On an Extension of Cesàro's Methods of Summability to Logarithmic Scales*

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Communicated by A. ZYGMUND

1. The arithmetic method $T(A_n)$. The methods we are going to consider are, like Cesàro's methods, special cases of Nörlund's (sometimes called Nörlund-Voronoi) methods of summability, and we begin by recalling basic definitions and properties of the latter. For the literature of the subject see section 6 below.

Consider a sequence of numbers $a_0, a_1, a_2, \dots$; let $A_n = a_0 + a_1 + \dots + a_n$. A sequence s_n ($n = 0, 1, 2, \dots$) is said to be limitable by the method $T(A_n)$ to the limit s if

$$\lim_{n \rightarrow +\infty} \frac{a_n s_0 + a_{n-1} s_1 + \dots + a_0 s_n}{A_n} = s,$$

and an infinite series $\sum_0^\infty u_n$ is said to be summable by the method $T(A_n)$ to the sum s , if the sequence of its partial sums $s_n = \sum_0^n u_n$ is limitable $T(A_n)$ to s . We assume that the A_n satisfy the following conditions:

I. The numbers A_n , with the possible exception of a finite number, are strictly positive, *i.e.*, $A_n > 0$ for $n > n_0$.

$$\text{II.} \quad \lim_{n \rightarrow \infty} \frac{a_n}{A_n} = 0, \quad \text{i.e.,} \quad \lim_{n \rightarrow \infty} \frac{A_{n-1}}{A_n} = 1.$$

$$\text{III.} \quad \sum_{n=0}^{\infty} A_n = +\infty.$$

These conditions are necessary if every convergent sequence is to be limitable $T(A_n)$ to the same limit. In some theorems we impose on the A_n some additional conditions, for example

$$\text{IV.} \quad \lim_{n \rightarrow \infty} A_n = +\infty.$$

Let

$$\sum_{n=0}^{\infty} A_n x^n = \phi(x).$$

* Summability $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$ was introduced in ZYGMUND [12], who also stated without proof a number of results. I am grateful to Professor ZYGMUND for the suggestion to develop the topic and for his helpful advice. The present paper is an abbreviated version of a PhD thesis submitted in September 1955 to the Department of Mathematics of the University of Chicago.

If the A_n ($n = 0, 1, 2, \dots$) satisfy the conditions I, II, III, we say that the function ϕ belongs to the class T , and by saying that the sequence s_n is limitable ϕ we shall mean that s_n is limitable $T(A_n)$.

Write

$$T_n = a_n s_0 + a_{n-1} s_1 + \dots + a_0 s_n.$$

In what follows we shall often write $\sum u_\nu$ for $\sum_{\nu=0}^{\infty} u_\nu$. If s_n are the partial sums of a series $\sum u_\nu$, then

$$\sum T_n x^n = \sum s_n x^n \cdot \sum a_n x^n = \sum s_n x^n \cdot (1-x) \sum A_n x^n = \sum u_n x^n \cdot \sum A_n x^n,$$

i.e.,

$$T_n = u_0 A_n + u_1 A_{n-1} + \dots + u_n A_0.$$

It is easy to see that if the sequence $s_0, s_1, s_2, \dots$ is limitable $T(A_n)$, then the sequences $s_1, s_2, s_3, \dots$ and $0, s_0, s_1, s_2, \dots$ are also limitable $T(A_n)$ and, moreover, all three limits are identical. Stated for series this means that if one of the three series $u_0 + u_1 + u_2 + \dots$, $0 + u_0 + u_1 + \dots$, $(u_0 + u_1) + u_2 + u_3 + \dots$ is summable $T(A_n)$ to sum s , the same holds for the remaining two series.

Theorem 1. *Let ϕ and ψ be functions of the class T . If the sequence $\{s_n\}$ is limitable ϕ , then it is also limitable $\phi\psi$ and to the same limit.*

Let

$$\phi(x) = \sum A_n x^n, \quad \psi(x) = \sum B_n x^n, \quad \phi(x)\psi(x) = \sum C_n x^n.$$

Then

$$C_n = A_0 B_n + A_1 B_{n-1} + \dots + A_n B_0.$$

Lemma. *Under the above hypotheses,*

$$(1) \quad \lim_{n \rightarrow \infty} \frac{C_n}{A_n} = \lim_{n \rightarrow \infty} \frac{C_n}{B_n} = +\infty.$$

Proof. Let $M > 0$. Choose an n_0 such that for $k \geq n_0$ we have

$$1^\circ \quad A_k, B_k > 0,$$

$$2^\circ \quad A_0 + A_1 + \dots + A_k > M, \quad B_0 + B_1 + \dots + B_k > M.$$

Then for $n > 2n_0$

$$\begin{aligned} C_n \geq B_n \left(A_0 + A_1 \frac{B_{n-1}}{B_n} + \dots + A_{n_0} \frac{B_{n-n_0}}{B_n} \right) \\ + A_n \left(B_0 + B_1 \frac{A_{n-1}}{A_n} + \dots + B_{n_0} \frac{A_{n-n_0}}{A_n} \right). \end{aligned}$$

By II and 2° the expressions in the parentheses tend to limits $> M$, hence,

for sufficiently large values of the index n , $C_n/A_n > M$, $C_n/B_n > M$, which by the arbitrariness of M implies (1).

An immediate consequence of (1) is

$$(2) \quad \lim_{n \rightarrow \infty} \frac{A_k B_{n-k} + A_{k+1} B_{n-k-1} + \cdots + A_{n-k} B_k}{A_0 B_n + A_1 B_{n-1} + \cdots + A_n B_0} = 1$$

for any k .

