Whitehead Products and Projective Spaces

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It has long been known how to express the homotopy groups of the classical projective spaces in terms of the homotopy groups of spheres. The object of this note is to provide expressions for the Whitehead products in similar terms. In particular, all Whitehead products are zero in the complex projective 3-space, although this is not an H -space.

1. The Samelson product. Let G be a topological group. Associated with a principal G -bundle over a space X is a transgression operator

$$\Delta : \pi_{r+1}(X) \rightarrow \pi_r(G), \quad r \geq 0.$$

Let $p, q \geq 1$ and let $[\xi, \eta] \in \pi_{p+q+1}(X)$ be the Whitehead product of $\xi \in \pi_{p+1}(X)$, $\eta \in \pi_{q+1}(X)$. We prove that

$$(1.1) \quad (-1)^p \Delta[\xi, \eta] = \langle \Delta\xi, \Delta\eta \rangle,$$

where the right hand side denotes the Samelson¹ product, defined, as in §1 of [15], by the use of commutators.

Instead of following [12] we prove (1.1) by a combination of results of Fox ([5]) and G. W. WHITEHEAD ([15]). Let Y^A denote the function space of base-point-preserving maps of a complex A into a space Y . We give G^A the group structure inherited from G . From the given G -bundle B over X we obtain a G^A -bundle B^A over X^A and a corresponding transgression homomorphism

$$\Delta^A : \pi_1(X^A) \rightarrow \pi_0(G^A).$$

When $A = S^r$, we can identify

$$\pi_1(X^A) = \pi_{r+1}(X), \quad \pi_0(G^A) = \pi_r(G),$$

so that Δ^A is transformed into Δ . If $A = B \times C$, we embed Y^B, Y^C in Y^A in the usual way ([2]).

Take $A = S^p \times S^q$; we can regard S^{p+q} as obtained from A by pinching

¹ The first mention of such a product appears to be in [12].

$S^p \vee S^q$ to a point. The pinching map induces homomorphisms² θ, φ such that the following diagram is commutative:

$$\begin{array}{ccc} \pi_1(X^A) & \xleftarrow{\theta} & \pi_{p+q+1}(X) \\ \Delta^A \downarrow & & \downarrow \Delta \\ \pi_0(G^A) & \xleftarrow{\varphi} & \pi_{p+q}(G) \end{array}$$

Let $\zeta \in \pi_1(X^A)$ be the commutator of the injections of $\xi \in \pi_{p+1}(X)$, $\eta \in \pi_{q+1}(X)$ in $\pi_1(X^A)$, and let $\gamma \in \pi_0(G^A)$ be the commutator of the injections of $\Delta\xi, \Delta\eta$ in $\pi_0(G^A)$. Then $\Delta^A\zeta = \gamma$ since Δ^A is a homomorphism. In §8 of [5] it is shown that $\zeta = (-1)^p\theta[\xi, \eta]$, and hence

$$(-1)^p\phi\Delta[\xi, \eta] = \Delta^A\zeta = \gamma = \phi\langle\Delta\xi, \Delta\eta\rangle,$$

by definition. Since ϕ is a monomorphism as is shown in [15], (1.1) follows.

2. The direct sum. We recall the construction of classifying spaces described by MILNOR in [11]. Let G be a topological group; MILNOR makes G act on B_m , the $(m+1)$ -fold join of G with itself, and denotes the orbit-space by X_m . Thus B_m is a principal G -bundle over X_m ; let $f_m : B_m \rightarrow X_m$ be the fibre map. MILNOR embeds B_m in a contractible subspace of B_{m+1} , so as to induce an embedding of X_m in X_{m+1} . Now X_1 is the suspension of G , and the suspension homomorphism

$$E : \pi_r(G) \rightarrow \pi_{r+1}(X_1)$$

can be selected to be a right inverse of the transgression Δ . Let

$$E_m : \pi_r(G) \rightarrow \pi_{r+1}(X_m)$$

denote the composition of $E_1 = E$ and the injection induced by the inclusion $X_1 \subset X_m$. The exact sequence of the fibration

$$\cdots \rightarrow \pi_{r+1}(B_m) \xrightarrow{\rho_m} \pi_{r+1}(X_m) \xrightarrow{\Delta} \pi_r(G) \xrightarrow{\sigma_m} \pi_r(B_m) \rightarrow \cdots$$

(where ρ_m is induced by f_m , and σ_m by the inclusion $G \subset B_m$) splits because $\Delta \circ E_m$ is the identity. This proves

Theorem (2.1). *There is a direct sum decomposition*

$$\pi_r(X_m) = \rho_m\pi_r(B_m) \oplus E_m\pi_{r-1}(G), \quad r \geq 2,$$

where ρ_m, E_m are monomorphisms. The injection of $\pi_r(X_m)$ in $\pi_r(X_{m+1})$ annihilates the first summand and maps the second isomorphically by $E_{m+1} \circ E_m^{-1}$.

