*Fracture of Closed Circular Pipes Under Internal Pressure and Axial Tension**

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FIGURE 1.

1. Introduction. The theory that a surface of strong instability is a rupture surface was proposed by T. Y. THOMAS and has been applied in studying several fracture problems.¹ This work will apply this theory to study the types of fracture of closed circular pipes (see Fig. 1) under internal pressure p and axial tension L . The material of the pipe will be assumed to be an ideal solid with sharp yield point. For the reasons mentioned in THOMAS' paper,² we have followed R. HILL³ in using the Tresca yield condition. Moreover, at plastic equilibrium, the deformation is assumed to be governed by HENCKY's stress-strain relations which were proposed by him in 1924.⁴ Many authors have later used these conditions to study elastic and plastic problems. MACGREGOR⁵ applied them to determine stress and strain of partially plastic thick-walled tubes, GLEYZAL⁶ to study the plastic deformation of a clamped circular diaphragm under pressure, and HODGE & WHITE⁷ to make a quantitative comparison of flow and deformation theories of plasticity. In the state of plastic flow, it is assumed in this paper that the material is governed by the von Mises stress and rate of strain relations. We also assume the material to be incompressible; more specifically, it will be considered that the density is independent of position and does not vary with time.

If both the internal pressure p and axial tensile load L are sufficiently small, the entire pipe will deform elastically. As L or p , or both, increases to a certain value, a part of the pipe will begin to become plastic, and the plastic portion will continue to grow as these loads increase until the entire pipe attains a state of plastic equilibrium. If these loads are maintained by the continued application of work upon the pipe, plastic flow and fracture may occur. Specifically,

¹ See THOMAS, T. Y., *Plastic flow and fracture in solids*, Jour. of Math. and Mech., **7** (1958), pp. 291-322; also *Rupture of round bars by tension*, *ibid.*, **8** (1959), pp. 1-25; and also *Fracture of circular pipes by internal pressure*, *ibid.*, **8** (1959), pp. 623-641.

² *Fracture of circular pipes by internal pressure*, *ibid.*

³ HILL, R., *The mathematical theory of plasticity*, Oxford, Clarendon Press (1950), p. 111.

⁴ HENCKY, H., *Zur Theorie plastischer Deformationen und hierdurch im Material hervorgerufenen Nachspannungen*, Zeit. ang. Math. und Mech., **4** (1924), pp. 323-334; also *Zur Theorie plastischer Deformationen und der hierdurch im Material hervorgerufenen Nebenspannungen*, Proc. First Int. Congress Appl. Mech., Delft (1924), pp. 312-317.

⁵ MACGREGOR, C. W., COFFIN, L. F., JR. & FISHER, J. C., *Partially plastic thick-walled tubes*, Jour. of the Franklin Inst., **245** (1948), pp. 135-158.

⁶ GLEYZAL, A., *Circular diaphragm under pressure*, Jour. of App. Mech., **15** (1948), pp. 288-296.

⁷ HODGE, P. G., JR. & WHITE, G. N., *A quantitative comparison of flow and deformation theories of plasticity*, *ibid.* (1950), pp. 180-184.

we assume that a slip surface may form, over which there will be a discontinuity in the velocity components of the material particles. This surface may (1) separate a region of plastic equilibrium and a region of plastic flow or (2) separate two regions of plastic flow. If the initially small discontinuity in the velocity components over the slip surface becomes indefinitely large within a finite time, we say that the slip surface is a surface of *strong instability*; fracture of the pipe will occur over such a surface in accordance with our basic hypothesis. Surfaces of strong instability or *fracture surfaces* are, however, exceptional since it can be shown that any small discontinuity in the velocity over an arbitrary surface will be damped out in consequence of the equations governing the behavior of the material.

A complete study of all possible distributions of stresses at plastic equilibrium under various axial tensile loads L and internal pressures p is made in Sects. 3–10. These results, which include the closed pipe under pure tension and under pure internal pressure as special cases, not only furnish the information needed for the study of surfaces of strong instability, but also lead us to the connection between the types of fracture and the applied loads. In Sect. 12, the following results are found: *Let Σ be the intersection of the pipe with the half plane through its axis and assume that Σ separates a region of plastic flow and a region of plastic equilibrium. Then Σ will be a surface of strong instability if, and only if, we have $\sigma_{22}/r^2 \geq \sigma_{33}$ for some value of r , where σ_{22} and σ_{33} are the angular and axial covariant components of stress respectively.* It is also shown that the intersection of the pipe with a plane perpendicular to its axis is not a surface of strong instability of the above type for any distribution of stress. In Sect. 13 it is proved that the intersection of the pipe with a plane normal to its central axis is a surface of strong instability in a region of plastic flow if, and only if, we have $\bar{\sigma}_{33}^* > 0$, where $\bar{\sigma}_{33}^*$ is the axial component of the stress deviator. On the other hand, the intersection Σ of the pipe with a half plane containing its axis is a slip surface in a region of plastic flow only when $\sigma_{33}^* = 0$, $\bar{\sigma}_{22}^*/r^2 > \bar{\sigma}_{11}^*$. Since fracture normal to the axis of the pipe takes place entirely in a region of plastic flow, it will be accompanied, in general, by local necking around the fracture section. This can be seen in pictures of actual fractured pipe.⁸

The connection between the type of fracture of pipes and the applied loads is established in Sect. 15, in which it is shown that one can predict the type of fracture from the values of the tensile load L and the pressure p . Finally, that there is good over-all agreement between the theoretical values of the loads which are just sufficient to cause complete plastic equilibrium of the pipe, *i.e.*, which produce fracture in accordance with the above theory, and the experimentally measured maximum loads before fracture is shown by the graphs in Figs. 2–14.

⁸ See DAVIS, E. A., *Yielding and fracture of medium carbon steel under combined stress*, Jour. of App. Mech., pp. A13–24.

FIGURE 2.

2. Cylindrical coordinates and some auxiliary formulas. Let the central axis of the pipe be the z -axis of a system of cylindrical coordinates and denote by r and θ the radial and angular coordinates of this system. The cylindrical coordinates r , θ and z will be denoted by x^1 , x^2 and x^3 , respectively.

In the cylindrical coordinate system the covariant components g_{ij} of the metric tensor are given by the matrix

$$(2.1) \quad ||g_{ij}|| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

where i indicates the number of the row and j indicates the number of the column where the element g_{ij} is located.

The corresponding contravariant components of this tensor are given by

$$(2.2) \quad ||g^{ij}|| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

The components of the affine connection based on this metric are defined by

$$\Gamma_{ik}^i = \frac{1}{2} g^{im} \left(\frac{\partial g_{mi}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^m} \right),$$

and from these equations we find that

$$(2.3) \quad \Gamma_{22}^1 = -r, \quad \Gamma_{12}^2 = \frac{1}{r}, \quad \Gamma_{ik}^i = 0 \quad \text{otherwise.}$$

Let u_i and σ_{ij} be the components of a covariant tensor; then their covariant

Marin and Kotalik⁹
Alcoa 24S-T

FIGURE 3.

Marin, Ulrich and Hughes¹⁰
Aluminum Alloy 75S-T6

FIGURE 4.

derivatives are given by

$$(2.4) \quad \begin{cases} u_{i,j} = \frac{\partial u_i}{\partial x^j} - u_m \Gamma_{ij}^m, \\ \sigma_{ij,k} = \frac{\partial \sigma_{ij}}{\partial x^k} - \sigma_{mj} \Gamma_{ik}^m - \sigma_{im} \Gamma_{jk}^m. \end{cases}$$

PLASTIC EQUILIBRIUM—ANALYSIS OF STRAIN

3. Basic relations of plastic deformation. Let the axial tension L and the internal pressure p be such that they produce plastic equilibrium everywhere in the pipe except at the heavily plugged ends (see Fig. 1). Let u_i , σ_{ij} be the covariant components of the displacement and stress tensors respectively. Furthermore, we assume the following relations to hold in the plastic equilibrium region:

$$(3.1) \quad \begin{cases} g^{ij} u_{i,j} = 0 & \text{(equation of incompressibility),} \\ \epsilon_{ij} = \psi \sigma_{ij}^* & \text{(Hencky's stress-strain relations),} \\ g^{ik} \sigma_{ij,k} = 0 & \text{(equations of equilibrium),} \\ \sigma_3^* - \sigma_1^* = J & \text{(Tresca yield condition),} \end{cases}$$

where the ϵ_{ij} are the components of the strain tensor and the σ_{ij}^* are the com-

⁹ MARIN, J. & KOTALIK, B. J., *Plastic biaxial stress-strain relations for Alcoa 24S-T subjected to variable-stress ratio*, Jour. of App. Mech. ASME (1950), pp. 372-376.

¹⁰ MARIN, J., ULRICH, B. H. & HUGHES, W. P., *Plastic stress-strain relations for 75S-T6 Aluminum alloy subjected to biaxial tensile stresses*, NACA TN2425, Aug. 1951.

Marin and Hu¹¹
Aluminum Alloy 14S-T4

FIGURE 5.

Marin and Hu¹²
Mild steel

FIGURE 6.

Cornet and Grassi¹³
Inoculated iron

FIGURE 7.

Grassi and Cornet¹⁴
Gray Cast iron

FIGURE 8.

¹¹ MARIN, J. & HU, L. W., *Plastic stress-strain for biaxial tension and variable stress ratio*, Proc. of ASTM 52 (1952), pp. 1098-1123.

¹² MARIN, J. & HU, L. W., *Biaxial plastic stress-strain relations of a mild steel for variable stress ratios*, TRANS. ASME (1956), pp. 499-509.

¹³ CORNET, I. & GRASSI, R. C., *Fracture of inoculated iron under biaxial stress*, Jour. of App. Mech. (1955), pp. 172-174.

¹⁴ GRASSI, R. C. & CORNET I., *Fracture of gray cast iron under biaxial stress*, Jour. of App. Mech. (1949), pp. 179-182.

Movikawa and Griffs¹⁵
0.23 % C steel

FIGURE 9.

Lessells and Gregor¹⁶
Ni-Cr-Mo Steel

FIGURE 10.

ponents of the stress deviator. These tensors are defined by

$$(3.2) \quad \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \sigma_{ij}^* = \sigma_{ij} - \frac{1}{3}(g^{ab}\sigma_{ab})g_{ij}.$$

Also, σ_1 , σ_2 and σ_3 are the principal stresses with the ordering $\sigma_3 \geq \sigma_2 \geq \sigma_1$, J is a material constant and ψ is a *positive* factor of proportionality. Hence we have

$$(3.3) \quad \psi = \frac{\epsilon_3 - \epsilon_1}{J}$$

from the Hencky stress-strain relations where ϵ_1 , ϵ_2 , ϵ_3 denote the principal strains.

4. Determination of displacements and strains. From circular symmetry the components u_i will be independent of the angular coordinate θ and the component u_2 must be zero. Since the two ends of the closed pipe are always kept parallel during the whole period of loading, the component u_3 may be assumed to be independent of the radial coordinate r . Also, neglecting the end effect, the component u_i is independent of the z coordinate. Hence, in the entire plastic equilibrium region, we have

$$(4.1) \quad u_1 = u_1(r), \quad u_2 = 0, \quad u_3 = u_3(z).$$

From the relations between the covariant and contravariant components of the displacement u , namely

$$u_i = g_{ij}u^j,$$

we see that

$$u_1 = u^1, \quad u_2 = u^2, \quad u_3 = u^3.$$

¹⁵ NADAI, A., *Theory of flow and fracture of solids*, McGraw-Hill (1950), p. 263.

¹⁶ LESSELLS, J. M. & MACGREGOR, C. W., *Combined stress experiments on a nickel-chrome-molybdenum steel*, Jour. of Franklin Inst. 230 (1940), pp. 163-181.

Thomsen and Dorn¹⁷
J-1 Mg. Alloy

FIGURE 11.

Thomsen and Dorn¹³
FS-1 Mg. Alloy

FIGURE 12.

Also, from (4.1) and the first formula (2.4) we find

$$(4.2) \quad ||u_{i,j}|| = \left\| \begin{array}{ccc} \frac{du_1}{dr} & 0 & 0 \\ 0 & ru_1 & 0 \\ 0 & 0 & \frac{du_3}{dz} \end{array} \right\| = ||\epsilon_{i,j}||.$$

Thomsen and Dorn¹⁹
0-1 Mg. Alloy

FIGURE 13.

Marin, Faupel, Dutton and Brassman²⁰
24S-T Al. Alloy

FIGURE 14.

¹⁷ THOMSEN, E. G. & DORN, J. E., *The effect of combined stresses on the ductility and rupture strength of magnesium-alloy extrusions*, Jour. of Aero. Science 11 (1944), pp. 125-136.

¹⁸ MARIN, J., FAUPEL, J. H., DUTTON, V. L. & BRASSMAN, M. W., *Biaxial plastic stress-strain relations for 24S-T aluminum alloy*, NACA Report TN, No. 1536 (1949).

On account of (4.2), the first equation (3.1) can now be written as

$$(4.3) \quad \frac{du_1}{dr} + \frac{u_1}{r} + \frac{du_3}{dz} = 0.$$

But the last term in the left member of this equation can depend at most on the coordinate z while the first two terms depend at most on the coordinate r . Hence (4.3) must be decomposed into the following two relations:

$$(4.4) \quad \frac{du_1}{dr} + \frac{u_1}{r} = 2M, \quad \frac{du_3}{dz} = -2M,$$

where M is a constant. Integrating (4.4) we obtain

$$u_1 = Mr + \frac{N}{r}, \quad u_3 = -2Mz + C,$$

in which N and C are additional constants. Evidently the component u_3 must vanish on some plane, perpendicular to the central axis, in order to avoid a separation of material. If we choose this plane to be the plane $z = 0$, then $C = 0$, so that

$$(4.5) \quad u_1 = Mr + \frac{N}{r}, \quad u_3 = -2Mz.$$

Substituting the expression (4.5) into (4.2), we find

$$(4.6) \quad ||u_{i,j}|| = \left\| \begin{array}{ccc} M - \frac{N}{r^2} & 0 & 0 \\ 0 & Mr^2 + N & 0 \\ 0 & 0 & -2M \end{array} \right\| = ||\epsilon_{i,j}||.$$

PLASTIC EQUILIBRIUM—ANALYSIS OF STRESS

5. General consideration for stresses. From (4.6) and the second system of equations in (3.1), we see

$$(5.1) \quad \sigma_{i,j}^* = \sigma_{i,j} = 0 \quad \text{for } i \neq j;$$

also we have

$$(5.2) \quad \begin{cases} M - \frac{N}{r^2} = \psi \sigma_{11}^* = \psi \left(\frac{2}{3} \sigma_{11} - \frac{1}{3} \frac{\sigma_{22}}{r^2} - \frac{1}{3} \sigma_{33} \right) \\ M + \frac{N}{r^2} = \psi \frac{\sigma_{22}^*}{r^2} = \psi \left(\frac{2}{3} \frac{\sigma_{22}}{r^2} - \frac{1}{3} \sigma_{33} - \frac{1}{3} \sigma_{11} \right), \\ -2M = \psi \sigma_{33}^* = \psi \left(\frac{2}{3} \sigma_{33} - \frac{1}{3} \sigma_{11} - \frac{1}{3} \frac{\sigma_{22}}{r^2} \right), \end{cases}$$

when we take $i, j = 1, 1; 2, 2; 3, 3$ and make use of the definition of σ_{ij}^* in (3.2).