Proof of the theorem. Denote

$$c_0 = C_0, \quad c_n = C_n - C_{n-1} \quad (n > 0);$$

we have to prove that $T_n^a/A_n \rightarrow s$ implies $T_n^c/C_n \rightarrow s$, where

$$T_n^a = a_n s_0 + a_{n-1} s_1 + \cdots + a_0 s_n, \quad T_n^c = c_n s_0 + c_{n-1} s_1 + \cdots + c_0 s_n.$$

From

$$\begin{aligned} \sum T_n^c x^n &= \sum c_n x^n \cdot \sum s_n x^n = (1-x) \sum C_n x^n \cdot \sum s_n x^n \\ &= (1-x) \sum A_n x^n \cdot \sum B_n x^n \cdot \sum s_n x^n \\ &= \sum B_n x^n \cdot \sum a_n x^n \cdot \sum s_n x^n = \sum B_n x^n \cdot \sum T_n^a x^n, \end{aligned}$$

we deduce

$$(3) \quad T_n^c = T_0^a B_n + T_1^a B_{n-1} + \cdots + T_n^a B_0.$$

Subtracting s from the s_n we reduce the general case to that of $s = 0$. Hence $T_n^a/A_n = \epsilon_n \rightarrow 0$. Then

$$\frac{T_n^c}{C_n} = \frac{A_0 B_n \epsilon_0 + A_1 B_{n-1} \epsilon_1 + \cdots + A_n B_0 \epsilon_n}{C_n},$$

and using (1) and (2) we easily prove that $T_n^c/C_n \rightarrow 0$, which is the theorem.

We remark that if ϕ and ψ are functions of the class T , then the function $\phi\psi$ is also of the same class. Indeed, the conditions $C_n/A_n \rightarrow +\infty$, $\sum A_n = +\infty$ imply $C_n > 0$ for $n > n_0$, $\sum C_n = +\infty$; we also have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{c_n}{C_n} &= \lim_{n \rightarrow \infty} \frac{C_n - C_{n-1}}{C_n} \\ &= \lim_{n \rightarrow \infty} \frac{B_0(A_n - A_{n-1}) + B_1(A_{n-1} - A_{n-2}) + \cdots + B_{n-1}(A_1 - A_0) + B_n A_0}{C_n} \\ &= \lim_{n \rightarrow \infty} \frac{B_0 a_n + B_1 a_{n-1} + \cdots + B_n a_0}{C_n} = 0. \end{aligned}$$

Theorem 2. If (a) $\sum u_n$ is summable $T(A_n)$ to s and (b) the series $\sum u_n r^n$ is convergent for $0 \leq r < 1$,* then

$$\lim_{r \rightarrow 1} \sum u_n r^n = s,$$

i.e., the series $\sum u_n$ is summable by Abel's method to s .

* Condition (b) in this theorem can be replaced by

$$\phi(z) = \sum A_n z^n \neq 0 \quad \text{for } |z| < 1.$$

Proof. Assume that for $\sum u_n$ we have $T_n/A_n \rightarrow s$. By condition II, $\sum A_n r^n$ converges for $0 \leq r < 1$, and so the same holds for $\sum T_n r^n$. Using the condition III, we have

$$\lim_{r \rightarrow 1} \frac{\sum T_n r^n}{\sum A_n r^n} = s,$$

and it is enough to observe that the ratio on the left is $\sum u_n r^n$.

Theorem 3. *Let*

$$\sum A_n r^n = \phi(r), \quad \frac{1}{\phi(r)} = \sum \bar{A}_n r^n.$$

If $\sum u_n$ is summable $T(A_n)$, where 1° $A_{n+1} \geq A_n$ for n sufficiently large and 2° the series $\sum |\bar{A}_n|$ is convergent, then

$$\lim_{n \rightarrow \infty} \frac{u_n}{A_n} = 0.$$

We may assume without loss of generality that $\sum u_n$ is $T(A_n)$ -summable to 0, i.e., $T_n = \epsilon_n A_n$ with $\epsilon_n \rightarrow 0$. Thus

$$\sum T_n r^n = \phi(r) \sum u_n r^n, \quad \sum u_n r^n = \frac{1}{\phi(r)} \sum T_n r^n = \sum \bar{A}_n r^n \cdot \sum T_n r^n,$$

and therefore

$$u_n = \bar{A}_n T_0 + \bar{A}_{n-1} T_1 + \cdots + \bar{A}_0 T_n,$$

$$\left| \frac{u_n}{A_n} \right| \leq \frac{|\bar{A}_n \epsilon_0 A_0| + |\bar{A}_{n-1} \epsilon_1 A_1| + \cdots + |\bar{A}_0 \epsilon_n A_n|}{A_n}.$$

There are two possibilities: Either (a) $\{A_n\}$ is bounded above, and hence $\lim_{n \rightarrow \infty} A_n = A > 0$, or (b) $\lim_{n \rightarrow \infty} A_n = +\infty$.

In case (a) there exist numbers M and n_0 such that $|A_n| < M$ ($n = 0, 1, 2, \dots$) and $A_n > \frac{1}{2}A$, $n > n_0$. Then, for $n > n_0$,

$$\frac{|u_n|}{A_n} \leq \frac{2M}{A} (|\bar{A}_n \epsilon_0| + |\bar{A}_{n-1} \epsilon_1| + \cdots + |\bar{A}_0 \epsilon_n|).$$

In case (b) there exists an n_0 such that $|A_n| \geq |A_m|$ for $n \geq m$, $n > n_0$, and for such n

$$\frac{|u_n|}{A_n} \leq |\bar{A}_n \epsilon_0| + |\bar{A}_{n-1} \epsilon_1| + \cdots + |\bar{A}_0 \epsilon_n|.$$

An application of a very well known fact about sequences to each of the two cases completes the proof.