In view of (2.1) we distinguish three basic kinds of Whitehead product in the homotopy groups of X_m ; thus, in $\pi_{p+q+1}(X_m)$ are

$$\left. \begin{array}{ll} \text{(i)} & [\rho_m\xi, \rho_m\eta] = \rho_m[\xi, \eta] \\ \text{(ii)} & [E_m\alpha, E_m\beta] \\ \text{(iii)} & [\rho_m\xi, E_m\beta] \end{array} \right\} \begin{array}{ll} \xi \in \pi_{p+1}(B_m), & \eta \in \pi_{q+1}(B_m), \\ \alpha \in \pi_p(G), & \beta \in \pi_q(G). \end{array}$$

² These map additive groups into multiplicative groups.

Any Whitehead product can be expressed as a linear combination of products of these three kinds.

The first kind are essentially Whitehead products in the homotopy groups of B_m , and there is nothing about these to discuss.

The second kind are related to the Samelson product in G , because

$$(2.2) \quad \Delta_m[E_m\alpha, E_m\beta] = (-1)^p \langle \alpha, \beta \rangle$$

by (1.1) and (2.1). When $m = 1$ it is possible for $[E\alpha, E\beta]$ to have a component in the image of ρ_1 (as when $G = S^3$):

$$[E\alpha, E\beta] = (-1)^p E \langle \alpha, \beta \rangle + \rho_1(\lambda).$$

However, when $m > 1$ it follows from the latter part of (2.1) that this term is zero, *i.e.*

$$(2.3) \quad [E_m\alpha, E_m\beta] = (-1)^p E_m \langle \alpha, \beta \rangle, \quad m \geq 2.$$

Notice also that, by the definition of E_m ,

$$E_m(\alpha \circ \lambda) = E_m\alpha \circ E\lambda, \quad E_m(\beta \circ \mu) = E_m\beta \circ E\mu$$

where $\lambda \in \pi_i(S^p)$, $\mu \in \pi_j(S^q)$; hence, by (3.59) of [14],

$$(2.4) \quad [E_m(\alpha \circ \lambda), E_m(\beta \circ \mu)] = [E_m\alpha, E_m\beta] \circ (\lambda * \mu),$$

where $(\lambda * \mu) = (-1)^{(i+1)(q-i)} E^{q+1}\lambda \circ E^{i+1}\mu$, by (2.4) of [3].

Finally, consider Whitehead products of the third kind. We say that

$$(2.5) \quad \Delta[\rho_m\xi, E_m\beta] = 0, \quad m \geq 1.$$

For Δ can be factored through the injection of $\pi_r(X_m)$ in $\pi_r(X_{m+1})$, which (by (2.1)) annihilates $\rho_m\xi$. Thus

$$(2.6) \quad [\rho_m\xi, E_m\beta] \in \rho_m\pi_{p+q+1}(B_m).$$

The third kind of product may be regarded as a pairing of $\pi_{p+1}(B_m)$ with $\pi_q(G)$ to $\pi_{p+q+1}(B_m)$. We examine this in the classical cases when G is a sphere.

3. The third kind of product in the classical case. Let F denote the field of real numbers, complex numbers, or quaternions, and let $G \subset F$ be the (multiplicative) group of elements with unit norm. The classical projective spaces can be obtained by applying MILNOR's construction to G , taking B_m to be the unit sphere in the $(m+1)$ -dimensional Euclidean space over F and X_m to be the corresponding projective space.