Now from the formula (2.4) for covariant differentiation and the relation (5.1) we calculate

$$(5.3) \quad \begin{cases} \sigma_{11,1} = \frac{\partial \sigma_{11}}{\partial r}, & \sigma_{12,2} = r\sigma_{11} - \frac{\sigma_{22}}{r}, & \sigma_{13,3} = 0, \\ \sigma_{21,1} = 0, & \sigma_{22,2} = \frac{\partial \sigma_{22}}{\partial \theta}, & \sigma_{23,3} = 0, \\ \sigma_{31,1} = 0, & \sigma_{32,2} = 0, & \sigma_{33,3} = \frac{\partial \sigma_{33}}{\partial z}. \end{cases}$$

Expanding the equilibrium equations in (3.1) we obtain

$$(5.4) \quad \begin{cases} \sigma_{11,1} + \frac{1}{r^2} \sigma_{12,2} + \sigma_{13,3} = 0, \\ \sigma_{21,1} + \frac{1}{r^2} \sigma_{22,2} + \sigma_{23,3} = 0, \\ \sigma_{31,1} + \frac{1}{r^2} \sigma_{32,2} + \sigma_{33,3} = 0. \end{cases}$$

Also, using (5.3) we can write (5.4) as

$$(5.5) \quad \frac{\partial \sigma_{11}}{\partial r} + \frac{1}{r} \left(\sigma_{11} - \frac{\sigma_{22}}{r^2} \right) = 0, \quad \frac{\partial \sigma_{22}}{\partial \theta} = 0, \quad \frac{\partial \sigma_{33}}{\partial z} = 0.$$

For convenience in the later investigation we consider the possible values of N and the relative magnitudes of σ_{11} , σ_{22}/r^2 and σ_{33} . Taking the difference of both sides of the first two equations in (5.2), we obtain

$$(5.6) \quad \frac{2N}{r^2} = \psi \left(\frac{\sigma_{22}}{r^2} - \sigma_{11} \right).$$

Suppose $N = 0$; then (5.6) implies $\sigma_{22}/r^2 - \sigma_{11} = 0$ at any point of the pipe. Hence, from the first equation of (5.5), we have

$$\frac{\partial \sigma_{11}}{\partial r} = 0$$

in the entire plastic equilibrium region of the pipe. Thus $\sigma_{11} = \sigma_{22}/r^2$ is independent of r . Now $\sigma_{11} = \sigma_{22}/r^2 = 0$ on the outer cylindrical surface of the pipe. Hence we must also have $\sigma_{11} = \sigma_{22}/r^2 = -p = 0$ at $r = a$ where a is the radius of the inner cylindrical surface of the pipe at plastic equilibrium. *This is the case for which the pipe is under pure axial tension with no internal pressure.* From this point on, we will assume $N \neq 0$ and keep in mind that $N = 0$ corresponds to pure tension.

Suppose now $N < 0$; since $\psi > 0$ for loading¹⁹ we have

$$\frac{\sigma_{22}}{r^2} - \sigma_{11} < 0,$$

in view of (5.6), and hence

$$(5.7) \quad \frac{\partial \sigma_{11}}{\partial r} < 0,$$

on account of (5.5). But from the boundary conditions we have

$$(5.8) \quad (\sigma_{11})_{r=b} = 0, \quad (\sigma_{11})_{r=a} = -p,$$

where $b > a$ is the radius of the outer cylindrical surface of the pipe at plastic equilibrium. But (5.7) contradicts (5.8) since $p \geq 0$. Thus $N > 0$ and, consequently, $\sigma_{22}/r^2 > \sigma_{11}$ everywhere. It was shown in the previous paper²⁰ that $\sigma_{33} = 0$ for the open pipe under pure radial pressure. It is obvious that $\sigma_{33} > 0$ over the ends of the closed pipe with $L \geq 0$. Hence $\sigma_{33} > 0$ everywhere on account of the third equation (5.5). Also $\sigma_{11} \leq 0$; this follows from the first equation (5.5) in view of the boundary conditions (5.8) and the fact that the parenthesis expression in (5.5) is negative. Hence, in particular, $\sigma_{33} - \sigma_{11} > 0$ everywhere in the pipe.

Based on the above considerations we must have one or the other of the following two primary cases, namely:

$$\text{Case A.} \quad \frac{\sigma_{22}}{r^2} \geq \sigma_{33} > \sigma_{11} \quad \text{everywhere,}$$

$$\text{Case B.} \quad \sigma_{33} > \frac{\sigma_{22}}{r^2} > \sigma_{11} \quad \text{everywhere.}$$

In addition, there will occur a mixed case *C* involving a combination of the above Cases A and B.

6. Case A ($\sigma_{22}/r^2 \geq \sigma_{33} > \sigma_{11}$ for all values of r). Since the entire pipe is in plastic equilibrium, we have

$$(6.1) \quad \frac{\sigma_{22}}{r^2} - \sigma_{11} = J,$$

in view of the Tresca yield condition. Substituting (6.1) into the first equation of (5.5), we obtain

$$\frac{\partial \sigma_{11}}{\partial r} = \frac{J}{r}.$$

¹⁹ PRAGER, W. & HODGE, P. G., Jr., *Theory of perfectly plastic solids*, J. Wiley and Sons, New York, 1951, p. 29; also HILL, R., *The mathematical theory of plasticity*, *ibid.*, p. 40.

²⁰ See THOMAS, T. Y., *Fracture of circular pipes under internal pressure*, *ibid.*

Integrating this equation, we get

$$\sigma_{11} = J \log r + f(\theta, z).$$

To determine the function $f(\theta, z)$, we simply apply the boundary condition, namely $(\sigma_{11})_{r=b} = 0$ for all values of the coordinates θ and z . That is,

$$0 = J \log b + f(\theta, z).$$

Thus $f(\theta, z) = -J \log b = \text{constant}$, and hence

$$(6.2) \quad \sigma_{11} = J \log \frac{r}{b}.$$

Applying the boundary condition $(\sigma_{11})_{r=a} = -p$ we have

$$(6.3) \quad p = J \log \frac{b}{a}.$$

This determines the value of p so that the entire pipe is in plastic equilibrium and $\sigma_{22}/r^2 \geq \sigma_{33} > \sigma_{11}$ everywhere. Also we have

$$(6.4) \quad \frac{\sigma_{22}}{r^2} = J \left(1 + \log \frac{r}{b} \right),$$

on account of (6.1) and (6.2). From (5.6) and (6.1) we have

$$\psi = \frac{2N}{Jr^2}.$$

Using this equation to eliminate ψ and the equations (6.2) and (6.4) to eliminate σ_{11} and σ_{22}/r^2 from the first equation (5.2), we can solve the resulting relation to obtain

$$(6.5) \quad \sigma_{33} = J \left(\log \frac{r}{b} - \frac{3M}{2N} r^2 + \frac{1}{2} \right).$$

It may be noted that for this case the $\sigma_{i,j}$ are independent of the coordinates θ and z ; consequently the last two equations in (5.5) are identically satisfied. From the definition (3.2) for σ_{ij}^* , we can now calculate

$$(6.6) \quad ||\sigma_{ij}^*|| = \left\| \begin{array}{ccc} \frac{J}{2} \left(\frac{Mr^2}{N} - 1 \right) & 0 & 0 \\ 0 & \frac{J}{2} \left(\frac{Mr^2}{N} + 1 \right) r^2 & 0 \\ 0 & 0 & -J \frac{Mr^2}{N} \end{array} \right\|.$$

In order to determine the value of the ratio M/N which occurs in the equation (6.5), we apply another boundary condition, namely

$$\iint \sigma_{33} ds = \pi a^2 p + L,$$

where the integration is taken over a plane section of the pipe perpendicular to the central axis. Substituting the expressions for σ_{33} and p given by (6.5) and (6.3) into the above equation, we obtain

$$J\pi \int_a^b 2\left(\log \frac{r}{b} - \frac{3M}{2N} r^2 + \frac{1}{2}\right)r \, dr = \pi a^2 \left(J \log \frac{b}{a} + l\right),$$

where $l = L/\pi a^2$. Performing the integration in the left member of this equation and simplifying we find

$$(6.7) \quad l = -\frac{3}{4} J \frac{M}{N} \left(\frac{b^4 - a^4}{a^2}\right).$$

Since $l \geq 0$, $N > 0$ and $b > a$, it follows from (6.7) that $M \leq 0$.

7. Necessary and sufficient conditions for Case A. If $\sigma_{22}/r^2 \geq \sigma_{33} > \sigma_{11}$ then, according to the preceding analysis, σ_{11} , σ_{22}/r^2 and σ_{33} are given by (6.2), (6.4) and (6.5). Thus it is necessary that

$$J \log \frac{r}{b} < J\left(\frac{1}{2} - \frac{3Mr^2}{2N} + \log \frac{r}{b}\right) \leq J\left(1 + \log \frac{r}{b}\right)$$

or

$$0 < \frac{1}{2} - \frac{3Mr^2}{2N} \leq 1.$$

These are equivalent to $-1 < -3Mr^2/N$ and $-1 \leq 3Mr^2/N$. Hence

$$(7.1) \quad -1 \leq \frac{3Mr^2}{N} < 1$$

is a necessary condition for Case A.

Conversely, assume that (7.1) holds. Since $N > 0$ the first inequality (7.1) implies $3M + N/r^2 \geq 0$. From this latter inequality and the last two equations of (5.2) we obtain

$$\psi\left(\frac{\sigma_{22}^*}{r^2} - \sigma_{33}^*\right) = \psi\left(\frac{\sigma_{22}}{r^2} - \sigma_{33}\right) \geq 0.$$

But $\psi > 0$, so that $\sigma_{22}/r^2 \geq \sigma_{33}$. From the first and third equation of (5.2) we obtain

$$\psi(\sigma_{33}^* - \sigma_{11}^*) = \psi(\sigma_{33} - \sigma_{11}) = -3M + \frac{N}{r^2}.$$

Using the second inequality (7.1) and the fact that $N > 0$, it follows that the right member of the above equation is positive. Hence $\sigma_{33} > \sigma_{11}$. Hence (7.1) is also a sufficient condition for Case A.

It will be convenient to approximate a and b by their initial values a_0 and b_0 .

Thus from (6.3) and (6.7) we have

$$(7.2) \quad p = J \log \frac{b_0}{a_0}, \quad l = -\frac{3}{4} J \frac{M}{N} \left(\frac{b_0^4 - a_0^4}{a_0^2} \right)$$

with

$$-1 \leq \frac{3Mb_0^2}{N} < 1.$$

Any point of the l, p curve represents the loads which produce the complete plastic equilibrium with stress distribution such that $\sigma_{22}/r^2 \geq \sigma_{33} > \sigma_{11}$ everywhere. From the equations (7.2) we see that the ratio M/N enters as a parameter in the determination of this curve.

8. Case B ($\sigma_{33} > \sigma_{22}/r^2 > \sigma_{11}$ for all values of r). Since the Tresca yield condition must be satisfied, we must have

$$(8.1) \quad \sigma_{33} - \sigma_{11} = J$$

everywhere in the pipe. In order to solve the first equation in (5.5) for σ_{11} , we have to find an explicit expression for $\sigma_{22}/r^2 - \sigma_{11}$. This can be done as follows: Taking the difference of both sides of the first and the last equation in (5.2), we obtain

$$3M - \frac{N}{r^2} = \psi(\sigma_{11} - \sigma_{33})$$

or

$$(8.2) \quad \psi = -\frac{3M - N/r^2}{J}.$$

Taking the difference of both sides of the second and the last equation of (5.2), we obtain

$$3M + \frac{N}{r^2} = \psi \left(\frac{\sigma_{22}}{r^2} - \sigma_{33} \right)$$

or

$$\frac{\sigma_{22}}{r^2} = \frac{3M + N/r^2}{\psi} + \sigma_{33}.$$

Eliminating ψ from this equation by means of (8.2) and using (8.1), we obtain

$$\frac{\sigma_{22}}{r^2} = -\frac{3M + N/r^2}{3M - N/r^2} J + \sigma_{11} + J$$

or

$$(8.3) \quad \frac{\sigma_{22}}{r^2} = \sigma_{11} + J \left(1 - \frac{3M + N/r^2}{3M - N/r^2} \right) = \sigma_{11} - \frac{2NJ}{3Mr^2 - N}.$$

Substituting this expression into the first equation of (5.5), we have

$$\frac{\partial \sigma_{11}}{\partial r} = -\frac{1}{r} \frac{2NJ}{3Mr^2 - N}.$$

Integrating we get

$$\sigma_{11} = J \log \frac{r^2}{N - 3Mr^2} + f(\theta, z),$$

where $f(\theta, z)$ is an arbitrary function of θ and z .

We now apply the boundary conditions:

$$(\sigma_{11})_{r=b} = 0 = J \log \frac{b^2}{N - 3Mb^2} + f(\theta, z).$$

Thus

$$f(\theta, z) = -J \log \frac{b^2}{N - 3Mb^2},$$

so that f is a constant. Hence

$$(8.4) \quad \sigma_{11} = J \log \frac{N - 3Mb^2}{N - 3Mr^2} \frac{r^2}{b^2}$$

and

$$(8.5) \quad -p = (\sigma_{11})_{r=a} = J \log \frac{N - 3Mb^2}{N - 3Ma^2} \frac{a^2}{b^2}.$$

From (8.3) and (8.4) we obtain

$$(8.6) \quad \frac{\sigma_{22}}{r^2} = J \left(\frac{2N}{N - 3Mr^2} + \log \frac{N - 3Mb^2}{b^2} \frac{r^2}{N - 3Mr^2} \right),$$

and finally from (8.1) and (8.4) we find that

$$(8.7) \quad \sigma_{33} = J \left(1 + \log \frac{N - 3Mb^2}{b^2} \frac{r^2}{N - 3Mr^2} \right).$$

Also, it may be recalled that the σ_{ii} were shown to be independent of the co-ordinates θ and z ; consequently the last two equations in (5.5) are satisfied identically. From the definition (3.2) for σ_{ii}^* we now calculate

$$(8.8) \quad ||\sigma_{ii}^*|| = \left\| \begin{array}{ccc} -J \frac{N - Mr^2}{N - 3Mr^2} & 0 & 0 \\ 0 & J \left(\frac{N + Mr^2}{N - 3Mr^2} \right) r^2 & 0 \\ 0 & 0 & -J \frac{2Mr^2}{N - 3Mr^2} \end{array} \right\|.$$

As before, we apply the boundary condition on σ_{33} to determine the value of M/N in terms of L and the radii a and b . Thus

$$\iint \sigma_{33} ds = \pi a^2 p + L,$$

where the integration is taken over a plane section of the pipe perpendicular to the central axis. Substituting the expressions for σ_{33} and p given in (8.7) and (8.5), respectively, into this equation, we find

$$2\pi J \int_a^b r \left(1 + \log \frac{N - 3Mb^2}{b^2} \frac{r^2}{N - 3Mr^2} \right) dr = \pi a^2 (p + l),$$

where $L = \pi a^2 l$ and a and b are the inner and outer radii of the pipe at the state under consideration. Performing the integration, we obtain

$$-a^2 \log \frac{a^2}{b^2} \frac{N - 3Mb^2}{N - 3Ma^2} + b^2 - a^2 + \frac{N}{3M} \log \frac{3Mb^2 - N}{3Ma^2 - N} = \frac{a^2}{J} (l + p).$$

Using (8.5) this equation can be written as

$$b^2 - a^2 + \frac{N}{3M} \log \frac{3Mb^2 - N}{3Ma^2 - N} = \frac{a^2 l}{J}.$$

Again applying (8.5) the above equation becomes

$$b^2 - a^2 - \frac{N}{3M} \left(\frac{p}{J} + \log \frac{a^2}{b^2} \right) = \frac{a^2 l}{J}.$$

Hence

$$\frac{N}{3M} = \frac{b^2 - a^2 - a^2 l/J}{p/J + \log(a^2/b^2)} = \frac{J(b^2 - a^2) - a^2 l}{p + J \log(a^2/b^2)}$$

or

$$(8.9) \quad l = \frac{J}{a^2} \left(b^2 - a^2 + \frac{N}{3M} \log \frac{3Mb^2 - N}{3Ma^2 - N} \right).$$

9. Necessary and sufficient condition for Case B. If $\sigma_{33} > \sigma_{22}/r^2 > \sigma_{11}$, then by the preceding analysis σ_{11} , σ_{22}/r^2 and σ_{33} are given by (8.4), (8.6) and (8.7). It is therefore necessary that

$$\begin{aligned} J \log \left(\frac{N - 3Mb^2}{N - 3Mr^2} \frac{r^2}{b^2} \right) &< J \left(\frac{2N}{N - 3Mr^2} + \log \frac{N - 3Mb^2}{N - 3Mr^2} \frac{r^2}{b^2} \right) \\ &< J \left(1 + \log \frac{N - 3Mb^2}{N - 3Mr^2} \frac{r^2}{b^2} \right) \end{aligned}$$

or

$$0 < \frac{2N}{N - 3Mr^2} < 1$$

or

$$0 < \frac{2}{1 - 3Mr^2/N} < 1.$$

That is,

$$0 < 1 - 3 \frac{Mr^2}{N}$$

and

$$2 < 1 - 3 \frac{Mr^2}{N}.$$

But the second inequality implies the first of these inequalities. Hence we obtain the single inequality

$$(9.1) \quad 3 \frac{Mr^2}{N} < -1$$

as a necessary condition for Case B.