Let $\sum u_n$ and $\sum v_n$ be two given series and $\sum w_n$ their Cauchy product:

$$w_n = u_0 v_n + u_1 v_{n-1} + \cdots + u_n v_0.$$

Theorem 4. *The product of the series $\sum u_n$ summable $T(A_n)$ to sum U , where $A_{n+1} \geq A_n$, $\ast \lim A_n = +\infty$, and of the absolutely convergent series $\sum v_n$ of sum V is summable $T(A_n)$ to the sum $U \cdot V$.*

Proof. Let

$$U_n = A_n u_0 + A_{n-1} u_1 + \cdots + A_0 u_n, \quad W_n = A_n w_0 + A_{n-1} w_1 + \cdots + A_0 w_n.$$

Then

$$\begin{aligned} \sum W_n x^n &= \sum A_n x^n \cdot \sum w_n x^n = \sum A_n x^n \cdot \sum u_n x^n \cdot \sum v_n x^n \\ &= \sum U_n x^n \cdot \sum v_n x^n, \end{aligned}$$

i.e.,

$$(4) \quad \frac{W_n}{A_n} = \frac{U_n v_0 + U_{n-1} v_1 + \cdots + U_0 v_n}{A_n}.$$

Assume temporarily that $U = 0$, i.e., $U_n = \epsilon_n A_n$, where $\epsilon_n \rightarrow 0$. Substituting this into the right side of (4), we obtain an expression which, as already shown in Theorem 3, tends to 0 (the v_n play here the role of the former $\bar{A}_n$). The case when $U \neq 0$ is easily reducible to that of $U = 0$ by considering the Cauchy product of $(u_0 - U) + u_1 + u_2 + \cdots$ and $\sum v_n$.

Theorem 5. *The product of the series $\sum u_n$ summable by the method ϕ and of the series $\sum v_n$ summable by the method ψ (where ϕ and ψ are functions of the class T) is summable $\phi\psi$ and has as its sum $U \cdot V$.*

Proof. Let

$$\begin{aligned} \phi(r) &= \sum A_n r^n, \quad \psi(r) = \sum B_n r^n, \quad \phi(r)\psi(r) = \sum A_n r^n \cdot \sum B_n r^n = \sum C_n r^n, \\ U_n &= \sum_{k=0}^n A_{n-k} u_k, \quad V_n = \sum_{k=0}^n B_{n-k} v_k, \quad W_n = \sum_{k=0}^n C_{n-k} w_k. \end{aligned}$$

Then

$$\sum W_n r^n = \sum C_n r^n \cdot \sum w_n r^n = \sum U_n r^n \cdot \sum V_n r^n,$$

i.e.,

$$\frac{W_n}{C_n} = \frac{\sum_{k=0}^n U_k V_{n-k}}{C_n}.$$

The general case is easily reducible to that of $U = V = 0$. Hence $U_n = \epsilon_n A_n$, $V_n = \eta_n B_n$, where $\epsilon_n \rightarrow 0$, $\eta_n \rightarrow 0$. In view of Lemma 1, the contribution of

* It suffices to have the validity of this inequality for $n > n_0$.

the terms with $0 \leq k \leq n_0$ and $n - n_0 \leq k \leq n$ tends to 0 as $n \rightarrow \infty$. The terms with $n_0 < k < n - n_0$ contribute less than

$$\frac{\delta^2 \sum_{k=n_0+1}^{n-n_0-1} A_k B_{n-k}}{C_n} = \delta^2 \frac{C_n}{C_n} + o(1) = \delta^2 + o(1),$$

and the theorem follows.

We shall use the following notation: We write $T(A_n) \rightarrow T(B_n)$ if, for every $\{s_n\}$, $s_n \rightarrow s(T(A_n))$ implies $s_n \rightarrow s(T(B_n))$; we write $T(A_n) \leftrightarrow T(B_n)$ if $T(A_n) \rightarrow T(B_n)$ and $T(B_n) \rightarrow T(A_n)$.

Consider two methods $T(A_n)$ and $T(B_n)$, where A_n and B_n satisfy conditions I, II, III. Denote

$$a(x) = \sum a_n x^n, \quad A(x) = \sum A_n x^n, \quad b(x) = \sum b_n x^n, \quad B(x) = \sum B_n x^n.$$

Since $A_{n-1}/A_n \rightarrow 1$, the series $A(x)$ converges for $|x| < 1$, and

$$a(x) = (1-x) \sum A_n x^n$$

converges to $(1-x)A(x)$ for $|x| < 1$. $B(x)$ and $b(x)$ are treated similarly. We have to assume in addition that a_0 and b_0 are $\neq 0$. The series

$$k(x) = \sum k_n x^n = \frac{b(x)}{a(x)} = \frac{B(x)}{A(x)}, \quad l(x) = \sum l_n x^n = \frac{a(x)}{b(x)} = \frac{A(x)}{B(x)}$$

are then convergent for small $|x|$, and

$$\sum_{i=0}^n k_i a_{n-i} = b_n, \quad \sum_{i=0}^n k_i A_{n-i} = B_n, \quad \sum_{i=0}^n l_i b_{n-i} = a_n, \quad \sum_{i=0}^n l_i B_{n-i} = A_n.$$

Inclusion Theorem I. Let $T(A_n)$ and $T(B_n)$ be two methods of summation. A necessary and sufficient condition for $T(A_n) \rightarrow T(B_n)$ is

$$(5) \quad k_n A_0 \epsilon_0 + \cdots + k_0 A_n \epsilon_n = o(B_n)$$

for every null-sequence $\{\epsilon_n\}$.

The essence of this theorem and of the next theorem is due to M. RIESZ [11] (see also HARDY [6]). We give proofs since we need the theorems in slightly more general forms than those given by RIESZ and HARDY.

Proof. Introduce the method $T(C_n)$ such that

$$\sum C_n x^n = \sum A_n x^n \cdot \sum B_n x^n, \quad C_n = A_0 B_n + A_1 B_{n-1} + \cdots + A_n B_0.$$

By the remark following Theorem 1, $A(x) \in T$ and $B(x) \in T$ imply $C(x) \in T$. According to (3),

$$\sum T_n^c x^n = \sum B_n x^n \cdot \sum T_n^a x^n, \quad \sum T_n^c x^n = \sum A_n x^n \cdot \sum T_n^b x^n,$$

hence

$$\sum A_n x^n \cdot \sum T_n^b x^n = \sum B_n x^n \cdot \sum T_n^a x^n,$$

and, for small $|x|$,

$$\sum T_n^b x^n = \frac{\sum B_n x^n}{\sum A_n x^n} \cdot \sum T_n^a x^n = \sum k_n x^n \cdot \sum T_n^a x^n,$$

$$T_n^b = k_n T_0^a + k_{n-1} T_1^a + \cdots + k_0 T_n^a.$$

Assume $T_n^a/A_n \rightarrow s$, i.e., $T_n^a = A_n(s + \epsilon_n)$, where $\epsilon_n \rightarrow 0$. Then

$$T_n^b = s \cdot B_n + (k_n A_0 \epsilon_0 + \cdots + k_0 A_n \epsilon_n),$$

and from this equality the necessity and sufficiency of the condition (5) follow immediately.

