[U' \frac{\partial \mathbf{H}'}{\partial \zeta} + \mathbf{N} \right] \cdot \frac{\partial}{\partial \dot{q}_i} \left(\frac{\partial \mathbf{H}'}{\partial \zeta} \right) .$$

The variation equation (5.5) then becomes

$$(6.6) \quad \delta V + \sum^i \delta q_i \frac{\partial D}{\partial \dot{q}_i} = \oint \theta \mathbf{n} \cdot \delta \mathbf{H}' dS ,$$

with

$$(6.7) \quad \frac{\partial D}{\partial \dot{q}_i} = \int_A \frac{1}{k'} \left[U' \frac{\partial \mathbf{H}'}{\partial \zeta} \mathbf{N} \right] \cdot \frac{\partial}{\partial \dot{q}_i} \left(\frac{\partial \mathbf{H}'}{\partial \zeta} \right) dy dz ,$$

and

$$(6.8) \quad D = \int \frac{\partial D}{\partial \dot{q}_i} d\dot{q}_i .$$

From the equation (6.2)

$$(6.9) \quad \oint \theta \mathbf{n} \cdot \delta \mathbf{H}' dS = \sum^i Q_i \delta q_i$$

where Q_i is a reduced generalized thermal force defined in the same way as a mechanical force, that is, as the work done by temperature θ on a virtual displacement $\delta \mathbf{H}'$. The variation in V can be expressed as

$$(6.10) \quad \delta V = \sum^i \left(\frac{\partial V}{\partial q_i} \right) \delta q_i .$$

The variational principle then gives n ordinary differential equations of Lagrangian type solvable for the field parameters q_i as functions of ζ ,

$$(6.11) \quad \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i , \quad i = 1, 2, \dots n .$$

Considering the variational invariant (5.10) and proceeding as above, the Lagrangian equations have the same form (6.11) with

$$(6.12) \quad \frac{\partial D}{\partial \dot{q}_i} = \frac{1}{k'} \int_A \left[\int U' \frac{\partial}{\partial y} \left(\frac{\partial \mathbf{H}'}{\partial \xi} \right) dy \right] \cdot \frac{\partial}{\partial \dot{q}_i} \left(\frac{\partial \mathbf{H}'}{\partial \xi} \right) dy dz.$$

The dissipation function can be obtained by integrating (6.12) with respect to $\dot{q}_i$.

7. Application to one-dimensional problems. Consider the slug flow between two semi-infinite parallel plates (axial heat conduction and viscous dissipation neglected). The differential equation for the fluid temperature is

$$(7.1) \quad k' \frac{\partial^2 \theta}{\partial y^2} = c' U_0 \frac{\partial \theta}{\partial x} = c' \frac{\partial \theta}{\partial \xi}.$$

The boundary conditions are

$$(7.2) \quad \begin{aligned} \theta &= 0 & \text{at } x &= 0, & 0 < y < 2l. \\ \theta &= \theta_0 & \text{at } x > 0, & y = 0, 2l. \end{aligned}$$

Therefore

$$(7.3) \quad \frac{\partial \theta}{\partial y} = 0 \quad \text{at } y = l.$$

Considering only the region between $y = 0, y = l$, the temperature distribution is assumed to be represented by a parabola and the phenomenon is considered in two phases. In the first phase which extends up to x_1 , the effect of the wall temperature penetrates only up to a depth q_1 (Fig. 1), the temperature in the shaded part being that of the incident fluid. During this phase the heat content distribution h , which is proportional to the temperature, is approximated by

$$(7.4) \quad \begin{aligned} h &= h_0 \left(1 - \frac{y}{q_1} \right)^2 & \text{for } y < q_1, \\ h &= 0 & \text{for } y > q_1. \end{aligned}$$

From equation (5.3) the thermal potential is

$$(7.5) \quad V = \frac{1}{2c'} \int_0^{q_1} h^2 dy = \frac{1}{10} \left(\frac{h_0^2}{c'} \right) q_1.$$

Equation (5.1) gives

$$(7.6) \quad -\frac{dH}{dy} = h,$$

which, on integration with the condition $H = 0$ at $y = q_1$, gives

$$(7.7) \quad H = \frac{1}{3} h_0 q_1 - h_0 y + h_0 \left(\frac{y^2}{q_1} \right) - \frac{1}{3} h_0 \left(\frac{y^3}{q_1} \right),$$

$$(7.8) \quad \dot{H} = \frac{1}{3} h_0 \dot{q}_1 - h_0 \left(\frac{y^2}{q_1} \right) \dot{q}_1 + \frac{2}{3} h_0 \left(\frac{y^3}{q_1} \right) \dot{q}_1,$$