Conversely, let us assume that (9.1) holds. Take the difference of both sides of the second and the last equation of (5.2); this gives

$$-3M - \frac{N}{r^2} = \psi \left(\sigma_{33} - \frac{\sigma_{22}}{r^2} \right).$$

But $N > 0$ and hence (9.1) implies $-3Mr^2 - N/r^2 > 0$. Also, for loading, $\psi > 0$; hence $\sigma_{33} > \sigma_{22}/r^2$ from the above equation. Taking the difference of both sides of the second and the first equation of (5.2) we obtain

$$2 \frac{N}{r^2} = \psi \left(\frac{\sigma_{22}^*}{r^2} - \sigma_{11}^* \right) = \psi \left(\frac{\sigma_{22}}{r^2} - \sigma_{11} \right).$$

Using the fact that $N > 0$ and $\psi > 0$ it now follows from the above equation that $\sigma_{11} < \sigma_{22}/r^2$. Consequently (9.1) is also a sufficient condition for Case B.

It will be convenient to approximate a and b by their initial values a_0 and b_0 . Thus from (8.5) and (8.9) we have

$$(9.2) \quad \begin{cases} -p = J \log \frac{1 - 3Mb_0^2/N}{1 - 3Ma_0^2/N} \frac{a_0^2}{b_0^2}, \\ l = \frac{J}{a_0^2} \left(b_0^2 - a_0^2 + \frac{N}{3M} \log \frac{1 - 3Mb_0^2/N}{1 - 3Ma_0^2/N} \right) \end{cases}$$

with $3Ma_0^2/N < -1$.

Any point of the l, p curve represents the loads which produce the complete plastic equilibrium with stress distribution such that $\sigma_{33} > \sigma_{22}/r^2 > \sigma_{11}$ everywhere. From the equation (9.2) we see that M/N enters as a parameter in the determination of this curve.

10. **Case C** ($\sigma_{22}/r^2 \geq \sigma_{33} > \sigma_{11}$ with $a \leq r \leq c$; $\sigma_{33} > \sigma_{22}/r^2 > \sigma_{11}$ with $c < r \leq b$ where $a \leq c < b$ and $3Mc^2/N = -1$). In this section we first prove that if a state of complete plastic equilibrium, produced by internal pressure and axial tension, does not belong either to Case A or to Case B, then it must belong to Case C defined as follows:

$$\begin{cases} \sigma_{33} > \frac{\sigma_{22}}{r^2} > \sigma_{11} & \text{for } b \geq r > c \\ \frac{\sigma_{22}}{r^2} \geq \sigma_{33} > \sigma_{11} & \text{for } c \geq r \geq a, \end{cases}$$

where $b > c \geq a$. Consequently this proof establishes that Cases A, B and C are the only possible cases of stress distribution for closed circular pipes under internal pressure and axial tension. We then proceed to determine the explicit expressions for the stress functions and the l, p curve.

In Sect. 7 we proved that $\sigma_{22}/r^2 \geq \sigma_{33} > \sigma_{11}$ for all values of r if, and only if, $-1 \leq 3Mr^2/N < 1$ for all values of r ; in particular,

$$(10.1) \quad -1 \leq 3 \frac{Mb^2}{N} < 1$$

is a necessary condition. If r is a real number such that $b \geq r \geq a$, then

$$-1 < 0 \leq 3 \frac{Mr^2}{N} \leq 3 \frac{Mb^2}{N} < 1 \quad \text{if } M > 0$$

and

$$-1 \leq 3 \frac{Mb^2}{N} \leq 3 \frac{Mr^2}{N} < 0 < 1 \quad \text{if } M < 0.$$

Hence (10.1) is a necessary and sufficient condition for $\sigma_{22}/r^2 \geq \sigma_{33} > \sigma_{11}$ for all values of r .

In Sect. 9 we proved that $\sigma_{33} > \sigma_{22}/r^2 > \sigma_{11}$ for all values of r if, and only if, $3Mr^2/N < -1$ for all values of r . In particular, we have that

$$(10.2) \quad 3Ma^2/N < -1$$

is a necessary condition. Conversely, for any real r with $r \geq a$ we have

$$3 \frac{Mr^2}{N} \leq 3 \frac{Ma^2}{N} < -1,$$

in view of the fact that $N > 0$ and $M < 0$ from (10.2). Hence (10.2) is a necessary and sufficient condition for $\sigma_{33} > \sigma_{22}/r^2 > \sigma_{11}$ for all values of r .

Consider now a deformation which gives complete plastic equilibrium of the pipe and which does not belong either to Case A or to Case B. Since (10.1) and (10.2) are sufficient conditions for Cases A and B, respectively, this means that the constants M and N , which completely determine the plastic deformation,

do not satisfy either of the inequalities (10.1) or (10.2). Hence we have either

$$(10.3) \quad 3 \frac{Ma^2}{N} \geq -1 \quad \text{and} \quad 3 \frac{Mb^2}{N} \geq 1$$

or

$$(10.4) \quad 3 \frac{Ma^2}{N} \geq -1 \quad \text{and} \quad 3 \frac{Mb^2}{N} < -1.$$

Since $N > 0$, we have $M > 0$ in (10.3). Consequently the second inequality implies the first one in (10.3). Now the second inequality in (10.3) implies

$$(10.5) \quad 3M - \frac{N}{b^2} \geq 0;$$

and, on the other hand, we have

$$(10.6) \quad 3M - \frac{N}{r^2} = \psi(\sigma_{11}^* - \sigma_{33}^*) = \psi(\sigma_{11} - \sigma_{33})$$

by taking the difference of both sides of the first and the third equations in (5.2). Since $\psi > 0$ for loading and $\sigma_{33} > \sigma_{11}$, as shown in Sect. 5, the equation (10.6) implies $3M - N/r^2 < 0$ for all values of r . In particular,

$$(10.7) \quad 3M - \frac{N}{b^2} < 0.$$

Thus (10.5) contradicts (10.7) and hence (10.3) cannot hold. Accordingly, (10.4) is the only possible case.

Since $N > 0$, the second inequality in (10.4) implies $M < 0$. Let

$$c = \left(-\frac{N}{3M}\right)^{\frac{1}{2}}.$$

Then

$$b^2 > c^2 = -\frac{N}{3M} \geq a^2,$$

in view of the inequalities (10.4). Hence

$$3 \frac{Mc^2}{N} = -1 \quad \text{and} \quad b > c \geq a.$$

It is now clear that

$$-1 \leq 3 \frac{Mr^2}{N} < 1 \quad \text{for} \quad c \geq r \geq a$$

and

$$3 \frac{Mr^2}{N} < -1 \quad \text{for} \quad b \geq r > c.$$

Reviewing the sufficiency proof given in Sects. 7 and 9, we see that the first set of the above inequalities implies that

$$\frac{\sigma_{22}}{r^2} \geq \sigma_{33} > \sigma_{11} \quad \text{for } c \geq r \geq a;$$

the second set of the above inequalities implies

$$\sigma_{33} > \frac{\sigma_{22}}{r^2} > \sigma_{11} \quad \text{for } b \geq r > c,$$

where $b > c \geq a$. This is Case C as above defined.

Proceeding in exactly the same way as in Sect. 6, we have, for $c \geq r \geq a$,

$$\sigma_{11} = J \log r + f(\theta, z).$$

Applying the boundary condition, namely

$$(\sigma_{11})_{r=a} = -p = J \log a + f(\theta, z),$$

we have

$$f(\theta, z) = -p - J \log a = \text{constant}.$$

Thus

$$(10.8) \quad \sigma_{11} = J \log \frac{r}{a} - p.$$

From the fact that $\sigma_{22}/r^2 - \sigma_{11} = J$ we have

$$(10.9) \quad \frac{\sigma_{22}}{r^2} = J \left(1 + \log \frac{r}{a} \right) - p.$$

Also, as shown in Sect. 6, we find

$$(10.10) \quad \sigma_{33} = J \left(\frac{1}{2} - \frac{3Mr^2}{2N} + \log \frac{r}{a} \right) - p.$$

For $b \geq r \geq c$, we proceed exactly in the same way as in Sect. 8 and obtain

$$(10.11) \quad \sigma_{11} = J \log \left(\frac{N - 3Mb^2}{b^2} \frac{r^2}{N - 3Mr^2} \right),$$

$$(10.12) \quad \frac{\sigma_{22}}{r^2} = J \left(\frac{2N}{N - 3Mr^2} + \log \frac{N - 3Mb^2}{b^2} \frac{r^2}{N - 3Mr^2} \right),$$

$$(10.13) \quad \sigma_{33} = J \left(1 + \log \frac{N - 3Mb^2}{b^2} \frac{r^2}{N - 3Mr^2} \right).$$

Since σ_{11} is a continuous function of the radial coordinate r , the value of σ_{11} given by (10.8) must be equal to that given by (10.11) at $r = c$. That is,

$$J \log \frac{N - 3Mb^2}{N - 3Mc^2} \frac{c^2}{b^2} \frac{a}{c} = -p$$

or

$$(10.14) \quad p = -J \log \frac{1 - 3Mb^2/N}{2} \frac{ca}{b^2},$$

where we have used the fact that $3Mc^2/N = -1$. Using the fact that p is given by (10.14), it is easy to see from (10.9), (10.10), (10.12) and (10.13) that σ_{22} and σ_{33} are continuous at $r = c$.

Substituting p given by (10.14) into (10.10) we obtain

$$(10.15) \quad \sigma_{33} = J \left(\frac{1}{2} - \frac{3Mr^2}{2N} + \log \frac{N - 3Mb^2}{2N} \frac{rc}{b^2} \right).$$

To find the relation between the ratio M/N and the values of l , p and the radii a and b , we apply the boundary condition, namely

$$\iint \sigma_{33} \, ds = \pi a^2 p + L,$$

where the integration is taken over a plane section of the pipe perpendicular to its central axis. Substituting the expressions for σ_{33} given by (10.13) and (10.15) into this equation and using the appropriate limits of integration for the above expressions for σ_{33} , we obtain

$$2\pi J \int_a^c r \left(\frac{1}{2} - \frac{3Mr^2}{2N} + \log \frac{N - 3Mb^2}{2Nb^2} cr \right) dr \\ + \int_c^b r \left(1 + \log \frac{N - 3Mb^2}{b^2} \frac{r^2}{N - 3Mr^2} \right) dr = \pi a^2 (p + l).$$

Performing the integration we obtain

$$-\frac{3}{4} \frac{M}{N} (c^4 - a^4) + (c^2 - a^2) \log \frac{N - 3Mb^2}{2Nb^2} c + (c^2 \log c - a^2 \log a) \\ - c^2 \log \frac{c^2}{b^2} \frac{N - 3Mb^2}{2N} + b^2 - c^2 + \frac{N}{3M} \log \frac{N - 3Mb^2}{2N} = a^2 \frac{p + l}{J}.$$

After applying the relation (10.14) and simplifying, we finally obtain

$$(10.16) \quad l = \frac{J}{a^2} \left\{ -\frac{3}{4} \frac{M}{N} (c^4 - a^4) + b^2 - c^2 + \frac{N}{3M} \log \frac{1 - 3Mb^2/N}{2} \right\}$$

with $b > c \geq a$ and $3Mc^2 = -1$. It may be observed that for $c = a$ and $c = b$ the above expression for l reduces to the expression for l given by (8.9) and (6.7), respectively.

If we approximate a , b and c by their initial values a_0 , b_0 and c_0 , then from (10.14) and (10.16) we have

$$(10.17) \quad \begin{cases} p = -J \log \frac{1 - 3Mb_0^2/N}{2} \frac{c_0 a_0}{b_0^2}, \\ l = \frac{J}{a_0^2} \left\{ -\frac{3}{4} \frac{M}{N} (c_0^4 - a_0^4) + (b_0^2 - c_0^2) + \frac{N}{3M} \log \frac{1 - 3Mb_0^2/N}{2} \right\} \end{cases}$$

with $b_0 > c_0 \geq a_0$ and $3Mc_0^2/N = -1$. Choosing any value of c_0 such that $b_0 > c_0 \geq a_0$, the equation $3Mc_0^2/N = -1$ determines the ratio M/N , after

which the equations (10.17) determine the values of p and l . Any point on the l, p curve defined by (10.17) represents the loads which produce complete plastic equilibrium with stress distribution such that

$$\frac{\sigma_{22}}{r^2} \geq \sigma_{33} > \sigma_{11} \quad \text{for } c_0 \geq r \geq a_0 ,$$

$$\sigma_{33} > \frac{\sigma_{22}}{r^2} > \sigma_{11} \quad \text{for } b_0 \geq r > c_0 .$$

SURFACE OF STRONG INSTABILITY—PLASTIC FLOW

11. Basic relations for plastic flow. In a region of plastic flow the following relations are assumed to hold:

$$(11.1) \quad \begin{cases} g^{ii} v_{i,i} = 0 & \text{(equation of incompressibility),} \\ \epsilon_{ii} = \psi \sigma_{ii}^* & \text{(von Mises stress and strain relations),} \\ g^{ik} \sigma_{ii,k} = \rho \left(\frac{\partial v_i}{\partial t} + v_{i,i} v^i \right) & \text{(equations of motion),} \\ \sigma_3 - \sigma_1 = J & \text{(Tresca yield condition),} \end{cases}$$

where v_i are the covariant components of velocity, σ_{ii}^* are the components of stress deviator as defined before and ϵ_{ii} are the components of the rate of strain tensor defined by

$$(11.2) \quad \epsilon_{ii} = \frac{1}{2}(v_{i,i} + v_{i,i}).$$

Also, σ_1 , σ_2 and σ_3 are the principal stresses with the ordering $\sigma_3 \geq \sigma_2 \geq \sigma_1$, J is the material constant as before, and ψ is a positive factor of proportionality.

For convenience in later work, we record here two dynamical conditions on discontinuity across a surface, namely

$$(11.3) \quad \begin{cases} \rho(G - v_n) = \bar{\rho}(G - \bar{v}_n), \\ [\sigma_{ii}]v_i = -\rho(G - v_n)[v_i], \end{cases}$$

where $[\sigma_{ii}]$, $[v_i]$ denote the discontinuities in stress and velocity components, respectively, ρ and $\bar{\rho}$ denote the densities on the two sides of a surface of discontinuity, v_n and $\bar{v}_n$ are the components of the velocities normal to the surface of discontinuity and the quantity G denotes the coordinate velocity of the discontinuity surface in the direction of the unit normal ν . In this problem we will have $G = 0$ and $v_n = \bar{v}_n$, so that the first condition (11.3) is satisfied identically and the second condition becomes

$$(11.4) \quad [\sigma_{ii}]v_i = 0.$$

These equations will be applied later in determining the stresses on the surface of discontinuity. Also, it may be noted that we may have continuity in the stress components without continuity in the velocity components as far as these dynamical conditions are concerned.