The theorem just proved can be written in another form, more useful for applications.

Inclusion Theorem II. A necessary and sufficient condition for $T(A_n) \rightarrow T(B_n)$ is that

$$(6) \quad |k_n A_0| + |k_{n-1} A_1| + \cdots + |k_0 A_n| \leq H |B_n|$$

for all n , where H is a constant (independent of n) and

$$(7) \quad \lim_{n \rightarrow \infty} k_n/B_n = 0.$$

Proof. It is well known (TOEPLITZ [6]) that the validity of (5) for every $\{\epsilon_n\} \rightarrow 0$ is equivalent to the two conditions

$$\lim_{n \rightarrow \infty} \frac{k_{n-r} A_r}{B_n} = 0 \quad (r = 0, 1, \cdots), \quad \frac{1}{B_n} \sum_{r=0}^n |k_{n-r} A_r| \leq H$$

with H independent of n , and, in view of the fact that $B_{n-r} \simeq B_n$ for each r , the first condition reduces to $k_n/B_n \rightarrow 0$.

Theorem 6. If $T(A_n) \leftrightarrow T(B_n)$, then

$$\sum |k_n| < \infty \quad \text{and} \quad \sum |l_n| < \infty.$$

Proof. If $T(A_n) \rightarrow T(B_n)$, then condition (6) implies that $|k_0 A_n| \leq H |B_n|$, i.e., the ratio A_n/B_n is bounded (observe that $k_0 \neq 0$ since $\sum l_n x^n = 1/\sum k_n x^n$ converges for small $|x|$). Similarly $T(B_n) \rightarrow T(A_n)$ implies that B_n/A_n is bounded. Dividing (6) by A_n , retaining only the last $r+1$ terms on the left and observing that $A_{n-j} \simeq A_n$ for each j , we obtain, after making $n \rightarrow \infty$, that

$$|k_0| + |k_1| + \cdots + |k_r| \leq H \sup |B_n/A_n|.$$

Hence $\sum |k_n|$ converges, and similarly we prove the convergence of $\sum |l_n|$.

If $\{A_n\}$ and $\{B_n\}$ monotonically increase for n large enough, Theorem 6 admits of a converse.

Theorem 7. If A_n and B_n are monotonically increasing for $n > n_0$ and tend to ∞ , and if $\sum |k_n| < \infty$ and $\sum |l_n| < \infty$, then $T(A_n) \leftrightarrow T(B_n)$.

Proof. Write $\sum |k_n| = K$, $\sum |l_n| = L$. Then

$$|A_n| \leq |B_0 l_n| + \cdots + |B_n l_0| \leq |B_n| \sum |l_\nu| = |B_n| L$$

for $n > n_0$. We may take n_0 so large that $|A_n| \geq |A_\nu|$ for $n \geq \nu \geq n_0$. Write

$$(8) \quad \sum_{\nu=0}^n |A_\nu k_{n-\nu}| = \sum_{\nu=0}^{n_0} + \sum_{\nu=n_0+1}^n = P + Q,$$

say. It is immediate that $Q \leq B_n L K$, and since we have $k_n \rightarrow 0$, it follows that $P \rightarrow 0$, and so $P = o(B_n)$. Hence the left side of (8) does not exceed $H B_n$ for a suitable H . That $k_n/B_n \rightarrow 0$ is obvious. Hence $T(A_n) \rightarrow T(B_n)$. By symmetry it follows that $T(B_n) \rightarrow T(A_n)$.

In the next section we shall apply the results just obtained to special methods of summation. Some of the special cases will not be covered by the general theorems, and will require additional argument.

2. The logarithmic scale. The classical method (C, α) of Cesàro is a special case of Nörlund's if the numbers $A_n = A_n^\alpha$ are defined by the function ϕ such that

$$\phi(x) = \phi(x; \alpha) = \sum A_n^\alpha x^n = (1-x)^{-(\alpha+1)}.$$

In this case

$$(1) \quad A_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1) \cdots (\alpha+n)}{n!} \simeq \frac{n^\alpha}{\Gamma(\alpha+1)},$$

$$a_n = A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1},$$

and conditions I, II, III hold if $\alpha > -1$. Condition IV holds for $\alpha > 0$.

Now let $\alpha_0, \alpha_1, \dots, \alpha_k$ be any set of $k+1$ real numbers. Write

$$(2) \quad \phi(x; \alpha_0, \alpha_1, \dots, \alpha_k) = \frac{\left(\log_1 \frac{e_1}{1-x}\right)^{\alpha_1} \left(\log_2 \frac{e_2}{1-x}\right)^{\alpha_2} \cdots \left(\log_k \frac{e_k}{1-x}\right)^{\alpha_k}}{(1-x)^{\alpha_0+1}},$$

where

$$\log_1 x = \log x, \quad \log_k x = \log(\log_{k-1} x),$$

$$e_1 = e, \quad e_j = e^{e^{j-1}} \quad (j = 2, 3, \dots).$$

We define numbers $A_n = A_n^{\alpha_0, \alpha_1, \dots, \alpha_k}$ by

$$(3) \quad \sum A_n^{\alpha_0, \alpha_1, \dots, \alpha_k} x^n = \phi(x; \alpha_0, \alpha_1, \dots, \alpha_k)$$

and denote the corresponding method $T(A_n)$ by $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$. If $k = 0$ the method reduces to (C, α) . In this section we prove a number of theorems about the methods $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$.