Let $q = 1, 2$ or 4 denote the dimension of F over the reals, and let $n = q(m+1)$. We can make the identifications

$$G = S^{q-1}, \quad X_1 = S^q, \quad B_m = S^{n-1}$$

in such a way that

$$E_1 = E : \pi_r(G) \rightarrow \pi_{r+1}(X_1)$$

is the usual suspension homomorphism. Furthermore, we can regard S^n as the result of pinching $X_m \subset X_{m+1}$ to a point; the pinching map will induce a homomorphism

$$\mu_m : \pi_r(X_{m+1}) \rightarrow \pi_r(S^n).$$

Let $\beta_m \in E_m \pi_{q-1}(S^{q-1})$, $\alpha_m \in \rho_m \pi_{n-1}(S^{n-1})$ be the images of the classes of the identity maps $G \rightarrow G$, $B_m \rightarrow B_m$, respectively, so that

$$(3.1) \quad \rho_m(\xi) = \alpha_m \circ \xi, \quad E_m(\lambda) = \beta_m \circ \lambda.$$

We show how $[\rho_m \pi_{p+1}(B_m), E_m \pi_r(G)]$ can be determined from the value of $[\alpha_m, \beta_m]$, and we shall determine this element. (2.6) proves the first part of

Lemma 3.2 *There is a $\gamma_m \in \pi_{n+q-2}(S^{n-1})$ such that*

$$[\alpha_m, \beta_m] = \rho_m(\gamma_m).$$

Moreover, the orientations can be chosen so that

$$E\gamma_m = \mu_m(\alpha_{m+1}).$$

This is the key lemma; $\mu_m \alpha_{m+1}$ will be determined later. Notice that E is an isomorphism in (3.2).

First, take the real case, where $m = n - 1$. The action of the fundamental group of X_m is given by the antipodal transformation $T: S^{n-1} \rightarrow S^{n-1}$ of degree $(-1)^n$. Therefore

$$[\alpha_m, \beta_m] = ((-1)^n - 1)\alpha_m,$$

and (3.2) follows at once.

Now let F be the complex or quaternionic field. Take an n -element E^n whose boundary is S^{n-1} and a class $\lambda_m \in \pi_n(X_{m+1}, X_m)$ of a relative homeomorphism

$$f'_m : (E^n, S^{n-1}) \rightarrow (X_{m+1}, X_m)$$

which coincides with f_m on S^{n-1} . Thus λ_m is a generator. Let f_m, f'_m induce homomorphisms ρ_m, ρ'_m , respectively, in the commutative diagram

$$\begin{array}{ccc} \pi_{n+q-1}(E^n, S^{n-1}) & \xrightarrow{\partial'} & \pi_{n+q-2}(S^{n-1}) \\ \rho'_m \downarrow & & \downarrow \rho_m \\ \pi_{n+q-1}(X_{m+1}, X_m) & \xrightarrow{\partial} & \pi_{n+q-2}(X_m). \end{array}$$

It follows from (2.1) and the exactness of the sequences of the pairs that the images of ρ_m and ∂ coincide. Also ρ_m is a monomorphism and ∂' is an isomorphism, while $\partial \circ \rho'_m = \rho_m \circ \partial'$. Therefore ρ'_m is a monomorphism, and ∂ maps its image isomorphically onto the image of ∂ . By (1.4) of [9] the image of ρ'_m is a direct summand of $\pi_{n+q-1}(X_{m+1}, X_m)$, and the complementary summand is cyclic infinite, generated by the relative Whitehead product $[\lambda_m, \beta_m]$. Therefore the kernel of ∂ is freely generated by

$$[\lambda_m, \beta_m] + \rho'_m \tau_m$$

for some $\tau_m \in \pi_{n+q-1}(E^n, S^{n-1})$. The kernel of ∂ is, however, the image of the homomorphism

$$j : \pi_{n+q-1}(X_{m+1}) \rightarrow \pi_{n+q-1}(X_{m+1}, X_m).$$

It follows from (2.1) that this image is generated by $j\alpha_{m+1}$; this proves

$$(3.3) \quad \pm j\alpha_{m+1} = [\lambda_m, \beta_m] + \rho'_m \tau_m.$$

Since $\partial \circ j = 0$, we deduce from this and (3.5) of [4] that

$$\partial \rho'_m \tau_m = \rho_m \partial' \tau_m = [\alpha_m, \beta_m];$$

hence $\gamma_m = \partial' \tau_m$ in (3.2).

Now β_m , and hence $[\lambda_m, \beta_m]$, is annihilated by the homomorphism

$$\mu'_m : \pi_{n+q-1}(X_{m+1}, X_m) \rightarrow \pi_{n+q-1}(S^n)$$

induced by pinching X_m to a point. Since

$$\mu_m = \mu'_m \circ j,$$

(3.3) implies

$$\pm \mu_m \alpha_{m+1} = \mu'_m \rho'_m \tau_m = \pm E \partial' \tau_m = \pm E \gamma_m,$$

as in §1 of [6]. Hence $\mu_m \alpha_{m+1} = \pm E \gamma_m$, and we can choose orientations so as to obtain (3.2).