12. Surface of strong instability as a boundary of a region of plastic equilibrium and a region of plastic flow. Denote by Σ the intersection of the pipe with a half plane through its central axis, and let us suppose that a disturbance or slip of material occurs along Σ , in consequence of which plastic flow is initiated on one side of this surface while the other side of the surface remains in a state of plastic equilibrium. It will be shown that the resulting discontinuity in the velocity becomes indefinitely large within a finite time, *i.e.*, Σ is a surface of strong instability, under the conditions of Case A or Case C. If the conditions of Case B apply, it will be shown that a discontinuity of this type is not possible.

First assume that the conditions of Case A apply, *i.e.*, $\sigma_{22}/r^2 \geq \sigma_{33} > \sigma_{11}$ for all values of r such that $a \leq r \leq b$. Corresponding to the condition that the two plugged ends of the plastically deformed pipe are kept parallel during the entire period of deformation, we assume that cross sections of the pipe are displaced into cross sections. It follows that the velocity component v_3 is independent of r and θ . A further simplifying assumption will be introduced by the requirement that, within the cross sections, radial lines are displaced into radial lines. Hence the angular velocity v_2/r^2 is independent of r . Also, from symmetry the components v_1 and v_2 of the velocity can be considered to be independent of the z coordinate. In order to avoid separation of material, v_2 must vanish on the plane Σ ; in fact, this is a consequence of the dynamical condition (11.3) since $G = 0$ and the material on one side of the plane Σ is in the equilibrium state. Hence we arrive at the following *conditions on the velocities*, namely:

$$(12.1) \quad v_1 = v_1(r, \theta, t), \quad \frac{v_2}{r^2} = \frac{v_2}{r^2}(\theta, t), \quad \bar{v}_2 = 0, \quad v_3 = v_3(z, t),$$

where a bar over v_2 indicates the value of v_2 evaluated on the plane Σ . Similar notation will be used later for other quantities.

From the rule for covariant differentiation and the relation (12.1), we obtain

$$(12.2) \quad ||v_{i,i}|| = \left\| \begin{array}{ccc} \frac{\partial v_1}{\partial r} & \frac{\partial v_1}{\partial \theta} - \frac{v_2}{r} & 0 \\ \frac{\partial v_2}{\partial r} - \frac{v_2}{r} & \frac{\partial v_2}{\partial \theta} + rv_1 & 0 \\ 0 & 0 & \frac{\partial v_3}{\partial z} \end{array} \right\|.$$

From the definition (11.2) for $\epsilon_{i,j}$ we have

$$(12.3) \quad ||\epsilon_{i,j}|| = \left\| \begin{array}{ccc} \frac{\partial v_1}{\partial r} & \frac{1}{2} \left(\frac{\partial v_1}{\partial \theta} + \frac{\partial v_2}{\partial r} - \frac{2v_2}{r} \right) & 0 \\ \frac{1}{2} \left(\frac{\partial v_1}{\partial \theta} + \frac{\partial v_2}{\partial r} - \frac{2v_2}{r} \right) & \frac{\partial v_2}{\partial \theta} + rv_1 & 0 \\ 0 & 0 & \frac{\partial v_3}{\partial z} \end{array} \right\|.$$

General solution for v_i . In view of (12.2) the first equation in (11.1) can be written as

$$(12.4) \quad \frac{\partial v_1}{\partial r} + \frac{v_1}{r} + \frac{1}{r^2} \frac{\partial v_2}{\partial \theta} + \frac{\partial v_3}{\partial z} = 0.$$

Differentiating this equation partially with respect to z we have

$$\frac{\partial^2 v_3}{\partial z^2} = 0,$$

in view of the relation (12.1). Thus

$$(12.5) \quad v_3 = f(t)z + g(t).$$

Assume that both of the heavily plugged ends of the pipe are held fixed and let $z = 0$ be the plane section of one end. Then $v_3 = 0$ for $z = 0$ at all times t . Hence $g(t) = 0$ and the above equation (12.5) for v_3 becomes $v_3 = f(t)z$. But the other end of the pipe is the plane $z = h$ where h is the length of the deformed pipe and hence $v_3 = f(t)h = 0$ for all t . Hence $f(t) = 0$ and $v_3 = 0$ everywhere.

Differentiating the equation (12.4) partially with respect to r we have

$$\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r} \frac{\partial v_1}{\partial r} - \frac{v_1}{r^2} = 0 \quad \text{or} \quad r^2 \frac{\partial^2 v_1}{\partial r^2} + r \frac{\partial v_1}{\partial r} - v_1 = 0,$$

since v_2/r^2 is function of the coordinate θ and time t only. This is a differential equation of Euler type. Its general solution is given by

$$(12.6) \quad v_1 = Q(\theta, t)r + \frac{S(\theta, t)}{r}.$$

On the plane Σ we have

$$\bar{v}_1 = \bar{Q}r + \frac{\bar{S}}{r}.$$

From equation (12.6) and the fact that $v_3 = 0$ the equation (12.4) becomes

$$\frac{1}{r^2} \frac{\partial v_2}{\partial \theta} = -2Q(\theta, t).$$

Hence

$$(12.7) \quad \frac{v_2}{r^2} = \int_0^\theta -2Q(\phi, t) d\phi,$$

where we have chosen the plane Σ to be the plane $\theta = 0$. Using these results the matrix (12.3) becomes

$$(12.8) \quad ||\bar{\epsilon}_{ij}|| = \begin{vmatrix} \bar{Q} - \frac{\bar{S}}{r^2} & \frac{1}{2} \left(\frac{\partial \bar{Q}}{\partial \theta} r + \frac{1}{r} \frac{\partial \bar{S}}{\partial \theta} \right) & 0 \\ \frac{1}{2} \left(\frac{\partial \bar{Q}}{\partial \theta} r + \frac{1}{r} \frac{\partial \bar{S}}{\partial \theta} \right) & -r^2 \bar{Q} + \bar{S} & 0 \\ 0 & 0 & 0 \end{vmatrix},$$

when evaluated on the plane Σ .

Determination of stress on the flow side of the plane Σ . The plane Σ is assumed to be a surface of discontinuity in velocity components. Applying the dynamical conditions (11.4) and noting that $v_1 = 0$, $v_2 = r$ and $v_3 = 0$ we obtain $[\sigma_{i2}] = 0$. That is,

$$(12.9) \quad \bar{\sigma}_{12} = \bar{\sigma}_{23} = 0, \quad \frac{\bar{\sigma}_{22}}{r^2} = J \left(1 + \log \frac{r}{b} \right),$$

when account is taken of the equations (5.1) and (6.4).

Consider now the stress and rate of strain relations in (11.1), namely

$$\epsilon_{ij} = \psi \sigma_{ij}^*.$$

Putting $i, j = 1, 3; 3, 3$ we have

$$(12.10) \quad \bar{\sigma}_{13} = 0 \quad \text{and} \quad \bar{\sigma}_{33}^* = 0,$$

in view of (12.8). Thus we have only $\bar{\sigma}_{11}$ left to be determined.

From (12.9) and (12.10) we know that $\bar{\sigma}_{ij} = 0$ for $i \neq j$; hence $\bar{\sigma}_{11}$, $\bar{\sigma}_{22}/r^2$ and $\bar{\sigma}_{33}$ are the principal stresses on the flow side of plane Σ . Moreover, the second equation in (12.10) implies

$$\bar{\sigma}_{33} = \frac{1}{2} \left(\frac{\bar{\sigma}_{22}}{r^2} + \bar{\sigma}_{11} \right).$$

This means that either

$$\frac{\bar{\sigma}_{22}}{r^2} \geq \bar{\sigma}_{33} \geq \bar{\sigma}_{11} \quad \text{or} \quad \bar{\sigma}_{11} \geq \bar{\sigma}_{33} \geq \frac{\bar{\sigma}_{22}}{r^2}$$

holds according as $\bar{\sigma}_{11} \leq \bar{\sigma}_{22}/r^2$ or $\bar{\sigma}_{11} \geq \bar{\sigma}_{22}/r^2$. In view of the above inequalities and the fact that the Tresca yield condition is assumed to hold in the plastic flow region, we must have

$$(12.11) \quad \left| \frac{\bar{\sigma}_{22}}{r^2} - \bar{\sigma}_{11} \right| = J.$$

Also, along the line of intersection of the outer cylindrical surface and the plane Σ we have

$$(12.12) \quad \frac{\bar{\sigma}_{22}}{r^2} - \bar{\sigma}_{11} = J - 0 = J$$

from the second equation (12.9) and the fact $\bar{\sigma}_{11} = 0$ from the boundary condition on the outer surface. Hence (12.12) must hold over the entire plane Σ on account of (12.11) and the continuity of $\bar{\sigma}_{11}$ and $\bar{\sigma}_{22}/r^2$ as functions of r . Hence we have

$$(12.13) \quad \bar{\sigma}_{11} = \frac{\bar{\sigma}_{22}}{r^2} - J = J \log \frac{r}{b}$$

over the plane Σ . Combining the above results we can write

$$(12.14) \quad ||\bar{\sigma}_{ij}|| = \begin{vmatrix} J \log \frac{r}{b} & 0 & 0 \\ 0 & J \left(1 + \log \frac{r}{b}\right) r^2 & 0 \\ 0 & 0 & J \left(\frac{1}{2} + \log \frac{r}{b}\right) \end{vmatrix}.$$

Finally, from the definition (3.2) for σ_{ij}^* we calculate

$$(12.15) \quad ||\bar{\sigma}_{ij}^*|| = \begin{vmatrix} -\frac{J}{2} & 0 & 0 \\ 0 & \frac{J}{2} r^2 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

Application of the equations of motion. From the rule for covariant differentiation and the relation (12.14) we calculate

$$(12.16) \quad \begin{cases} \bar{\sigma}_{11,1} = \frac{\partial \bar{\sigma}_{11}}{\partial r}, & \bar{\sigma}_{12,2} = \frac{\partial \bar{\sigma}_{12}}{\partial \theta} + r \bar{\sigma}_{11} - \frac{\bar{\sigma}_{22}}{r}, & \bar{\sigma}_{13,3} = 0, \\ \bar{\sigma}_{21,1} = 0, & \bar{\sigma}_{22,2} = \frac{\partial \bar{\sigma}_{22}}{\partial \theta}, & \bar{\sigma}_{23,3} = 0, \\ \bar{\sigma}_{31,1} = 0, & \bar{\sigma}_{32,2} = 0, & \bar{\sigma}_{33,3} = 0. \end{cases}$$

Using (12.16) and expanding the equations of motion in (11.1) we obtain

$$(12.17) \quad \begin{cases} \frac{\partial \bar{\sigma}_{11}}{\partial r} + \frac{1}{r} \left(\bar{\sigma}_{11} - \frac{\bar{\sigma}_{22}}{r^2} \right) + \frac{1}{r^2} \frac{\partial \bar{\sigma}_{12}}{\partial \theta} = \bar{\rho} \left\{ \frac{\partial \bar{v}_1}{\partial t} + \bar{v}_{1,1} \bar{v}^1 + \bar{v}_{1,2} \bar{v}^2 + \bar{v}_{1,3} \bar{v}^3 \right\}, \\ \frac{1}{r^2} \frac{\partial \bar{\sigma}_{22}}{\partial \theta} = \bar{\rho} \left\{ \frac{\partial \bar{v}_2}{\partial t} + \bar{v}_{2,1} \bar{v}^1 + \bar{v}_{2,2} \bar{v}^2 + \bar{v}_{2,3} \bar{v}^3 \right\}, \\ 0 = \bar{\rho} \left\{ \frac{\partial \bar{v}_3}{\partial t} + \bar{v}_{3,1} \bar{v}^1 + \bar{v}_{3,2} \bar{v}^2 + \bar{v}_{3,3} \bar{v}^3 \right\}. \end{cases}$$

The last equation in (12.17) is satisfied identically on account of the fact that $\bar{v}_2 = 0$ and $\bar{v}_3 = 0$. The second equation in (12.17) asserts that $\partial \bar{\sigma}_{22} / \partial \theta = 0$,

because the right hand side vanishes identically. Substituting the expressions (12.13) and (12.9) for $\bar{\sigma}_{11}$ and $\bar{\sigma}_{22}/r^2$, respectively, and the expressions (12.1) and (12.6) for the velocity components into the first equation of (12.17), we obtain

$$(12.18) \quad \begin{aligned} \frac{1}{r^3} \frac{\partial \bar{\sigma}_{22}}{\partial \theta} &= \bar{\rho} \left\{ \frac{\partial \bar{Q}}{\partial t} r + \frac{1}{r} \frac{\partial \bar{S}}{\partial t} + \left(\bar{Q} - \frac{\bar{S}}{r^2} \right) \left(\bar{Q} + \frac{\bar{S}^2}{r} \right) r \right\} \\ &= \bar{\rho} \left\{ \frac{\partial \bar{Q}}{\partial t} r + \frac{1}{r} \frac{\partial \bar{S}}{\partial t} + \bar{Q}^2 r - \frac{\bar{S}^2}{r^3} \right\}. \end{aligned}$$

To find the expression for $\partial \bar{\sigma}_{12}/\partial \theta$, we consider the equation

$$\epsilon_{12} = \psi \sigma_{12}.$$

Differentiating this equation partially with respect to θ and evaluating it on the plane Σ , we find

$$\frac{\partial \bar{\sigma}_{12}}{\partial \theta} = \frac{1}{\bar{\psi}} \frac{\partial \bar{\epsilon}_{12}}{\partial \theta}.$$

Using (12.3), (12.6) and (12.7) we find

$$\frac{\partial \bar{\epsilon}_{12}}{\partial \theta} = \frac{1}{2} \left(\frac{\partial^2 \bar{v}_1}{\partial \theta^2} + \frac{\partial^2 \bar{v}_2}{\partial r \partial \theta} - \frac{2}{r} \frac{\partial \bar{v}_2}{\partial \theta} \right) = \frac{1}{2} \left(\frac{\partial^2 \bar{Q}}{\partial \theta^2} r + \frac{1}{r} \frac{\partial^2 \bar{S}}{\partial \theta^2} \right).$$

But from the relation

$$\bar{\psi} \left(\frac{\bar{\sigma}_{22}}{r^2} - \bar{\sigma}_{11} \right) = \frac{\bar{\epsilon}_{22}}{r^2} - \bar{\epsilon}_{11},$$

the yield condition (12.13) and the matrix (12.8) we obtain

$$\bar{\psi} = \frac{-2(\bar{Q} - \bar{S}/r^2)}{J} > 0.$$

Hence

$$(12.19) \quad \frac{\partial \bar{\sigma}_{12}}{\partial \theta} = -\frac{J}{4(\bar{Q} - \bar{S}/r^2)} \left(\frac{\partial^2 \bar{Q}}{\partial \theta^2} r + \frac{1}{r} \frac{\partial^2 \bar{S}}{\partial \theta^2} \right).$$

Substituting the expression (12.19) for $\partial \bar{\sigma}_{12}/\partial \theta$ into (12.18) we find

$$-\frac{J}{4r^2(\bar{Q} - \bar{S}/r^2)} \left(r \frac{\partial^2 \bar{Q}}{\partial \theta^2} + \frac{1}{r} \frac{\partial^2 \bar{S}}{\partial \theta^2} \right) = \bar{\rho} \left\{ \frac{\partial \bar{Q}}{\partial t} r + \bar{Q}^2 r + \frac{1}{r} \frac{\partial \bar{S}}{\partial t} - \frac{\bar{S}^2}{r^3} \right\}$$

or

$$\begin{aligned} -\frac{J}{4} \left(\frac{1}{r} \frac{\partial^2 \bar{Q}}{\partial \theta^2} + \frac{1}{r^3} \frac{\partial^2 \bar{S}}{\partial \theta^2} \right) &= \bar{\rho} \left\{ r \left(\bar{Q} \frac{\partial \bar{Q}}{\partial t} + \bar{Q}^3 \right) + \frac{1}{r} \left(\bar{Q} \frac{\partial \bar{S}}{\partial t} - \bar{S} \frac{\partial \bar{Q}}{\partial t} - \bar{Q}^3 \bar{S} \right) \right. \\ &\quad \left. - \frac{1}{r^3} \left(\bar{Q} \bar{S}^2 + \bar{S} \frac{\partial \bar{S}}{\partial t} \right) - \frac{\bar{S}^3}{r^3} \right\}. \end{aligned}$$