We shall write $(\alpha_0, \alpha_1, \dots, \alpha_k) \succ (\beta_0, \beta_1, \dots, \beta_k)$ if the first non-zero difference $\alpha_0 - \beta_0, \alpha_1 - \beta_1, \dots, \alpha_k - \beta_k$ is positive; if either

$$(\alpha_0, \alpha_1, \dots, \alpha_k) \succ (\beta_0, \beta_1, \dots, \beta_k)$$

or

$$(\alpha_0, \alpha_1, \dots, \alpha_k) = (\beta_0, \beta_1, \dots, \beta_k)$$

we write

$$(\alpha_0, \alpha_1, \dots, \alpha_k) \succeq (\beta_0, \beta_1, \dots, \beta_k).$$

It is well known that the method (C, α) is interesting only if $\alpha > -1$, and it will turn out that for the method $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$ the corresponding condition is $(\alpha_0, \alpha_1, \dots, \alpha_k) \succ (-1, 0, 0, \dots, 0)$.

The main difficulty in operating with the method $(C, \alpha_0, \dots, \alpha_k)$ is that for the $A_n^{\alpha_0, \dots, \alpha_k}$ we do not have simple formulas analogous to (1). We only know their asymptotic behavior. It is well known that

$$(4) \quad A_n^{\alpha, \beta} \simeq \frac{n^\alpha (\log n)^\beta}{\Gamma(\alpha + 1)}$$

if $\alpha \neq -1, -2, \dots$, and

$$(5) \quad A_n^{-m, \beta} \simeq (-1)^{m-1} \beta (m-1)! n^{-m} (\log n)^{\beta-1} \quad (\beta \neq 0; m = 1, 2, \dots).$$

Similarly

$$A_n^{\alpha, \beta, \gamma} \simeq \frac{n^\alpha (\log n)^\beta (\log \log n)^\gamma}{\Gamma(\alpha + 1)}$$

if $\alpha \neq -1, -2, \dots$; and

$$A_n^{-m, \beta, \gamma} \simeq (-1)^{m-1} \beta (m-1)! n^{-m} (\log n)^{\beta-1} (\log \log n)^\gamma$$

if $\beta \neq 0$; and

$$A_n^{-m, 0, \gamma} \simeq (-1)^{m-1} \gamma (m-1)! n^{-m} (\log n)^{-1} (\log \log n)^{\gamma-1} \quad (\gamma \neq 0).$$

Analogous formulas hold for $A_n^{\alpha_0, \alpha_1, \dots, \alpha_k}$. We only mention the fact that

$$A_n^{\alpha_0, \alpha_1, \dots, \alpha_k} \simeq \frac{n^{\alpha_0}}{\Gamma(\alpha_0 + 1)} (\log n)^{\alpha_1} (\log_2 n)^{\alpha_2} \dots (\log_k n)^{\alpha_k}$$

provided $\alpha_0 \neq -1, -2, \dots$. If α_0 is one of the numbers $m = -1, -2, \dots$, the asymptotic expression for $A_n^{\alpha_0, \alpha_1, \dots, \alpha_k}$ is obtained by differentiating $(\log n)^{\alpha_1} (\log_2 n)^{\alpha_2} \dots (\log_k n)^{\alpha_k} - m$ times with respect to n . The asymptotic formulas just stated indicate that for each $(\alpha_0, \dots, \alpha_k)$ the sequence $\{A_n^{\alpha_0, \dots, \alpha_k}\}$ is monotone from some place on.

Though the theorems in this section are stated for the general method $(C, \alpha_0, \dots, \alpha_k)$, the proofs will, for the sake of simplicity, be given in the case $k = 1$ only; the general case does not require new ideas.

We first note that (2) implies

$$(6) \quad \sum_{j=0}^n A_j^{\alpha_0, \alpha_1, \dots, \alpha_k} = A_n^{\alpha_0+1, \alpha_1, \dots, \alpha_k},$$

$$A_n^{\alpha_0, \alpha_1, \dots, \alpha_k} - A_{n-1}^{\alpha_0, \alpha_1, \dots, \alpha_k} = A_n^{\alpha_0-1, \alpha_1, \dots, \alpha_k}.$$

For a given sequence $\{s_n\}$, or series $\sum u_n$ whose partial sums are s_n , we shall write

$$(7) \quad s_n^{\alpha_0, \alpha_1, \dots, \alpha_k} = \sum_{j=0}^n A_j^{\alpha_0-1, \alpha_1, \dots, \alpha_k} s_{n-j} = \sum_{j=0}^n A_j^{\alpha_0, \alpha_1, \dots, \alpha_k} u_{n-j},$$

$$\sigma_n^{\alpha_0, \alpha_1, \dots, \alpha_k} = s_n^{\alpha_0, \alpha_1, \dots, \alpha_k} / A_n^{\alpha_0, \alpha_1, \dots, \alpha_k}$$

and say that $\{s_n\}$ (or $\sum u_n$) is summable $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$ to sum s if $\lim_{n \rightarrow \infty} \sigma_n^{\alpha_0, \alpha_1, \dots, \alpha_k} = s$. Clearly

$$\begin{aligned} \sum s_n^{\alpha_0, \alpha_1, \dots, \alpha_k} x^n &= \phi(x; \alpha_0 - 1, \alpha_1, \dots, \alpha_k) \sum s_n x^n \\ &= \phi(x; \alpha_0, \alpha_1, \dots, \alpha_k) \sum u_n x^n. \end{aligned}$$

It follows that

$$(8) \quad \sum_{j=0}^n s_j^{\alpha_0, \alpha_1, \dots, \alpha_k} = s_n^{\alpha_0+1, \alpha_1, \dots, \alpha_k},$$

$$s_n^{\alpha_0, \alpha_1, \dots, \alpha_k} - s_{n-1}^{\alpha_0, \alpha_1, \dots, \alpha_k} = s_n^{\alpha_0-1, \alpha_1, \dots, \alpha_k},$$

also

$$\sum_{j=0}^n A_{n-j}^{\alpha_0, \alpha_1, \dots, \alpha_k} s_j^{\beta_0, \beta_1, \dots, \beta_k} = s_n^{\alpha_0+\beta_0+1, \alpha_1+\beta_1, \dots, \alpha_k+\beta_k}$$

and

$$\sum_{j=0}^n A_{n-j}^{\alpha_0, \alpha_1, \dots, \alpha_k} A_j^{\beta_0, \beta_1, \dots, \beta_k} = A_n^{\alpha_0+\beta_0+1, \alpha_1+\beta_1, \dots, \alpha_k+\beta_k}.$$