FIGURE 1. Fluid Flow between Parallel Walls.

where the dot denotes differentiation with respect to ζ . From (5.11) the dissipation function is

$$(7.9) \quad D = \frac{1}{2k'} \int_0^{q_1} (\dot{H})^2 dy = \frac{1}{2} \left(\frac{13}{315} \right) \frac{h_0^2}{k'} q_1 \dot{q}_1^2.$$

Considering the virtual displacement δH at $y = 0$, (7.7) gives

$$(7.10) \quad \delta H = \frac{1}{3} h_0 \delta q_1.$$

The generalized force Q_1 is therefore given by

$$(7.11) \quad Q_1 \delta q_1 = \theta_0 \delta H.$$

Thus

$$(7.12) \quad Q_1 = \frac{1}{3c'} h_0^2.$$

The differential equation (6.11) for the generalized coordinate q_1 yields

$$(7.13) \quad \frac{1}{10} \left(\frac{h_0^2}{c'} \right) + \frac{13}{315} \left(\frac{h_0^2}{k'} \right) q_1 \dot{q}_1 = \frac{1}{3c'} h_0^2.$$

The solution of (7.13) with the initial condition $q_1 = 0$ for $\zeta = 0$ is

$$(7.14) \quad q_1 = 2l(3.36) \left(\frac{x}{2lP\epsilon} \right)^{\frac{1}{2}}, \quad P\epsilon = \frac{2lc'U_0}{k'}.$$

The temperature distribution during the first phase therefore is

$$(7.15) \quad \theta_1 = \theta_0 \left[1 - \frac{y}{2l(3.36) \left(\frac{x}{2lP\epsilon} \right)^{\frac{1}{2}}} \right]^2.$$

This phase ends at $x = x_1$, where x_1 is obtained from (7.14) by putting $q_1 = l$,

$$(7.16) \quad x_1 = 2l(0.022125) P\epsilon.$$

In the second phase, using the heat content $c'\theta$ at $y = l$ as generalized coordinate q_2 and approximating the heat content distribution by a parabola

$$(7.17) \quad h = (h_0 - q_2)\left(1 - \frac{y}{l}\right)^2 + q_2,$$

the thermal potential is

$$(7.18) \quad V = \frac{1}{2c'} \int_0^l h^2 dy = \left(\frac{h_0^2 l}{2c'}\right) + \frac{4l}{15c'} (q_2 - h_0)^2 + \frac{2lh_0}{3c'} (q_2 - h_0).$$

The heat flow vector found by integrating (7.6) with the condition $H = 0$ at $y = l$, is

$$(7.19) \quad H = (q_2 - h_0)\left[y - \frac{y^2}{l} + \frac{y^3}{3l^2} - \frac{l}{3}\right] - q_2(y - l),$$

$$(7.20) \quad \dot{H} = \dot{q}_2\left[\frac{2l}{3} - \frac{y^2}{l} + \frac{y^3}{3l^2}\right].$$

The dissipation function is

$$(7.21) \quad D = \frac{1}{2k'} \int_0^l \dot{H}^2 dy = \frac{34}{315} \frac{l^3}{k'} \dot{q}_2^2,$$

and the generalized force is

$$(7.22) \quad Q_2 = \frac{2}{3} \left(\frac{h_0}{c'}\right) l.$$

The differential equation for q_2 gives, after simplification,

$$(7.23) \quad q_2 + 4.67 \frac{x_1}{U_0} \dot{q}_2 = h_0.$$

Its solution with the condition $q_2 = 0$ at $x = x_1$ is

$$(7.24) \quad \frac{q_2}{h_0} = 1 - \exp\left\{-0.214\left(\frac{x}{x_1} - 1\right)\right\}.$$

The temperature distribution in the second phase therefore is given by

$$(7.25) \quad \frac{\theta_2}{\theta_0} = \left\{\exp\left[-0.214\left(\frac{x}{2l P \dot{\epsilon}} (0.022125) - 1\right)\right]\right\} \left[\left(1 - \frac{y}{l}\right)^2 - 1\right] + 1.$$

For the first phase the mean mixed temperature may be defined as

$$(7.26) \quad T_{M_1} = \int_0^{q_1} U \theta_0 \left(1 - \frac{y}{q_1}\right)^2 dy / \int_0^l U dy = 2.24 \theta_0 \left(\frac{x}{2l P \dot{\epsilon}}\right)^{\frac{1}{2}}.$$