This equation holds for all values of the radial coordinate r ; hence the coefficients of the different powers of r must vanish since the coefficients are independent of r . Equating to zero the coefficients of the various powers of r we therefore have

$$\begin{aligned} r: \quad & \bar{Q} \frac{\partial \bar{Q}}{\partial t} + \bar{Q}^3 = 0, \\ r^{-1}: \quad & \frac{J}{4} \frac{\partial^2 \bar{Q}}{\partial \theta^2} + \bar{p} \left(\bar{Q} \frac{\partial \bar{S}}{\partial t} - \bar{S} \frac{\partial \bar{Q}}{\partial t} - \bar{Q}^2 \bar{S} \right) = 0, \\ r^{-3}: \quad & \frac{J}{4} \frac{\partial^2 \bar{S}}{\partial \theta^2} - \bar{p} \left(\bar{Q} \bar{S}^2 + \bar{S} \frac{\partial \bar{S}}{\partial t} \right) = 0, \\ r^{-5}: \quad & \bar{S}^3 = 0. \end{aligned}$$

These equations are equivalent to

$$\bar{S} = 0 = \frac{\partial^2 \bar{S}}{\partial \theta^2} = \frac{\partial^2 \bar{Q}}{\partial \theta^2}$$

and

$$\bar{Q} \left(\frac{\partial \bar{Q}}{\partial t} + \bar{Q}^2 \right) = 0.$$

If $\bar{Q} = 0$ then $\bar{v}_1 = 0$ from (12.6) and the above result that $\bar{S} = 0$. Hence all components $\bar{v}_i = 0$ since $\bar{v}_2 = 0$ by hypothesis and v_3 has been shown to vanish everywhere. But this contradicts the hypothesis that the plane Σ is a surface of discontinuity in velocity components. Hence assume $\bar{Q} \neq 0$; then, integrating the equation

$$\frac{\partial \bar{Q}}{\partial t} + \bar{Q}^2 = 0,$$

we obtain

$$\frac{1}{\bar{Q}} = t - H \quad \text{or} \quad \bar{Q} = \frac{1}{t - H},$$

where H is a constant. Now on the flow side of the plane Σ we have

$$(12.20) \quad \bar{v}_1 = \frac{r}{t - H}, \quad \bar{v}_2 = 0, \quad \bar{v}_3 = 0.$$

Hence

$$||\bar{v}_{i,i}|| = \left\| \begin{array}{ccc} \frac{1}{t - H} & 0 & 0 \\ 0 & -\frac{r^2}{t - H} & 0 \\ 0 & 0 & 0 \end{array} \right\|.$$

To determine the sign of the constant H , we consider the above equation for $\bar{\psi}$, which now becomes

$$\bar{\psi} = \frac{-2\bar{Q}}{J} = -\frac{2}{J(t-H)} > 0$$

for all times t . In particular, we have $\bar{\psi} = 2/JH > 0$ for $t = 0$. Thus $H > 0$. Hence $\bar{v}_1 \rightarrow -\infty$ as $t \rightarrow H$. Accordingly, the plane Σ is a surface of strong instability which separates the material into a region of plastic equilibrium and a region of plastic flow. The plastically deformed pipe will be fractured over this plane in view of our basic hypothesis.

Case B. $\sigma_{33} > \sigma_{22}/r^2 > \sigma_{11}$ for all values of r at complete plastic equilibrium. In this case the intersection Σ of the pipe with a half plane through its central axis cannot be a slip surface. Consequently it is not a surface of strong instability. Suppose the plane Σ were a slip surface; then the velocity components on the flow side of the plane Σ are still given by (12.5), (12.6) and (12.7) because the underlying conditions on the velocity components are the same for Case B as for Case A. Accordingly, the rate of strain tensor is given by the matrix (12.8).

From the dynamical conditions (11.4) we have $[\sigma_{i2}] = 0$ and from the equation $\bar{\psi}\bar{\sigma}_{13} = \bar{\epsilon}_{13}$ and the matrix (12.8) we have $\bar{\sigma}_{13} = 0$. Thus

$$(12.21) \quad \bar{\sigma}_{ii} = 0 \quad \text{for } i \neq j,$$

and hence $\bar{\sigma}_{11}$, $\bar{\sigma}_{22}/r^2$ and $\bar{\sigma}_{33}$ are the principal stresses on the flow side of the plane Σ .

Putting $i, j = 1, 1; 2, 2; 3, 3$ in the second system of equations in (11.1), we have

$$(12.22) \quad \begin{cases} \bar{\psi}\bar{\sigma}_{11}^* = \bar{Q} + \frac{\bar{S}}{r^2}, \\ \bar{\psi}\frac{\bar{\sigma}_{22}^*}{r^2} = -\bar{Q} + \frac{\bar{S}}{r^2}, \\ \bar{\psi}\bar{\sigma}_{33}^* = 0, \end{cases}$$

where we have made use of the matrix (12.8). Since $\bar{\psi}$ is assumed to be positive, the third equation implies $\bar{\sigma}_{33}^* = 0$. This equation implies

$$\bar{\sigma}_{33} = \frac{1}{2}\left(\frac{\bar{\sigma}_{22}}{r^2} + \bar{\sigma}_{11}\right).$$

Thus, on the flow side of the plane Σ we must have either

$$\frac{\bar{\sigma}_{22}}{r^2} \geq \bar{\sigma}_{33} \geq \bar{\sigma}_{11} \quad \text{or} \quad \bar{\sigma}_{11} \geq \bar{\sigma}_{33} \geq \frac{\bar{\sigma}_{22}}{r^2}.$$

In either case, we must have

$$(12.23) \quad \left| \frac{\bar{\sigma}_{22}}{r^2} - \bar{\sigma}_{11} \right| = J$$

if the Tresca yield condition is to hold. But on the intersection lines of the plane Σ with the outer and the inner surface of the pipe we have

$$(12.24) \quad \begin{cases} 0 < \frac{\bar{\sigma}_{22}}{r^2} - \bar{\sigma}_{11} = \frac{\sigma_{22}}{r^2} - \sigma_{11} < \sigma_{33} = J & \text{at } r = b, \\ 0 < \frac{\bar{\sigma}_{22}}{r^2} - \bar{\sigma}_{11} = \frac{\sigma_{22}}{r^2} - \sigma_{11} < \sigma_{33} - \sigma_{11} = J & \text{at } r = a, \end{cases}$$

where use is made of the fact that σ_{22}/r^2 is continuous across Σ from (11.4) and that $\bar{\sigma}_{11} = \sigma_{11} = 0$ at $r = b$ and $\bar{\sigma}_{11} = \sigma_{11} = -p$ at $r = a$ from the boundary conditions; application is also made of the equations (8.4) and (8.7). Now the equations in (12.24) contradict the equation (12.23). Hence the original assumption that the plane Σ is a slip surface is not true; in fact, the plane Σ for this case cannot separate the material into a region of plastic equilibrium and a region of plastic flow since if the velocity components are continuous across Σ , so that Σ is not a slip surface, we likewise have a contradiction because then the equation (12.6) implies $\bar{Q} = \bar{S} = 0$ and hence $\bar{\sigma}_{11}^* = \bar{\sigma}_{22}^*/r^2 = \bar{\sigma}_{33}^* = 0$ from (12.22).

Case C. $\sigma_{22}/r^2 \geq \sigma_{33} > \sigma_{11}$ for $c \geq r \geq a$ and $\sigma_{33} > \sigma_{22}/r^2 > \sigma_{11}$ for $b \geq r > c$ at complete plastic equilibrium, where $b > c \geq a$ and $3Mc^2/N = -1$. In this case an analysis made for Case A is applicable to the part of the plane Σ on which

$$\sigma_{22}/r^2 \geq \sigma_{33} > \sigma_{11} \quad \text{for } c \geq r \geq a.$$

Hence if a discontinuity of the velocity components occurs on this part of the plane Σ , then it will become indefinitely large within finite time. Consequently fracture will take place on this part of the plane Σ . Once fracture is initiated it will propagate along the plane Σ to cause fracture over the entire plane. This gives rise to a problem in crack propagation which we shall not attempt to treat in this paper.

Remark. The intersection of the pipe with a plane normal to its central axis cannot be such a surface of strong instability for any distribution of stresses, i.e., such a plane Σ cannot be a slip surface separating the pipe into a region of plastic equilibrium and a region of plastic flow. Suppose the plane Σ were a slip surface of this character. From circular symmetry the components v_1 will be independent of the angular coordinate θ and the component v_2 must be zero. It will be assumed that the component v_3 is independent of the radial coordinate r ; this condition must obviously be satisfied over the ends of the pipe since the ends are kept parallel during the entire period of loading. Also,

$\bar{v}_3$ must be zero in order to avoid separation of material. Hence we arrive at the following conditions on the velocities:

$$(12.25) \quad v_1 = v_1(r, z, t), \quad v_2 = 0, \quad v_3 = v_3(z, t), \quad \bar{v}_3 = 0.$$

From the equations, the expression (2.3) and the definition (2.4) for co-variant differentiation, we calculate

$$(12.26) \quad ||v_{i,j}|| = \begin{vmatrix} \frac{\partial v_1}{\partial r} & 0 & \frac{\partial v_1}{\partial z} \\ 0 & rv_1 & 0 \\ 0 & 0 & \frac{\partial v_3}{\partial z} \end{vmatrix}.$$

From this matrix and the first equation of (11.1) we have

$$(12.27) \quad \frac{\partial v_1}{\partial r} + \frac{v_1}{r} + \frac{\partial v_3}{\partial z} = 0.$$

Differentiating the equation (12.27) partially with respect to r we obtain

$$\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r} \frac{\partial v_1}{\partial r} - \frac{v_1}{r^2} = 0.$$

The general solution of this equation is given by

$$(12.28) \quad v_1 = Q(t, z)r + \frac{S(t, z)}{r}.$$

From the equations (12.27) and (12.28) we obtain

$$(12.29) \quad v_3 = -2 \int_0^z Q(t, \xi) d\xi,$$

where we have chosen the plane Σ to be the plane $z = 0$. Using the expressions (12.28) and (12.29) to evaluate the matrix (12.26) on the plane Σ , we obtain

$$(12.30) \quad ||\bar{v}_{i,j}|| = \begin{vmatrix} \bar{Q} - \frac{\bar{S}}{r^2} & 0 & \frac{\partial \bar{Q}}{\partial z} r + \frac{\partial \bar{S}}{\partial z} \frac{1}{r} \\ 0 & \bar{Q}r^2 + \bar{S} & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

Assume the stress components σ_{ij} to be continuous across the surface Σ , i.e.,

$$(12.31) \quad [\sigma_{ij}] = 0.$$

Let us consider the second system of equations in (11.1) on the flow side of the plane Σ , namely

$$\bar{\Psi} \bar{\sigma}_{ij}^* = \bar{\epsilon}_{ij}.$$

From the matrix (12.30) and the fact that $\sigma_{i,j} = 0$ for $i \neq j$ it is clear that the above equations are satisfied identically for $i, j = 1, 2; 2, 3$. Also, for $i, j = 1, 3$ we have

$$\frac{\partial \bar{v}_1}{\partial z} = \frac{\partial \bar{Q}}{\partial z} r + \frac{\partial \bar{S}}{\partial z} \frac{1}{r} = 0.$$

The other equations with $i = j$ will be considered in the following different cases.

Case A. $\sigma_{22}/r^2 \geq \sigma_{33} > \sigma_{11}$ for all values of r at the state of complete plastic equilibrium. From the matrices (6.6) and (12.30) we have

$$(12.32) \quad \begin{cases} \frac{J\bar{\psi}}{2} \left(\frac{Mr^2}{N} - 1 \right) = \bar{Q} - \frac{\bar{S}}{r^2}, \\ \frac{J\bar{\psi}}{2} \left(\frac{Mr^2}{N} + 1 \right) = \bar{Q} + \frac{\bar{S}}{r^2}, \\ -J\bar{\psi} \frac{Mr^2}{N} = -2\bar{Q}, \end{cases}$$

by putting $i, j = 1, 1; 2, 2; 3, 3$ in the second system of equations in (11.1). The last equation of (12.32) can be written as

$$(12.33) \quad \bar{\psi} = 2 \frac{\bar{Q}N}{JMr^2}.$$

Substituting the expression (12.33) into the second equation of (12.32) we have

$$(12.34) \quad \bar{Q} = \frac{M\bar{S}}{N}.$$

Using the relation (12.31), the expressions (2.3), (6.2), (6.4) and (6.5) and the definition (2.4) for covariant differentiation, we calculate

$$(12.35) \quad \begin{cases} \bar{\sigma}_{11,1} = \frac{J}{r}, & \bar{\sigma}_{12,2} = Jr, & \bar{\sigma}_{13,3} = \frac{\partial \bar{\sigma}_{13}}{\partial z}, \\ \bar{\sigma}_{21,1} = 0, & \bar{\sigma}_{22,2} = 0, & \bar{\sigma}_{23,3} = \frac{\partial \bar{\sigma}_{23}}{\partial z}, \\ \bar{\sigma}_{31,1} = 0, & \bar{\sigma}_{32,2} = 0, & \bar{\sigma}_{33,3} = \frac{\partial \bar{\sigma}_{33}}{\partial z}. \end{cases}$$

Using the expressions (12.35), the matrix (12.30) and the expressions (12.25) and (12.28) to expand the equation of motion in (11.1), we obtain

$$(12.36) \quad \begin{cases} \frac{\partial \bar{\sigma}_{13}}{\partial z} = \bar{\rho} \left\{ \frac{\partial \bar{Q}}{\partial t} r + \frac{\partial \bar{S}}{\partial t} \frac{1}{r} + \bar{Q}^2 r - \bar{S}^2 \frac{1}{r^3} \right\}, \\ \frac{\partial \bar{\sigma}_{23}}{\partial z} = 0; & \frac{\partial \bar{\sigma}_{33}}{\partial z} = 0. \end{cases}$$

To find the expression for $\partial\bar{\sigma}_{13}/\partial z$ we consider the relation

$$\psi\sigma_{13} = \epsilon_{13}.$$

Differentiating this relation partially with respect to z and evaluating on the plane Σ , we have

$$\psi \frac{\partial\bar{\sigma}_{13}}{\partial z} = \frac{1}{2} \left(\frac{\partial^2 \bar{v}_1}{\partial z^2} + \frac{\partial^2 \bar{v}_3}{\partial z \partial r} \right) = \frac{1}{2} \left(\frac{\partial^2 \bar{Q}}{\partial z^2} r + \frac{\partial^2 \bar{S}}{\partial z^2} \frac{1}{r} \right),$$

in view of the equation (12.28) and the fact that $\bar{\sigma}_{13} = 0$ and $v_3 = v_3(t, z)$. Hence

$$\frac{\partial\bar{\sigma}_{13}}{\partial z} = \frac{1}{2\bar{\psi}} \left(\frac{\partial^2 \bar{Q}}{\partial z^2} r + \frac{\partial^2 \bar{S}}{\partial z^2} \frac{1}{r} \right).$$

Substituting the expression (12.33) for $\bar{\psi}$ into this equation we get

$$(12.37) \quad \frac{\partial\bar{\sigma}_{13}}{\partial z} = \frac{JM}{4N} \frac{1}{\bar{Q}} \left(\frac{\partial^2 \bar{Q}}{\partial z^2} r^3 + \frac{\partial^2 \bar{S}}{\partial z^2} r \right).$$

Using the expression (12.37) for $\partial\bar{\sigma}_{13}/\partial z$, the first equation of (12.36) becomes

$$(12.38) \quad \frac{JM}{4N\bar{Q}} \left(\frac{\partial^2 \bar{Q}}{\partial z^2} r^3 + \frac{\partial^2 \bar{S}}{\partial z^2} r \right) = \bar{p} \left\{ \frac{\partial \bar{Q}}{\partial t} r + \frac{\partial \bar{S}}{\partial t} \frac{1}{r} + \bar{Q}^2 r - \bar{S}^2 \frac{1}{r^3} \right\}.$$

This equation holds for all values of r . This is possible if, and only if, the coefficients of the different powers of r vanish identically since the coefficients are independent of r . In particular, the coefficient of r^{-3} vanishes identically. That is, $\bar{S}^2 = 0$. Then (12.34) asserts $\bar{Q} = 0$ because $N > 0$ in this case. Thus

$$\bar{v}_1 = \bar{v}_2 = \bar{v}_3 = 0,$$

in view of the expressions (12.25) and (12.28); hence the plane Σ is not a slip surface. Consequently it is not a surface of strong instability.