Theorem 1. If $\sum u_n$ is summable $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$ to sum s , where

$$(\alpha_0, \alpha_1, \dots, \alpha_k) \succ (-1, 0, \dots, 0),$$

and if $(\beta_0, \beta_1, \dots, \beta_k) \succ (0, 0, \dots, 0)$, then $\sum u_n$ is summable

$$(C, \alpha_0 + \beta_0, \alpha_1 + \beta_1, \dots, \alpha_k + \beta_k) \text{ to } s.$$

Theorem 2. If $\sum u_n$ is summable $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$ and

$$(\alpha_0, \alpha_1, \dots, \alpha_k) \succ (-1, 0, \dots, 0),$$

then $u_n = o(A_n^{\alpha_0, \alpha_1, \dots, \alpha_k})$.

Theorem 3. If $\sum u_n$ is summable $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$ to sum s and

$$(\alpha_0, \alpha_1, \dots, \alpha_k) \succ (-1, 0, \dots, 0),$$

then $\sum u_n$ is summable A to sum s .

Theorem 4. If $\sum u_n$ is summable $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$ to sum s and $\sum v_n$ is summable $(C, \beta_0, \beta_1, \dots, \beta_k)$ to sum t , and if

$$(\alpha_0, \alpha_1, \dots, \alpha_k) \succ (-1, 0, \dots, 0), \quad (\beta_0, \beta_1, \dots, \beta_k) \succ (-1, 0, \dots, 0),$$

then the Cauchy product $\sum w_n$ of $\sum u_n$ and $\sum v_n$ is summable

$$(C, \alpha_0 + \beta_0 + 1, \alpha_1 + \beta_1, \dots, \alpha_k + \beta_k)$$

to sum st.

Theorem 5. If $\sum u_n$ is summable $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$, where

$$(\alpha_0, \alpha_1, \dots, \alpha_k) \succ (0, 0, \dots, 0),$$

and if $\sum v_n$ converges absolutely, then $\sum w_n$ is summable $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$.

Theorem 1 is a corollary of Theorem 1 of section 1.

Theorem 2 follows from Theorem 3 of section 1 when $(\alpha_0, \alpha_1, \dots, \alpha_k) \succ (0, 0, \dots, 0)$, since then condition 1° of the latter theorem is satisfied and (with the notation of that theorem) $\bar{A}_n = A_n^{-\alpha_0-2, -\alpha_1, \dots, -\alpha_k}$ so that $\sum |\bar{A}_n| < \infty$. The proof of Theorem 2 in the case $(0, 0, \dots, 0) \succ (\alpha_0, \alpha_1, \dots, \alpha_k) \succ (-1, 0, \dots, 0)$ requires a slight modification. It is based on the immediately verifiable fact that if $\gamma_n \rightarrow 0$ and $\sum |\delta_n| < \infty$, then the terms of the Cauchy product of $\sum \gamma_n$ and $\sum \delta_n$ tend to 0. Write

$$A_n = A_n^{\alpha_0, \alpha_1, \dots, \alpha_k}, \quad s_n = s_n^{\alpha_0, \alpha_1, \dots, \alpha_k}$$

and let $\bar{A}_n$ be defined as above. We may assume that $\sum u_n$ is summable to 0, so that $s_n = \epsilon_n A_n$, where $\epsilon_n \rightarrow 0$. Then

$$\frac{u_n}{A_n} = \frac{1}{A_n} \sum_{\nu=0}^n \bar{A}_\nu s_{n-\nu} = \frac{1}{A_n} \sum_{\nu=0}^{[\frac{1}{2}n]} + \frac{1}{A_n} \sum_{\nu=[\frac{1}{2}n]+1}^n = I'_n + I''_n,$$

say. Let $M = \text{Max } |\epsilon_\nu|$. For n large enough,

$$|I'_n| \leq \frac{A_{n-[\frac{1}{2}n]}}{A_n} \cdot \text{Max}_{\nu \geq [\frac{1}{2}n]} |\epsilon_\nu| \sum_{\nu=0}^{\infty} |\bar{A}_\nu| = o(1),$$

$$|I''_n| \leq \frac{M}{A_n} |\bar{A}_{[\frac{1}{2}n]}| \sum_{\nu=0}^n |A_\nu| = O\left(\frac{A_n^{-\alpha_0-2, -\alpha_1, \dots, -\alpha_k}}{A_n^{\alpha_0, \alpha_1, \dots, \alpha_k}} \cdot A_n^{\alpha_0+1, \alpha_1, \dots, \alpha_k}\right) = o(1),$$

so that $u_n/A_n \rightarrow 0$.

Theorem 3 follows from this and Theorem 2 of section 1. Theorems 4 and 5 are corollaries of Theorems 5 and 4 of section 1 respectively.

Return to the definition of summability $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$. The definition of the function $\phi(x; \alpha_0, \alpha_1, \dots, \alpha_k)$ has a certain element of arbitrariness due to the introduction of the constants $e_1, e_2, \dots, e_k$. We shall now show that the strength of the methods $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$ depends only on the numbers $\alpha_0, \alpha_1, \dots, \alpha_k$ and not on the values of other parameters appearing in the definition of ϕ .

Theorem 6. Let $b_1, b_2, \dots, b_k$ be any real-valued constants such that the functions

$$\log \frac{b_1}{1-z}, \quad \log_2 \frac{b_2}{1-z}, \quad \dots, \quad \log_k \frac{b_k}{1-z}$$

are regular and distinct from 0 in the unit circle $|z| \leq 1$ minus the point $z = 1$. Then, if $(\alpha_0, \alpha_1, \dots, \alpha_k) \succ (-1, 0, \dots, 0)$, the method T defined by the function

$$\psi(x) = (1-x)^{-\alpha_0-1} \left(\log \frac{b_1}{1-x} \right)^{\alpha_1} \cdots \left(\log_k \frac{b_k}{1-x} \right)^{\alpha_k}$$

is equivalent to $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$.