4. Conclusions. We can now carry out our intention of expressing Whitehead products in the classical projective spaces in terms of operations on the homotopy groups of spheres. We ignore products of the first kind since B_m is itself a sphere, and $m = 1$ since X_1 is itself a sphere.

Case 1. $F = \text{real numbers}$. Here $B_m = S^m$, $\rho_m : \pi_r(S_m) \approx \pi_r(X_m)$ for $r \geq 2$. If $T : S^m \rightarrow S^m$ is the antipodal map (of degree $(-1)^{m+1}$), the operation of $\beta_m \in \pi_1(X_m)$ on $\theta \in \pi_r(X_m)$ produces $\rho_m T_* \rho_m^{-1}(\theta)$. This determines products of the third kind:

$$(4.1) \quad \begin{aligned} [\rho_m \xi, \beta_m] &= (-1)^r [\beta_m, \rho_m \xi] = (-1)^r \{ \rho_m (-\iota)^{m+1} \circ \xi - \rho_m \xi \} \\ &= \begin{cases} 0 & m : \text{odd} \\ (-1)^r (-2\rho_m \xi + \rho_m [\iota, \iota] \circ H_0 \xi) & m : \text{even} \end{cases} \end{aligned}$$

(using Theorem 6.9 of [7]). Products of the second kind are trivial: $[\beta_m, \beta_m] = 0$.

Case 2. $F = \text{complex numbers}$. Here, again, products of the second kind are zero, for G is abelian and the Samelson products vanish. Also $B_m = S^{2m+1}$, an odd sphere.

To determine $\mu_m \alpha_{m+1}$ in (3.2), we examine the Steenrod squares in X_{m+1} . This shows that $\mu_m \alpha_{m+1}$ is trivial, and hence $[\alpha_m, \beta_m] = 0$ if, and only if, m is odd. When m is odd, it follows from (7.4) of [1], or from (2.1) of [8], that all

products of the third kind are zero. This can be proved directly from the standard representation of X_{2k+1} as a 2-sphere bundle over the quaternionic projective k -space. Notice, in particular, that $B_3 = S^7$, and Whitehead products of all kinds are trivial in X_3 , though it is not an H -space.

When m is even, $[\alpha_m, \beta_m] = \alpha_m \circ \eta$, where $\eta \neq 0 \in \pi_n(S^{n-1})$. Therefore

$$[\alpha_m[\alpha_m, \beta_m]] = \alpha_m \circ [\iota, \iota] \circ \eta, \quad [\alpha_m[\alpha_m[\alpha_m, \beta_m]]] = 0,$$

and by (3.1) and (7.4) of [1]

$$(4.2) \quad \begin{aligned} [\rho_m \xi, \beta_m] &= [\alpha_m \circ \xi, \beta_m] = \alpha_m \circ \eta \circ E\xi + \alpha_m \circ [\iota, \iota] \circ \eta \circ EH_0\xi \\ &= \rho_m(\eta \circ E\xi + [\iota, \iota] \circ \eta \circ EH_0\xi); \end{aligned}$$

while $[\rho_m \xi, \beta_m \circ \nu] = 0$ for any ν in any stem other than the 0-stem.

Case 3. $F = \text{quaternions}$. G. W. WHITEHEAD has shown that if ι_3 generates $\pi_3(S^3)$, then $\langle \iota_3, \iota_3 \rangle$ generates³ $\pi_6(S^3)$; the methods of §9 of [13] can be used. The products of the second kind are given by (2.3), (2.4).

By (2.10a) of [10], we have

$$\mu_{m-1}\alpha_m = \pm mE^{4m-4}\gamma,$$

where γ is the Hopf class in $\pi_7(S^4)$. Therefore

$$(4.3) \quad [\alpha_m, \beta_m] = \pm(m+1)\rho_mE^{4m-1}\gamma$$

by (3.2), and the general product of the third kind can be deduced from this and (7.4) of [1] as before, though attention must be paid to the sign of the first term in this case.

The stable group of the 3-stem is cyclic of order 24, generated by the suspensions of γ . It follows that all products of the third kind are zero when $m \equiv -1 \pmod{24}$.

The study of Whitehead products in the Cayley projective plane is much more awkward, since (2.1) breaks down.

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³ In fact, — $\langle \iota_3, \iota_3 \rangle$ is the Blakers-Massey element.

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