Case B. $\sigma_{33} > \sigma_{22}/r^2 > \sigma_{11}$ for all values of r at the state of complete plastic equilibrium. From the matrices (8.8) and (12.30) we obtain

$$(12.39) \quad \begin{cases} -\bar{\psi}J \frac{N - Mr^2}{N - 3Mr^2} = \bar{Q} - \frac{\bar{S}}{r^2}, \\ \bar{\psi}J \frac{N + Mr^2}{N - 3Mr^2} = \bar{Q} + \frac{\bar{S}}{r^2}, \\ -\bar{\psi}J \frac{2Mr^2}{N - 3Mr^2} = -2\bar{Q}, \end{cases}$$

by putting $i, j = 1, 1; 2, 2; 3, 3$ in the second system of equations in (11.1). The last equation in (12.39) can be written as

$$(12.40) \quad \bar{\psi} = \frac{(N - 3Mr^2)\bar{Q}}{JMr^2}.$$

Substituting the expression (12.40) into the second equation of (12.39), we have

$$(12.41) \quad \bar{Q} = \frac{M}{N} \bar{S}.$$

Using the relations (12.31), the expressions (2.3), (8.4), (8.6) and (8.7) and the definition (2.4) for covariant differentiation, we find

$$(12.42) \quad \begin{cases} \bar{\sigma}_{11,1} = \frac{2NJ}{(N-3Mr^2)r}, & \bar{\sigma}_{12,2} = \frac{2NJr}{N-3Mr^2}, & \bar{\sigma}_{13,3} = \frac{\partial \bar{\sigma}_{13}}{\partial z}, \\ \bar{\sigma}_{21,1} = 0, & \bar{\sigma}_{22,2} = 0, & \bar{\sigma}_{23,3} = \frac{\partial \bar{\sigma}_{23}}{\partial z}, \\ \bar{\sigma}_{31,1} = 0, & \bar{\sigma}_{32,2} = 0, & \bar{\sigma}_{33,3} = \frac{\partial \bar{\sigma}_{33}}{\partial z}. \end{cases}$$

Using the expressions in (12.42), the matrix (12.30) and the expressions in (12.25) to expand the equations of motion in (11.1), we obtain

$$(12.43) \quad \begin{cases} \frac{\partial \bar{\sigma}_{13}}{\partial z} = \bar{\rho} \left\{ \frac{\partial \bar{Q}}{\partial t} r + \frac{\partial \bar{S}}{\partial t} \frac{1}{r} + \bar{Q} r^2 - \frac{\bar{S}}{r^3} \right\}, \\ \frac{\partial \bar{\sigma}_{23}}{\partial z} = 0; \quad \frac{\partial \bar{\sigma}_{33}}{\partial z} = 0. \end{cases}$$

By the same method as used in Case A, we find

$$\frac{\partial \bar{\sigma}_{13}}{\partial z} = \frac{1}{2\bar{\psi}} \left(\frac{\partial^2 \bar{Q}}{\partial z^2} r + \frac{\partial^2 \bar{S}}{\partial z^2} \frac{1}{r} \right).$$

Substituting the expression (12.40) into this equation, we have

$$(12.44) \quad \frac{\partial \bar{\sigma}_{13}}{\partial z} = \frac{JM}{2\bar{Q}(N-3Mr^2)} \left(\frac{\partial^2 \bar{Q}}{\partial z^2} r^3 + \frac{\partial^2 \bar{S}}{\partial z^2} r \right).$$

Using the expression (12.44) for $\partial \bar{\sigma}_{13} / \partial z$, the first equation in (12.43) becomes

$$\frac{JM}{2\bar{Q}} \left(\frac{\partial^2 \bar{Q}}{\partial z^2} r^3 + \frac{\partial^2 \bar{S}}{\partial z^2} r \right) = \bar{\rho} \left\{ \frac{\partial \bar{Q}}{\partial t} r + \frac{\partial \bar{S}}{\partial t} \frac{1}{r} + \bar{Q} r^2 - \bar{S} \frac{1}{r^3} \right\} (N - Mr^2).$$

This equation holds for all values of r . Hence the coefficients of the different powers of r must vanish identically since the coefficients are independent of r . In particular, the coefficient of r^{-3} vanishes. That is, $\bar{S}^2 = 0$. Then (12.41) asserts that $\bar{Q} = 0$ provided $N \neq 0$, *i.e.*, excluding the case of pure tension. Thus if $N \neq 0$ we have

$$\bar{v}_1 = \bar{v}_2 = \bar{v}_3 = 0,$$

in view of the expressions (12.25) and (12.28). Hence the plane Σ cannot be a slip surface. Consequently it is not a surface of strong instability.

Case C. Since this case is just a combination of Cases A and B, the plane Σ cannot be a slip surface according to the reasons given for Case A and Case B.

13. Surface of strong instability in plastic flow region. Let Σ be the intersection of the pipe with a plane normal to its central axis. If the material of the pipe on both sides of the plane Σ is in the state of plastic flow such that the stress components σ_{ij} and their partial derivatives are continuous, but the velocity components v_i are discontinuous across the plane Σ , we are going to show that the plane Σ will be a surface of strong instability in the plastic flow region, provided $\bar{\sigma}_{33}^* > 0$, where as before the bar denotes evaluation on Σ . That is, these discontinuities in velocity components will become indefinitely large within finite time. Hence the pipe will be subjected to fracture over the plane Σ .

Now the component v_2 must be zero and the components v_1 and v_3 independent of the angular coordinates θ from the symmetry of the problem. Corresponding to the condition that the two ends of the pipe are always kept parallel during the entire period of loading, we assume that the cross sections of the pipe are displaced into cross sections. It follows that the component v_3 is independent of the radial coordinates r . Assuming the incompressibility condition for plastic flow to be satisfied by the special requirement that $\rho = \text{constant}$ everywhere, it follows from the first dynamical condition (11.3) that the component v_3 is continuous across Σ . Hence $\bar{v}_3$ must be zero, in view of the continuity of the stress components across Σ and the second equation in (11.3). Hence we arrive at the same conditions on the velocity components as in the remark of Sect. 12, namely

$$(13.1) \quad v_1 = v_1(r, t, z), \quad v_2 = 0, \quad v_3 = v_3(t, z), \quad \bar{v}_3 = 0.$$

These general equations are valid for material on either side of the plane Σ . Consequently we will use the same notation for the components on each side of the plane Σ until we have to distinguish between them.

Using the expressions (13.1), (2.3) and the definition (2.4) for covariant differentiation, we calculate

$$(13.2) \quad ||v_{i,j}|| = \begin{vmatrix} \frac{\partial v_1}{\partial r} & 0 & \frac{\partial v_1}{\partial z} \\ 0 & rv_1 & 0 \\ 0 & 0 & \frac{\partial v_3}{\partial z} \end{vmatrix}.$$

Using the definition for ϵ_{ij} we have

$$(13.3) \quad ||\epsilon_{ij}|| = \begin{vmatrix} \frac{\partial v_1}{\partial r} & 0 & \frac{1}{2} \frac{\partial v_1}{\partial z} \\ 0 & rv_1 & 0 \\ \frac{1}{2} \frac{\partial v_1}{\partial z} & 0 & \frac{\partial v_3}{\partial z} \end{vmatrix}.$$

Using the matrix (13.2), the equation of continuity in (11.1) can be written as

$$(13.4) \quad \frac{\partial v_1}{\partial r} + \frac{v_1}{r} + \frac{\partial v_3}{\partial z} = 0.$$

Differentiating this equation partially with respect to r we get

$$r^2 \frac{\partial^2 v_1}{\partial r^2} + r \frac{\partial v_1}{\partial r} - v_1 = 0,$$

in view of the equations in (13.1). The general solution of this equation is given by

$$(13.5) \quad v_1 = Q(t, z)r + \frac{S(t, z)}{r}.$$

Substituting the expression (13.5) for v_1 into (13.4), we find

$$(13.6) \quad v_3 = -\int_0^z 2Q(t, \xi) d\xi,$$

where we have assumed the plane Σ to be given by $z = 0$.

From now on we will use a bar over a quantity to denote the value of that quantity on the plane Σ , and use $\bar{f}_{i,}$ to denote the value of the function $f(r, t, z)$ on the side i of the plane Σ with $i = 1$ or $i = 2$. Using the expressions (13.5) and (13.6) for v_1 and v_3 , the matrix (13.2) can now be written as

$$(13.7) \quad ||\bar{v}_{i,}| = \begin{vmatrix} \bar{Q} - \frac{\bar{S}}{r^2} & 0 & \frac{\partial \bar{Q}}{\partial z} r + \frac{\partial \bar{S}}{\partial z} \frac{1}{r} \\ 0 & \bar{Q}r^2 + \bar{S} & 0 \\ 0 & 0 & -2\bar{Q} \end{vmatrix},$$

when evaluated on the plane Σ .

From the above assumption of the continuity of the stresses and their partial derivatives across the plane Σ and from the equations of motion in (11.1), we obtain

$$(13.8) \quad 0 = \frac{\partial [v_i]}{\partial t} + [v_{i,} v^i].$$

Since v_2 and v_3 vanish on each side of the plane Σ , we see that the equation in (13.8) is satisfied identically for $i = 2$ and 3. Putting $i = 1$, we have

$$(13.9) \quad 0 = \frac{\partial [v_1]}{\partial t} + \left[\frac{\partial v_1}{\partial r} v^1 \right].$$

Using the expression (13.5) and the matrix (13.7), the equation (13.9) can be written as

$$\frac{d[Q]}{dt} r + \frac{d[S]}{dt} \frac{1}{r} + \left[\left(Q - \frac{S}{r^2} \right) \left(Q + \frac{S}{r^2} \right) r \right] = 0$$

or

$$\left(\frac{d[Q]}{dt} + [Q^2] \right) r + \frac{d[S]}{dt} \frac{1}{r^2} - [S^2] \frac{1}{r^3} = 0,$$

in which we observe that the functions $Q(t, z)$ and $S(t, z)$ become functions $Q(t)$ and $S(t)$ of t alone over the plane Σ , *i.e.*, the plane $z = 0$. This equation holds for all values of r ; hence it is necessary that the coefficients of the different powers of r vanish since the coefficients are independent of r . Consequently we have

$$(13.10) \quad \frac{d[Q]}{dt} = -[Q^2], \quad \frac{d[S]}{dt} = 0, \quad [S^2] = 0.$$

The last equation of (13.10) asserts that either

$$[S] = 0 \quad \text{or} \quad \bar{S}_{(1)} = -\bar{S}_{(2)}.$$

Now consider the second system of equations in (11.1). Evaluating these equations on the side 1 and side 2 of the plane Σ we obtain

$$(13.11) \quad \begin{cases} \bar{\psi}_{(1)} \bar{\sigma}_{11}^* = \bar{Q}_{(1)} - \frac{\bar{S}_{(1)}}{r^2}, & \bar{\psi}_{(1)} \bar{\sigma}_{12} = 0, \\ \bar{\psi}_{(1)} \bar{\sigma}_{13} = \frac{1}{2} \left(\frac{\partial \bar{Q}_{(1)}}{\partial z} r + \frac{\partial \bar{S}_{(1)}}{\partial z} \frac{1}{r} \right), & \bar{\psi}_{(1)} \frac{\bar{\sigma}_{22}^*}{r^2} = \bar{Q}_{(1)} + \frac{\bar{S}_{(1)}}{r^2}, \\ \bar{\psi}_{(1)} \bar{\sigma}_{23} = 0, & \bar{\psi}_{(1)} \bar{\sigma}_{33}^* = -2\bar{Q}_{(1)}; \end{cases}$$

$$(13.12) \quad \begin{cases} \bar{\psi}_{(2)} \bar{\sigma}_{11}^* = \bar{Q}_{(2)} - \frac{\bar{S}_{(2)}}{r^2}, & \bar{\psi}_{(2)} \bar{\sigma}_{12} = 0, \\ \bar{\psi}_{(2)} \bar{\sigma}_{13} = \frac{1}{2} \left(\frac{\partial \bar{Q}_{(2)}}{\partial z} r + \frac{\partial \bar{S}_{(2)}}{\partial z} \frac{1}{r} \right), & \bar{\psi}_{(2)} \frac{\bar{\sigma}_{22}^*}{r^2} = \bar{Q}_{(2)} + \frac{\bar{S}_{(2)}}{r^2}, \\ \bar{\psi}_{(2)} \bar{\sigma}_{23} = 0, & \bar{\psi}_{(2)} \bar{\sigma}_{33}^* = -2\bar{Q}_{(2)}, \end{cases}$$

respectively, where we have made use of the matrix (13.7). To determine the strong instability property of the slip surface Σ , we have the following three possible cases to be considered.

Case 1. $\bar{\sigma}_{33}^* = 0$ at time $t = 0$ for some value of r where $b \geq r \geq a$. From the sixth equations in (13.11) and (13.12) we see that

$$\bar{Q}_{(1)} = \bar{Q}_{(2)} = 0$$

since $\bar{\psi}_{(1)} > 0$ and $\bar{\psi}_{(2)} > 0$. If $[S] = 0$ it follows from (13.5) that $[v_1] = 0$. Hence from (13.1) the components v_i are continuous across Σ and hence Σ is not a slip surface. Hence assume $\bar{S}_{(1)} = -\bar{S}_{(2)}$. Then the first equation in (13.11) and the first equation in (13.12) become

$$\bar{\psi}_{(1)} \bar{\sigma}_{11}^* = -\frac{\bar{S}_{(1)}}{r^2}, \quad \bar{\psi}_{(2)} \bar{\sigma}_{11}^* = \frac{\bar{S}_{(1)}}{r^2}.$$

This is impossible for $\bar{\sigma}_{11}^* > 0$ or $\bar{\sigma}_{11}^* < 0$. If $\bar{\sigma}_{11}^* = 0$ then $\bar{S}_{(1)} = \bar{S}_{(2)} = 0$. Thus we always have $[Q] = 0$ and $[S] = 0$ in this case. Accordingly, the plane Σ cannot be a slip surface in view of the expressions (13.1) and (13.5).