Therefore

$$(7.27) \quad \frac{T_{M_1} - \theta_0}{-\theta_0} = \bar{\theta}_1 = \left[1 - 2.24\left(\frac{x}{2l P \dot{\epsilon}}\right)^{\frac{1}{2}}\right].$$

TABLE I

Slug Flow	θ/θ_0 (Present solution)					θ/θ_0 (exact solution)				
$x/2l \text{ } P\acute{e} \rightarrow$	.005	.01	.1	.2	1	.005	.01	.1	.2	1
$y/2l$ $\downarrow$ 0	1	1	1	1	1	1	1	1	1	1
0.05	0.620	0.724	0.911	0.970	0.99998	0.620	0.724	0.915	0.970	0.99999
0.10	0.340	0.490	0.831	0.936	0.99997	0.320	0.480	0.833	0.940	0.99998
0.20	0.030	0.164	0.700	0.885	0.99995	0.043	0.168	0.720	0.890	0.99996
0.30	0.0020	0.012	0.605	0.850	0.99993	0.0022	0.021	0.610	0.852	0.99994
0.50	0	0	0.529	0.821	0.99992	0	0.0008	0.526	0.823	0.99993

Similarly for the second phase

$$(7.28) \quad \frac{T_{M_2} - \theta_0}{-\theta_0} = \bar{\theta}_2 = \frac{2}{3} \exp \left\{ -0.214 \left(\frac{x}{2l \text{ } P\acute{e} (0.022125)} - 1 \right) \right\}.$$

The distributions θ/θ_0 and $\bar{\theta}_{1,2}$ are tabulated (Tables I, II) and plotted as functions of $y/2l$ and $x/2l \text{ } P\acute{e}$ respectively (Figs. 2, 3). The dotted lines show the results obtained by the FOURIER series expansion method. The temperature distribution

$$(7.29) \quad \frac{\theta}{\theta_0} = 1 - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{\pi y(2n-1)}{2l} \exp \left\{ -\frac{(2n-1)^2 \pi^2 x}{2l \text{ } P\acute{e}} \right\},$$

TABLE II

$x/2l \text{ } P\acute{e}$	$\theta_{1,2}$ (slug flow)		$\theta_{1,2}$ (Parabolic flow)	
	Present solution	Exact solution	Present solution	Exact solution
0	1	1	1	1
0.005	0.841	0.840	0.915	0.92
0.01	0.776	0.774	0.868	0.87
0.025			0.764	0.76
0.05	0.509	0.496	0.631	0.63
0.1	0.314	0.306	0.431	0.44
0.2	0.119	0.117		
0.25			0.119	0.14
0.5	0.0063	0.0058	0.02	0.02
1	0.00005	0.00005		
∞	0	0	0	0

FIGURE 2. Temperature in Slug Flow.

and the dimensionless mean mixed temperature

$$(7.30) \quad \bar{\theta}_M = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \exp \left\{ -\frac{(2n-1)^2 \pi^2 x}{2l Pe} \right\} / (2n-1)^2.$$

The agreement between the present and the exact solution is seen to be quite satisfactory.

8. Heat transfer between parallel plates for parabolic flow. Assuming again the parabolic heat content distribution (7.4), the thermal potential V and the reduced heat flow vector H' are given by (7.5) and (7.7) respectively. The expression (6.12) for calculating the dissipation function becomes

$$(8.1) \quad \frac{\partial D}{\partial q_1} = \frac{1}{k'} \int_0^{y_1} \left[\int_{a_1}^y U' \frac{\partial}{\partial y} \left(\frac{\partial H'}{\partial \zeta} \right) dy \right] \frac{\partial}{\partial q_1} \left(\frac{\partial H'}{\partial \zeta} \right) dy.$$

The local fluid velocity is given by

$$(8.2) \quad U = U' U_0 = \frac{3}{l} \left(y - \frac{y^2}{2l} \right) U_0.$$

The equation (8.1) then yields

$$(8.3) \quad \frac{\partial D}{\partial q_1} = \frac{3h_0^2}{k'l} q_1 \left[\frac{59}{2520} q_1^2 - \frac{113q_1^3}{15120l} \right].$$

FIGURE 3. Mean Mixed Temperature.