Proof. Write

$$\psi(x) = \sum B_n^{\alpha_0, \alpha_1, \dots, \alpha_k} x^n.$$

The coefficients $B_n^{\alpha_0, \alpha_1, \dots, \alpha_k}$ are known to satisfy the same asymptotic formulas as the $A_n^{\alpha_0, \alpha_1, \dots, \alpha_k}$. Consider the function

$$(9) \quad k(z) = \frac{\psi(z; \alpha_0, \alpha_1, \dots, \alpha_k)}{\phi(z; \alpha_0, \alpha_1, \dots, \alpha_k)} = \left\{ \frac{\log \frac{b_1}{1-z}}{\log \frac{e_1}{1-z}} \right\}^{\alpha_1} \cdots \left\{ \frac{\log_k \frac{b_k}{1-z}}{\log_k \frac{e_k}{1-z}} \right\}^{\alpha_k} = \sum k_n z^n$$

and its reciprocal $l(z) = 1/k(z) = \sum l_n z^n$. Both functions are regular in $|z| < 1$ and are of bounded variation on $|z| = 1$; the latter fact will follow if we show that each ratio which is enclosed in braces in (9) has a derivative which is integrable over $|z| = 1$. We omit the computation, which is straightforward but a bit awkward. Using now the theorem (due to HARDY & LITTLEWOOD ([7], [16])) that if a power series $\sum c_n z^n$ is of bounded variation on $|z| = 1$, then $\sum |c_n|$ is finite, we see that Theorem 6 is a corollary of Theorem 7 of section 1 if we assume that $(\alpha_0, \alpha_1, \dots, \alpha_k) \succ (0, 0, \dots, 0)$.

We now consider the case $(-1, 0, \dots, 0) \prec (\alpha_0, \alpha_1, \dots, \alpha_k) \prec (0, 0, \dots, 0)$; it requires a slightly more elaborate argument. We assume for simplicity that $k = 1$ and consider two subcases: (i) $-1 < \alpha_0 \leq 0$ and (ii) $\alpha_0 = -1$.

In case (i) we shall base the proof on the Inclusion Theorem II of section 1 and the fact that the Fourier coefficients of a function of bounded variation are $O(1/n)$. Hence k_n and l_n are $O(1/n)$ and, in view of the formula (6) of section 1, it is enough to show that

$$\sum_{j=0}^n |k_j A_{n-j}^{\alpha_0, \alpha_1}| = O(B_n^{\alpha_0, \alpha_1}).$$

We split the sum on the left into two, P_n and Q_n , extended over $j \leq [\frac{1}{2}n]$ and $[\frac{1}{2}n] < j \leq n$ respectively. $|P_n|$ is majorized by

$$A_{[\frac{1}{2}n]}^{\alpha_0, \alpha_1} \cdot \sum |k_n| = O(A_n^{\alpha_0, \alpha_1}) = O(B_n^{\alpha_0, \alpha_1}),$$

and

$$|Q_n| \leq O\left(\frac{1}{n}\right) \sum_{v=0}^n |A_v^{\alpha_0, \alpha_1}| = O\left(\frac{1}{n}\right) O(A_n^{\alpha_0+1, \alpha_1}) = O(B_n^{\alpha_0, \alpha_1}).$$

In case (ii) the inequality for P_n remains unchanged. If we show that k_n

satisfies a sharper estimate, namely that $k_n = O(1/n \log n)$, then (since $\alpha_1 > 0$)

$$\begin{aligned} |Q_n| &\leq O(1/n \log n) \sum_{\nu=0}^n |A_{\nu}^{-1, \alpha_1}| = O(1/n \log n) \sum_{\nu=0}^n O(\nu^{-1} (\log \nu)^{\alpha_1-1}) \\ &= O(1/n \log n) O((\log n)^{\alpha_1}) = O(n^{-1} (\log n)^{\alpha_1-1}) = O(B_n^{-1, \alpha_1}). \end{aligned}$$

It remains to show that $k_n = O(1/n \log n)$. The k_n are the Fourier coefficients of the function

$$f(\theta) = \left[\frac{\log \frac{b}{1 - e^{i\theta}}}{\log \frac{e}{1 - e^{i\theta}}} \right]^{\alpha_1}$$

which is of bounded variation. We have

$$(10) \quad k_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{in\theta} d\theta = -\frac{1}{2n\pi} \int_{-\pi}^{\pi} f'(\theta) e^{in\theta} d\theta.$$

Now it is easy to verify that

$$f'(\theta) = \frac{1}{\theta(\log \theta)^2} g(\theta)$$

near $\theta = 0$, where $g(\theta)$ is bounded and monotone in the right hand and left hand neighborhood of $\theta = 0$. Since f' is of bounded variation over any interval not comprising 0, the contribution of the last integral (10) over any such interval is $O(1/n)$ and, using the second Mean-Value Theorem, it is enough to show that, for a fixed h , $0 < h < 1$, we have

$$\int_0^h \frac{e^{in\theta}}{\theta(\log \theta)^2} d\theta = O(1/\log n).$$

This is a familiar fact, easy to prove (we split the integral into two, extended respectively over $(0, 1/n)$ and $(1/n, h)$; in the first integral we replace $e^{in\theta}$ by 1, and in the second we first integrate by parts and then replace $e^{in\theta}$ by 1).*

3. Further theorems on the summability $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$. In this section we extend to the logarithmic scale some of the theorems well known for summability (C, α) . As in the preceding section, one of the difficulties of the proofs is that we know only the asymptotic behavior and the monotonicity properties of the $A_n^{\alpha_0, \alpha_1, \dots, \alpha_k}$ and not their actual values.