Case 2. $\bar{\sigma}_{33}^* > 0$ at time $t = 0$ for all values of r with $b \geq r \geq a$. From the first and sixth equations in (13.11) and (13.12) we have

$$\frac{\bar{Q}_{(1)} - \bar{S}_{(1)}/r^2}{-2\bar{Q}_{(1)}} = \frac{\bar{\psi}_{(1)}\bar{\sigma}_{11}^*}{\bar{\psi}_{(1)}\bar{\sigma}_{33}^*} = \frac{\bar{\sigma}_{11}^*}{\bar{\sigma}_{33}^*} = \frac{\bar{\psi}_{(2)}\bar{\sigma}_{11}^*}{\bar{\psi}_{(2)}\bar{\sigma}_{33}^*} = \frac{\bar{Q}_{(2)} - \bar{S}_{(2)}/r^2}{-2\bar{Q}_{(2)}}$$

or

$$(13.13) \quad \frac{\bar{S}_{(1)}}{\bar{Q}_{(1)}} = \frac{\bar{S}_{(2)}}{\bar{Q}_{(2)}}.$$

First, if $\bar{S}_{(1)} = \bar{S}_{(2)} \neq 0$ then $\bar{Q}_{(1)} = \bar{Q}_{(2)}$, in view of the equation (13.13). Consequently from the expressions (13.1) and (13.5) we have

$$[v_1] = [v_2] = [v_3] = 0.$$

But this contradicts our hypothesis that the plane Σ is a slip surface. Secondly, if $\bar{S}_{(1)} = -\bar{S}_{(2)} \neq 0$ then $\bar{Q}_{(1)} = -\bar{Q}_{(2)}$; but under the condition that $\bar{\sigma}_{33}^* > 0$ this is also impossible, in view of the sixth equations in (13.11) and (13.12), respectively. Hence in either case we must have $\bar{S}_{(1)} = \bar{S}_{(2)} = 0$. From the above result $[S] = 0$ at $t = 0$. But from the second equation (13.10) $[S]$ is independent of t . Hence $[s] = 0$ for all times t .

Without loss of generality, but for definiteness, we may assume

$$\bar{Q}_{(2)}(0) < \bar{Q}_{(1)}(0) < 0,$$

so that

$$[Q(0)] = \bar{Q}_{(2)}(0) - \bar{Q}_{(1)}(0) < 0.$$

Since

$$[Q(t)^2] = (\bar{Q}_{(2)}(t))^2 - (\bar{Q}_{(1)}(t))^2 = [Q(t)]^2 + 2\bar{Q}_{(1)}(t)[Q(t)],$$

the first equation in (13.10) can be written as

$$(13.14) \quad \frac{d[Q(t)]}{dt} = -[Q(t)]^2 - 2\bar{Q}_{(1)}(t)[Q(t)].$$

In order to establish the strong instability of the surface Σ , we first prove two important facts, namely (1) $\bar{Q}_{(i)}(t) < 0$ for all time t and (2) if $[Q(0)] < 0$ then $[Q(t)] < 0$ for all time t .

First we prove that the statement (2) implies the statement (1). Consequently we assume the statement (2) holds. Suppose the statement (1) is not true. This means that one of the $\bar{Q}_{(i)}(t)$, at least, vanishes or becomes positive at some instant t . Hence suppose that $\bar{Q}_{(1)}(\tau) \geq 0$. If $\bar{Q}_{(1)}(\tau) > 0$ then from the continuity of $\bar{Q}_{(1)}(t)$ as function of t we can find an instant τ_0 such that $\bar{Q}_{(1)}(\tau_0) = 0$ with $\tau_0 < \tau$. Hence we may assume that $\bar{Q}_{(1)}(\tau) = 0$. From the sixth equations in (13.11) and (13.12) we see that $\bar{Q}_{(1)}(\tau) = 0$ implies $\bar{Q}_{(2)}(\tau) = 0$. Thus $[Q(\tau)] = 0$. This contradicts the statement (2); so (1) must hold if (2) holds.

Now we prove the statement (2). Suppose the statement is false. This means

that there is a *first* instant $t = \tau$ such that $[Q(\tau)] = 0$. There are only three possibilities for the values of $\bar{Q}_{(1)}(\tau)$, namely

$$\text{Case (a)} \quad \bar{Q}_{(1)}(\tau) = \bar{Q}_{(2)}(\tau) > 0,$$

$$\text{Case (b)} \quad \bar{Q}_{(1)}(\tau) = \bar{Q}_{(2)}(\tau) = 0,$$

$$\text{Case (c)} \quad \bar{Q}_{(1)}(\tau) = \bar{Q}_{(2)}(\tau) < 0.$$

Case (a) cannot occur. If $\bar{Q}_{(1)}(\tau) > 0$ then there is a τ_0 such that $\bar{Q}_{(1)}(\tau_0) = 0$ with $\tau_0 < \tau$ since $\bar{Q}_{(1)}(0) < 0$ from the sixth equation in (13.11) and $\bar{Q}_{(1)}(t)$ is assumed to be a continuous function of the time t . But $\bar{Q}_{(1)}(\tau_0) = 0$ implies $\bar{Q}_{(2)}(\tau_0) = 0$ as observed above. Hence $[Q(\tau_0)] = 0$ with $\tau_0 < \tau$. This contradicts the assumption that τ is the *first* instant such that $[Q(\tau)] = 0$.

Suppose Case (b) occurs. Then $\tau > 0$, otherwise $[Q(0)] = 0$, which contradicts the assumption that $[Q(0)] < 0$. Since $[Q(t)]$ is a continuous and differentiable function of the time t defined on the interval $\langle 0, \tau \rangle$, it follows from the mean value theorem that there is a τ_0 such that

$$(13.15) \quad \frac{[Q(\tau)] - [Q(0)]}{\tau - 0} = \left(\frac{d[Q(t)]}{dt} \right)_{t=\tau_0} = -[Q(\tau_0)]^2 - 2\bar{Q}_{(1)}(\tau_0)[Q(\tau_0)],$$

with $\tau > \tau_0 > 0$. Since τ is the *first* instant such that $\bar{Q}_{(1)}(\tau) = \bar{Q}_{(2)}(\tau) = 0$, we must have $\bar{Q}_{(1)}(\tau_0) < 0$ and $[Q(\tau_0)] < 0$. Consequently the right hand side of (13.15) is less than zero, while the left hand side in (13.15) is greater than zero. This contradiction shows that Case (b) cannot occur.

Suppose now Case (c) occurs. This means that there is a *first* instant τ as above such that (13.15) holds with $\tau > \tau_0 > 0$. Since τ is the *first* instant such that $[Q(\tau)] = 0$, we must have $[Q(\tau_0)] < 0$. Hence we cannot have $\bar{Q}_{(1)}(\tau_0) = \bar{Q}_{(2)}(\tau_0) = 0$. If $\bar{Q}_{(1)}(\tau_0) > 0$, then from the continuity of $\bar{Q}_{(1)}(t)$ and the fact that $\bar{Q}_{(1)}(0) < 0$ we can find a τ_1 such that $\bar{Q}_{(1)}(\tau_1) = 0$ with $\tau_0 > \tau_1 > 0$. But then $\bar{Q}_{(1)}(\tau_1) = \bar{Q}_{(2)}(\tau_1) = 0$, in view of the sixth equations in (13.11) and (13.12). Consequently $[Q(\tau_1)] = 0$ with $\tau > \tau_0 > \tau_1 > 0$. This is also a contradiction. Thus $\bar{Q}_{(1)}(\tau_0) < 0$. Now the right hand side in (13.15) is less than zero while the left hand side is greater than zero. This contradiction shows that Case (c) cannot occur.

What we have now proved is that there is no first instant τ such that $[Q(\tau)] = 0$. Hence $[Q(t)] < 0$ for all t .

We can now prove the strong instability of the plane Σ as follows. Since $[Q(t)] < 0$ and $\bar{Q}_{(1)}(t) < 0$ for all time t , we have $-2\bar{Q}_{(1)}(t)[Q(t)] < 0$ for all time t . Hence the equation (13.14) implies

$$\frac{d[Q(t)]}{dt} < -[Q(t)]^2 \quad \text{or} \quad \frac{d[Q(t)]}{-[Q(t)]^2} > dt.$$

Integrating this inequality from the time $t = 0$ to time t , we obtain

$$\frac{1}{[Q(t)]} - \frac{1}{[Q(0)]} > t \quad \text{or} \quad \frac{1}{[Q(t)]} > t + \frac{1}{[Q(0)]}.$$

Multiplying the second inequality on the right by $[Q(t)]$ we get

$$1 < \left\{ t + \frac{1}{[Q(0)]} \right\} [Q(t)].$$

For times t such that $t + 1/[Q(0)] < 0$, we have

$$(13.16) \quad [Q(t)] < \frac{1}{t + 1/[Q(0)]}.$$

Now the right hand side of (13.16) approaches $-\infty$ as t approaches $-1/[Q(0)]$. Hence the left hand side $[Q(t)] \rightarrow -\infty$ as t approaches some value of t which is less than or equal to $-1/[Q(0)]$. Thus in this case the plane Σ is a surface of strong instability. Consequently the pipe will be fractured over this surface by our basic hypothesis.

Case 3. $\bar{\sigma}_{33}^* < 0$ at time $t = 0$ for all values of r with $b \geq r \geq a$. Using the same arguments as used in Case 2 we will also have $\bar{S}_{(1)} = \bar{S}_{(2)} = 0$ at $t = 0$. Without loss of generality, but for convenience, we assume $\bar{Q}_{(2)} > \bar{Q}_{(1)} > 0$, so that $[Q(0)] = \bar{Q}_{(2)}(0) - \bar{Q}_{(1)}(0) > 0$. Assume $t = \tau$ is the *first* time for which $[Q(\tau)] = 0$. Hence $[Q(t)] > 0$ for $t < \tau$ since $[Q(0)] > 0$ and $[Q(t)]$ is a continuous function of t . Similarly, $\bar{Q}_{(2)}(t) > 0$. Hence the right hand side of (13.14) is negative for $t < \tau$ and it follows that $[Q(t)]$ decreases continuously to zero as $t \rightarrow \tau$. If there is no first time $t = \tau$ for which the condition $[Q(\tau)] = 0$ is satisfied then $[Q(t)] > 0$ for all times t . It follows from the sixth equations in (13.11) and (13.12) that $\bar{Q}_{(1)}(t) > 0$ for all times t . Hence the right hand side of (13.14) is negative and thus $[Q(t)]$ is a continuously decreasing function of t . Hence Σ is not a surface of strong instability.

Remark. Let Σ be the intersection of the pipe with a half plane through its central axis. If the material of the pipe on both sides of the plane Σ is in the state of plastic flow such that the components of the stress and their partial derivatives are continuous across the plane Σ , we are going to show that the plane Σ can be a slip surface only when $\sigma_{33}^* = 0$ everywhere and $\bar{\sigma}_{22}^*/r^2 > \bar{\sigma}_{11}^*$ for all values of r .

With the same underlying assumptions as in Case A in Sect. 12, we arrive at the following general conditions:

$$(13.17) \quad v_1 = v_1(r, \theta, t), \quad \frac{v_2}{r^2} = \frac{v_2}{r^2}(\theta, t), \quad v_3 = v_3(z, t).$$

These general conditions are valid for the flow on either side of the plane Σ . Consequently we will use the same notation for the velocity components on each side of the plane Σ until we have to distinguish between them. For the same reasons as in the beginning of this section, we have

$$(13.18) \quad \bar{v}_2 = 0.$$

Using the expressions (2.3) and (13.17) and the definition (2.4) for covariant differentiation, we calculate

$$(13.19) \quad ||v_{i,i}|| = \begin{vmatrix} \frac{\partial v_1}{\partial r} & \frac{\partial v_1}{\partial \theta} - \frac{v_2}{r} & 0 \\ \frac{\partial v_2}{\partial r} - \frac{v_2}{r} & rv_1 + \frac{\partial v_2}{\partial \theta} & 0 \\ 0 & 0 & \frac{\partial v_3}{\partial z} \end{vmatrix}.$$

From the matrix (13.19) and the first equation in (11.1) we obtain

$$(13.20) \quad \frac{\partial v_1}{\partial r} + \frac{v_1}{r} + \frac{1}{r^2} \frac{\partial v_2}{\partial \theta} + \frac{\partial v_3}{\partial z} = 0.$$

Differentiating this equation partially with respect to r and then integrating, we obtain

$$(13.21) \quad v_1 = Q(\theta, t)r + \frac{S(\theta, t)}{r}.$$

When evaluated on the plane Σ , the matrix (13.19) can therefore be written as

$$(13.22) \quad ||\bar{v}_{i,i}|| = \begin{vmatrix} \bar{Q} - \frac{\bar{S}}{r^2} & \frac{\partial \bar{Q}}{\partial \theta} r + \frac{\partial \bar{S}}{\partial \theta} \frac{1}{r} & 0 \\ 0 & \bar{Q}r^2 + \bar{S} + \frac{\partial \bar{v}_2}{\partial \theta} & 0 \\ 0 & 0 & \frac{\partial \bar{v}_3}{\partial z} \end{vmatrix},$$

when we make use of the equation (13.21).

From the continuity of the components of the stress and their partial derivatives across the plane Σ and the equations of motion, we find

$$(13.23) \quad 0 = \frac{\partial [v_i]}{\partial t} + [v_{i,i}v^i].$$

Putting $i = 2, 3$ in (13.23), we have

$$\frac{\partial [v_2]}{\partial t} + [v_{2,1}v^1] + [v_{2,2}v^2] + [v_{2,3}v^3] = 0,$$

$$\frac{\partial [v_3]}{\partial t} + [v_{3,1}v^3] + [v_{3,2}v^2] + [v_{3,3}v^3] = 0.$$

These two equations are satisfied identically, in view of (13.17) and (13.18) and the matrix (13.22). Putting $i = 1$ in (13.23), it becomes

$$\frac{\partial [v_1]}{\partial t} + [v_{1,1}v^1] + [v_{1,2}v^2] + [v_{1,3}v^3] = 0$$

or

$$\frac{\partial[Q]}{\partial t} r + \frac{\partial[S]}{\partial t} \frac{1}{r} + [Q^2]r + [S^2] \frac{1}{r^3} = 0,$$

when use is made of the equation (13.21). This equation holds for all values of r . Since the coefficients of the different powers of r are independent of r , these coefficients must vanish identically. That is,

$$(13.24) \quad \frac{\partial[Q]}{\partial t} + [Q^2] = 0, \quad \frac{\partial[S]}{\partial t} = 0, \quad [S^2] = 0.$$

The last equation of (13.24) implies that either

$$(13.25) \quad \bar{S}_{(1)} = \bar{S}_{(2)} \quad \text{or} \quad \bar{S}_{(1)} = -\bar{S}_{(2)}.$$

Consider now the value of $\bar{\sigma}_{33}^*$ at any point on the plane Σ at the time $t = 0$. If $\bar{\sigma}_{33}^* \neq 0$ at any point P on Σ , then $\bar{\sigma}_{33}^* \neq 0$ in some neighborhood N_p of P since $\bar{\sigma}_{33}^*$ is a continuous function on the plane Σ . But from the stress-strain relation in (11.1) we have

$$[\psi]\bar{\sigma}_{33}^* = \frac{\partial[v_3]}{\partial z} = 0,$$

and hence we see that $[\psi] = 0$ in the neighborhood N_p . Consequently the relation

$$[\psi]\bar{\sigma}_{11}^* = [Q] - \frac{[S]}{r^2} = 0$$

holds for all values of r in N_p . This is possible if, and only if,

$$[Q] = [S] = 0$$

at the time $t = 0$, since $[Q]$ and $[S]$ are independent of r and hence $[v_i] = 0$. Thus, if the plane Σ is a slip surface, we must have $\bar{\sigma}_{33}^* = 0$ on Σ . Hence we assume $\bar{\sigma}_{33}^* = 0$ in the following discussion.

The relations $\bar{\sigma}_{33}^* = 0$ and $\bar{\psi}\bar{\sigma}_{33}^* = \partial\bar{v}_3/\partial z$ imply $\bar{v}_3 = f(t)$. But the component v_3 must vanish or have opposite algebraic sign at the two ends of the pipe; so $\bar{v}_3 = f(t) = 0$. Since v_3 is independent of θ we must therefore have $v_3 = 0$ everywhere. Consequently the relations

$$\psi\sigma_{33}^* = \frac{\partial v_3}{\partial z} = 0 \quad \text{and} \quad \psi > 0$$

imply that $\sigma_{33}^* = 0$ everywhere. The matrix (12.22) now becomes

$$(13.26) \quad ||\bar{v}_{i,j}|| = \begin{vmatrix} \bar{Q} - \frac{\bar{S}}{r^2} & \frac{\partial\bar{Q}}{\partial\theta} r + \frac{\partial\bar{S}}{\partial\theta} \frac{1}{r} & 0 \\ 0 & -\bar{Q}r^2 + \bar{S} & 0 \\ 0 & 0 & 0 \end{vmatrix},$$

when we make use of the fact that $v_a = 0$ everywhere and the equations (13.20) and (13.21).