The reduced generalized thermal force obtained by considering the virtual displacement $\delta H'$ at $y = 0$ is

$$(8.4) \quad Q_1 = \frac{h_0^2}{3c'}.$$

The Lagrangian equation for q_1 gives

$$(8.5) \quad \frac{h_0^2}{10c'} + \frac{3h_0^2}{k'l} \dot{q}_1 \left[\frac{59}{2520} q_1^2 - \frac{113}{15120} \frac{q_1^3}{l} \right] = \frac{h_0^2}{3c'}.$$

The solution of (8.5) with the initial condition $q_1 = 0$ for $\zeta = 0$ is

$$(8.6) \quad \frac{q_1^3}{l^3} - 0.24 \frac{q_1^4}{l^4} = M \quad \left(M = \frac{39.88x}{2lPe} \right),$$

which is solved graphically for q_1/l (Fig. 4) for values of $x/2lPe \leq 0.01905$. This phase ends when $q_1 = l$, giving

$$(8.7) \quad x_1 = 0.0381lPe.$$

In the second phase,

$$(8.8) \quad h = (h_0 - q_2) \left(1 - \frac{y}{l} \right)^2 + q_2,$$

$$(8.9) \quad V = \frac{1}{2c'} \int_0^l h^2 dy = \left(\frac{h_0^2 l}{2c'} \right) + \frac{4l}{15c'} (q_2 - h_0)^2 + \frac{2lh_0}{3c'} (q_2 - h_0),$$

FIGURE 4. Graphical Solution of q_1/l (Equation 8.6).

$$(8.10) \quad H' = (q_2 - h_0) \left[y - \frac{y^2}{l} + \frac{y^3}{3l^2} - \frac{l}{3} \right] + q_2(y - l).$$

From (6.12)

$$(8.11) \quad \frac{\partial D}{\partial q_2} = \frac{1}{k'} \int_0^l \left[\int_l^y U' \frac{\partial}{\partial y} \left(\frac{\partial H'}{\partial \zeta} \right) dy \right] \frac{\partial}{\partial q_2} \left(\frac{\partial H'}{\partial \zeta} \right) dy = \frac{88}{315} \frac{l^3}{k'} \dot{q}_2.$$

The reduced generalized force obtained by considering the value of $\delta H'$ at $y = 0$, is

$$(8.12) \quad Q_2 = \frac{2}{3} \frac{h_0 l}{c'}.$$

The differential equation for q_2 becomes

$$(8.13) \quad q_2 + 6.874 x_1 \dot{q}_2 / u_0 = h_0.$$

The solution of (8.13) with the condition $q_2 = 0$ at $x = x_1$ is

$$(8.14) \quad \frac{q_2}{h_0} = 1 - \exp \left\{ -0.1455 \left(\frac{x}{x_1} - 1 \right) \right\}.$$

With this value of q_2/h_0 the temperature distribution in the second phase is given by

$$(8.15) \quad \frac{\theta_2}{\theta_0} = \exp \left\{ -0.1455 \left(\frac{x}{2l \text{ Pr } 0.01905} - 1 \right) \right\} \left[\left(1 - \frac{y}{l} \right)^2 - 1 \right] + 1.$$

For the first phase the dimensionless mean mixed temperature

$$(8.16) \quad \frac{T_{M_1} - \theta_0}{-\theta_0} = \bar{\theta}_1 = \left[1 - \left(\frac{1}{4} \cdot \frac{q_1^2}{l^2} - \frac{1}{20} \cdot \frac{q_1^3}{l^3} \right) \right].$$

For the second phase

$$(8.17) \quad \frac{T_{M_2} - \theta_0}{-\theta_0} = \bar{\theta}_2 = \frac{4}{5} \exp \left\{ -0.1455 \left(\frac{x}{2l Pe (0.01905)} - 1 \right) \right\}.$$

The distributions $\bar{\theta}_{1,2}$ are tabulated (Table II) and plotted as functions of $x/2l Pe$ (Fig. 3). It is found that the results agree with those obtained in [6].

In conclusion I thank Dr. S. D. NIGAM for helpful suggestions.

APPENDIX

The equation of steady heat convection (axial conduction neglected) for channel flow is

$$(A1) \quad c'U' \frac{\partial \theta}{\partial \zeta} = \text{div} (k' \text{grad } \theta) + \Phi$$

where $\Phi(y, z)$ is the viscous dissipation function. A heat flow vector $\mathbf{H}(y, z, \zeta)$ is defined such that

$$(A2) \quad \frac{\partial \mathbf{H}}{\partial \zeta} = -k' \text{grad } \theta - \mathbf{L}, \quad \text{div } \mathbf{L} = \Phi.$$

This gives (5.1).

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