Evaluating the second system of equations in (11.1) on the two sides of the plane Σ , we obtain

$$(13.27) \quad \begin{cases} \bar{\psi}_{(1)} \bar{\sigma}_{11}^* = \bar{Q}_{(1)} - \frac{\bar{S}_{(1)}}{r^2}, & \bar{\psi}_{(1)} \bar{\sigma}_{12} = \frac{1}{2} \left(\frac{\partial \bar{v}_1}{\partial \theta} \right)_{(1)}, \\ \bar{\psi}_{(1)} \bar{\sigma}_{13} = 0, & \bar{\psi}_{(1)} \frac{\bar{\sigma}_{22}^*}{r^2} = -\bar{Q}_{(1)} + \frac{\bar{S}_{(1)}}{r^2}, \\ \bar{\psi}_{(1)} \bar{\sigma}_{23} = 0, & \bar{\psi}_{(1)} \bar{\sigma}_{33}^* = 0; \end{cases}$$

$$(13.28) \quad \begin{cases} \bar{\psi}_{(2)} \bar{\sigma}_{11}^* = \bar{Q}_{(2)} - \frac{\bar{S}_{(2)}}{r^2}, & \bar{\psi}_{(2)} \bar{\sigma}_{12} = \frac{1}{2} \left(\frac{\partial \bar{v}_1}{\partial \theta} \right)_{(2)}, \\ \bar{\psi}_{(2)} \bar{\sigma}_{13} = 0, & \bar{\psi}_{(2)} \frac{\bar{\sigma}_{22}^*}{r^2} = -\bar{Q}_{(2)} + \frac{\bar{S}_{(2)}}{r^2}, \\ \bar{\psi}_{(2)} \bar{\sigma}_{23} = 0, & \bar{\psi}_{(2)} \bar{\sigma}_{33}^* = 0. \end{cases}$$

Suppose now that $\bar{\sigma}_{11}^* = 0$ at $r = r_0$ with $b \geq r_0 \geq a$. Then the first equations in (13.27) and (13.28) become

$$(\psi_{(1)} \bar{\sigma}_{11}^*)_{r=r_0} = \bar{Q}_{(1)} - \frac{\bar{S}_{(1)}}{r_0^2} = 0,$$

$$(\psi_{(2)} \bar{\sigma}_{11}^*)_{r=r_0} = \bar{Q}_{(2)} - \frac{\bar{S}_{(2)}}{r_0^2} = 0.$$

These two equations show that if $\bar{S}_{(1)} = \bar{S}_{(2)}$, then $\bar{Q}_{(1)} = \bar{Q}_{(2)}$ and hence Σ will not be a slip surface. Hence the second condition in (13.25) must apply, and it follows from the above equations that $\bar{Q}_{(1)} = -\bar{Q}_{(2)}$. Hence the first equations (13.27) and (13.28) become

$$\bar{\psi}_{(1)} \bar{\sigma}_{11}^* = \bar{Q}_{(1)} - \frac{\bar{S}_{(1)}}{r^2},$$

$$\bar{\psi}_{(2)} \bar{\sigma}_{11}^* = \bar{Q}_{(2)} - \frac{\bar{S}_{(2)}}{r^2} = -\left(\bar{Q}_{(1)} - \frac{\bar{S}_{(1)}}{r^2} \right).$$

Adding the corresponding members of these equations, we have

$$(\bar{\psi}_{(1)} + \bar{\psi}_{(2)}) \bar{\sigma}_{11}^* = 0.$$

But $\bar{\psi}_{(1)} + \bar{\psi}_{(2)} > 0$ and hence $\bar{\sigma}_{11}^* = 0$ everywhere on the plane Σ . Consequently $\bar{Q}_{(1)} = \bar{S}_{(1)}/r^2$ from the first equation in (13.27), and this implies that $\bar{Q}_1 = \bar{S}_{(1)}/r^2 = 0$. Similarly, $\bar{Q}_{(2)} = \bar{S}_{(2)}/r^2 = 0$. Hence $[Q] = 0$ and $[S] = 0$ and hence Σ is not a slip surface. Thus we can conclude that if the plane Σ is a slip surface in the plastic flow region, then $\sigma_{33}^* = 0$ and $\bar{\sigma}_{11}^* \neq 0$ for all values of r .

From the continuity of $\bar{\sigma}_{11}^*$ as a function of r we know now that either $\bar{\sigma}_{11}^* > 0$

for all values of r or $\bar{\sigma}_{11}^* < 0$ for all values of r if the plane Σ is a slip surface. Evidently $\bar{\sigma}_{11} \leq 0$ at $r = a$ since $\bar{\sigma}_{11} = -p$ at $r = a$ and $\bar{\sigma}_{33} \geq 0$ at $r = a$ near the ends of the pipe. Moreover, from the equations of motion we have

$$\sigma_{31,1} + \frac{1}{r^2} \sigma_{32,2} + \sigma_{33,3} = \rho \left(\frac{\partial v_3}{\partial t} + v_{3,1}v^1 + v_{3,2}v^2 + v_{3,3}v^3 \right) = 0$$

or

$$\frac{\partial \sigma_{13}}{\partial r} + \frac{1}{r^2} \frac{\partial \sigma_{32}}{\partial \theta} + \frac{\partial \sigma_{33}}{\partial z} = \frac{\partial \sigma_{33}}{\partial z} = 0,$$

where use is made of the matrix (13.19) and the fact that $v_3 = 0$. This equation shows that σ_{33} is independent of z . Hence $\bar{\sigma}_{22}/r^2 > \bar{\sigma}_{33} > \bar{\sigma}_{11}$ at $r = a$ since $\bar{\sigma}_{33}^* = 0$. That is,

$$\frac{\bar{\sigma}_{22}^*}{r^2} - \bar{\sigma}_{11}^* = \frac{\bar{\sigma}_{22}}{r^2} - \bar{\sigma}_{11} > 0$$

at $r = a$. From the continuity of $\bar{\sigma}_{22}^*/r^2 - \bar{\sigma}_{11}^*$ as a function of r and the fact that

$$\bar{\sigma}_{11}^* \neq 0, \quad \frac{\bar{\sigma}_{22}^*}{r^2} + \bar{\sigma}_{11}^* = -\bar{\sigma}_{33}^* = 0, \quad \frac{\bar{\sigma}_{22}^*}{r^2} > \bar{\sigma}_{11}^* \quad \text{at } r = a,$$

we know that $\bar{\sigma}_{22}^*/r^2 > \bar{\sigma}_{11}^*$ for all values of r . Hence we reach the conclusion that the plane Σ can be a slip surface in the plastic flow region only when $\sigma_{33}^* = 0$ everywhere and $\bar{\sigma}_{22}^*/r^2 > \bar{\sigma}_{11}^*$ for all values of r .

FRACTURE IN PIPES

14. Determination of fracture. Let Σ_1 be the intersection of the pipe with a half plane through its central axis and Σ_2 be the intersection of the pipe with a plane normal to its central axis. It was found in Sect. 12 that a slip surface Σ_1 is a surface of strong instability separating the pipe into a region of plastic equilibrium and a region of plastic flow if Case A or C at complete plastic equilibrium holds and it cannot be such a slip surface if Case B holds; on the other hand, a surface Σ_2 cannot be a slip surface of this type for any distribution of stresses at plastic equilibrium. In Sect. 13 it was found that a slip surface Σ_2 in a plastic flow region is a surface of strong instability if, and only if, $\bar{\sigma}_{33}^* > 0$; if $\bar{\sigma}_{33}^* < 0$ a slip surface Σ_2 will not be a surface of strong instability and if $\bar{\sigma}_{33}^* = 0$ the surface Σ_2 cannot be a slip surface. It was furthermore shown in Sect. 13 that a surface Σ_1 can be a slip surface in the plastic flow region only when $\sigma_{33}^* = 0$ and $\bar{\sigma}_{22}^*/r^2 > \bar{\sigma}_{11}^*$ for all values of r .

We say that a fracture surface Σ is of the type 1 if it separates the pipe into a region of plastic equilibrium and a region of plastic flow and a fracture surface Σ is of the type 2 if the material of the pipe on both sides of Σ is in the state of plastic flow. If both types of fracture surface are permissible for certain

given loads, in view of the laws governing the material, then the fracture over a surface of type 2 will require greater total work than the fracture over a surface of type 1. Hence we conclude that fracture over a surface of type 1 will have precedence over fracture along a surface of type 2 if both types of fracture are permissible.

If Case A or Case C holds at complete plastic equilibrium of the pipe, it therefore follows that the pipe will fracture over a surface of type 1. If Case B holds at complete plastic equilibrium, we have

$$\sigma_{33}^* = \frac{2}{3} \sigma_{33} - \frac{1}{3} \left(\frac{\sigma_{22}}{r^2} + \sigma_{11} \right) = \frac{1}{3} \left\{ \left(\sigma_{33} - \frac{\sigma_{22}}{r^2} \right) + (\sigma_{33} - \sigma_{11}) \right\} > 0.$$

In the absence of a slip surface Σ of the type 1 under the conditions of Case B we are led to assume the continuity of the stress components in the transition from complete plastic equilibrium to complete plastic flow in the pipe. But this means that the above inequality $\sigma_{33}^* > 0$ remains valid when the material passes from the equilibrium state under the conditions of Case B to the state of complete plastic flow. Hence under the conditions of Case B the pipe will fracture over a surface Σ_2 of type 2.

To state the above results in a more convenient form, we recall that

$$(14.1) \quad \left\{ \begin{array}{l} \frac{\sigma_{22}}{r^2} \geq \sigma_{33} > \sigma_{11} \text{ with } a \leq r \leq b \text{ if, and only if, } -1 \leq 3 \frac{Mb^2}{N} < 1; \\ \sigma_{33} > \frac{\sigma_{22}}{r^2} > \sigma_{11} \text{ with } a \leq r \leq b \text{ if, and only if, } 3 \frac{Ma^2}{N} < -1; \\ \frac{\sigma_{22}}{r^2} \geq \sigma_{33} > \sigma_{11} \text{ with } a \leq r \leq c \text{ if, and only if, } -1 \leq 3 \frac{Mr^2}{N} < 1; \\ \sigma_{33} > \frac{\sigma_{22}}{r^2} > \sigma_{11} \text{ with } c < r \leq b \text{ if, and only if, } 3 \frac{Mr^2}{N} < -1, \end{array} \right.$$

where $c = (-N/3M)^{\frac{1}{2}}$ as shown in Sect. 10. If a deformation which is produced both by internal pressure and axial tension and which gives complete plastic equilibrium does not belong to Case B, then it must belong either to Case A or Case C since Cases A, B and C give the only possible distributions of stresses at complete plastic equilibrium. Hence the pipe will be fractured over a plane Σ_1 of type 1 if, and only if, $3Ma^2/N \geq -1$ and it will be fractured over a plane Σ_2 of type 2 if, and only if, $3Ma^2/N < -1$. These results show that the minimum value of $3Ma^2/N$ for fracture over a plane Σ_1 of type 1 is -1 and the maximum value of $3Ma^2/N$ for fracture over a plane Σ_2 of type 2 is arbitrarily close to -1 . That is, the fracture surface changes abruptly from a plane Σ_1 of type 1 to a plane Σ_2 of type 2 at the value $3Ma^2/N = -1$. But then $a = c$ since $3Mc^2/N = -1$. The deformation for which $3Ma^2/N = 3Mc^2/N = -1$ does not belong to Case A, in view of the first inequalities in (14.1), and it does not belong to Case B, in view of the second inequalities in (14.1). Hence it must

belong to Case C. Thus the deformation for which $3Ma^2/N = 3Mc^2/N = -1$ is produced by the loads given by the equations (10.14) and (11.16) with $a = c$.

15. Maximum loads before fracture. Since fracture is the direct result of the applied internal pressure p and axial tension l , the values of the loads which are just sufficient to produce complete plastic equilibrium and their effects upon the type of fracture will be determined. Hence we can predict in advance whether the pipe will be fractured over a surface Σ_1 or a surface Σ_2 from the knowledge of the applied loads. Moreover, for ideal solids without strain hardening, the loads just sufficient to produce complete plastic equilibrium of the pipe are the maximum loads before fracture. For convenience we approximate the radii a and b by their initial values a_0 and b_0 . This approximation will give some loss of exactness; but it has the advantage that the graph of the l, p curve for loads just sufficient to produce complete plastic equilibrium will be determined for a given pipe when we know the material constant J . For $-1 \leq 3Mb^2/N < 1$ the l, p curve is given by equations in (7.2); for $-1 \leq 3Mr^2/N < 1$ with $a \leq r \leq c$ and for $3Mr^2/N < -1$ with $c < r \leq b$ and $3Mc^2/N = -1$ the l, p curve is given by the equations in (10.17); and for $3Ma^2/N < -1$ the l, p curve is given by the equation in (9.2). Each point on this curve represents the loads which are just sufficient to produce complete plastic equilibrium. Those points corresponding to the value of M/N for which $3Ma^2/N < -1$ represent the loads required to produce fracture over a surface Σ_2 and all other points represent the loads required to produce fracture over a surface Σ_1 .

TABLE
Values of Equilibrium Loads

$\sigma_{22}/r^2 \geq \sigma_{33} > \sigma_{11}$ for all values of r					
$0 \leq l \leq 0.151J$ and $p = 0.149J$					
$\sigma_{22}/r^2 \geq \sigma_{33} > \sigma_{11}$ with $a \leq r \leq c$ $\sigma_{33} > \sigma_{22}/r^2 > \sigma_{11}$ with $c < r \leq b$ $a \leq c < b$, $3Mc^2/N = -1$					
c/a	1.13	1.1	1.06	1.03	
l	0.162J	0.163J	0.170J	0.177J	
P	0.148J	0.144J	0.140J	0.135J	
$\sigma_{33} > \sigma_{22}/r^2 > \sigma_{11}$ for all values of r					
$3Ma^2/N$	-1	-2	-4	-10	-20
l	0.187J	0.242J	0.285J	0.318J	0.332J
p	0.140J	0.0907J	0.0536J	0.0237J	0.0149J

Using the equations (7.2), (10.17) and (9.2), values of l and p corresponding to $a_0 = 0.625''$ and $b_0 = 0.725''$ are calculated as shown in the table of values of equilibrium loads. The corresponding l, p curve is plotted in Fig. 2. The arrow on the curve indicates the increasing direction of M/N . At the points H and G the values of M/N are given by

$$3 \frac{Ma_0^2}{N} = -1 \quad \text{and} \quad 3 \frac{Mb_0^2}{N} = -1,$$

respectively. The portion FG of the curve represents the loads with $-1 \leq 3Mb_0^2/N < 1$; GH represents the loads with $-1 \leq 3Ma_0^2/N$ and $3Mb_0^2/N < 1$; and HK represents the loads with $3Ma_0^2/N < -1$. Consequently, the loads represented by the point H and the points on the curve below H will produce fracture over a surface Σ_1 while the loads represented by points on the curve above the point H will produce fracture over a surface Σ_2 .

Comparison of the l, p curves with experimentally measured maximum loads before fracture is shown in Figs. 3-14, where the curves represent the theoretical values of the loads which are just sufficient to produce complete plastic equilibrium of the pipe and the points represent the experimentally measured maximum loads before fracture. Data have been selected from combined stress test on isotropic or nearly isotropic material, as assumed in our theory, for comparison with calculated results.

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