We begin with a generalization of the theorem of KOGNETLIANTZ-HAUSDORFF ([9], [8]) which asserts that if to the (C, α) means of a sequence (or series) we

* The reader interested in general formulas for the asymptotic behavior of Fourier coefficients may consult ZYGMUND [16], Chapter IV, §2. These formulas contain as a special case the result we needed here.

apply the method (C, β) , $\alpha > -1$, $\beta > -1$, the resulting expressions define a method of summability equivalent to $(C, \alpha + \beta)$, in symbols

$$(C, \beta)(C, \alpha) \leftrightarrow (C, \alpha + \beta); \quad \alpha > -1, \beta > -1, \alpha + \beta > -1.$$

Theorem 1. If $\alpha_0 > -1$, $\beta_0 > -1$ and $\alpha_0 + \beta_0 > -1$, then

$$(C, \beta_0, \dots, \beta_k)(C, \alpha_0, \dots, \alpha_k) \leftrightarrow (C, \alpha_0 + \beta_0, \dots, \alpha_k + \beta_k).$$

Proof. We assume for simplicity that $k = 1$; the proof in the general case follows the same pattern. We show that (i) summability $(C, \alpha_0 + \beta_0, \alpha_1 + \beta_1)$ implies summability (C, β_0, β_1) (C, α_0, α_1) and that (ii) the converse also holds.

The argument which follows shows that (i) holds if $\alpha_0 > -1$, $\beta_0 > -1$ and $(\alpha_0 + \beta_0, \alpha_1 + \beta_1) \succ (-1, 0)$; similarly (ii) is true if $(\alpha_0, \alpha_1) \succ (-1, 0)$, $(\beta_0, \beta_1) \succ (-1, 0)$ and $\alpha_0 + \beta_0 > -1$.

(i) We shall use the fact that

$$(1) \quad \frac{A_n^{\alpha_0 + \beta_0, \alpha_1 + \beta_1}}{A_n^{\alpha_0, \alpha_1} A_n^{\beta_0, \beta_1}} = O(1) \quad (\alpha_0, \beta_0 > -1).$$

By (7) of section 2,*

$$\begin{aligned} (C, \beta_0, \beta_1)(C, \alpha_0, \alpha_1)s_n &= \frac{1}{A_n^{\beta_0, \beta_1}} \sum_{\nu=0}^n A_{n-\nu}^{\beta_0-1, \beta_1} \frac{s_\nu^{\alpha_0, \alpha_1}}{A_\nu^{\alpha_0, \alpha_1}} \\ &= \frac{1}{A_n^{\beta_0, \beta_1} A_n^{\alpha_0, \alpha_1}} \sum_{\nu=0}^n A_{n-\nu}^{\beta_0-1, \beta_1} s_\nu^{\alpha_0, \alpha_1} \\ &\quad + \frac{1}{A_n^{\beta_0, \beta_1}} \sum_{\nu=0}^n A_{n-\nu}^{\beta_0-1, \beta_1} s_\nu^{\alpha_0, \alpha_1} \left[\frac{1}{A_\nu^{\alpha_0, \alpha_1}} - \frac{1}{A_n^{\alpha_0, \alpha_1}} \right] \\ &= P_n + Q_n, \end{aligned}$$

say. If we assume, as we may, that the $(C, \alpha_0 + \beta_0, \alpha_1 + \beta_1)$ limit of $\{s_n\}$ is 0, it is enough to show that P_n and Q_n are $o(1)$. Clearly

$$P_n = \frac{\sigma_n^{\alpha_0 + \beta_0, \alpha_1 + \beta_1} A_n^{\alpha_0 + \beta_0, \alpha_1 + \beta_1}}{A_n^{\alpha_0, \alpha_1} A_n^{\beta_0, \beta_1}} = o(1).$$

To estimate Q_n we shall use the formula

$$A_n^{\alpha_0, \alpha_1} - A_\nu^{\alpha_0, \alpha_1} = \sum_{i=\nu+1}^n A_i^{\alpha_0-1, \alpha_1},$$

so that

$$(2) \quad A_n^{\alpha_0, \alpha_1} - A_\nu^{\alpha_0, \alpha_1} = O(n - \nu) A_n^{\alpha_0-1, \alpha_1} \quad \text{for } [\tfrac{1}{2}n] \leq \nu \leq n.$$

Suppose first that $(\beta_0, \beta_1) \preceq (0, 0)$. We split Q_n into two parts Q'_n and Q''_n extended respectively over $0 \leq \nu \leq [\tfrac{1}{2}n]$ and $[\tfrac{1}{2}n] < \nu \leq n$. Since summability

* The case when some of the early $A_\nu^{\alpha_0, \alpha_1}$ are 0 and therefore we cannot divide by $A_\nu^{\alpha_0, \alpha_1}$ is easily handled by omitting the corresponding terms in the sum and treating them separately.

$(C, \alpha_0 + \beta_0, \alpha_1 + \beta_1)$ implies summability (C, α_0, α_1) , we have $s_v^{\alpha_0, \alpha_1} = o(A_v^{\alpha_0, \alpha_1})$ and

$$\begin{aligned} |Q'_n| &\leq \frac{1}{|A_n^{\beta_0, \beta_1}|} \sum_{\nu=0}^{[\frac{1}{2}n]} |A_{n-\nu}^{\beta_0-1, \beta_1}| o(|A_\nu^{\alpha_0, \alpha_1}|) \left[\frac{1}{|A_\nu^{\alpha_0, \alpha_1}|} + \frac{1}{|A_n^{\alpha_0, \alpha_1}|} \right] \\ &= \frac{1}{|A_n^{\beta_0, \beta_1}|} \sum_{\nu=0}^n |A_{n-\nu}^{\beta_0-1, \beta_1}| o(1) + \frac{1}{A_n^{\alpha_0, \alpha_1} A_n^{\beta_0, \beta_1}} \sum_{\nu=0}^n |A_{n-\nu}^{\beta_0-1, \beta_1}| o(A_\nu^{\alpha_0, \alpha_1}) \\ &= o\left(\frac{A_n^{\beta_0, \beta_1}}{A_n^{\beta_0, \beta_1}}\right) + o\left(\frac{A_n^{\alpha_0+\beta_0, \alpha_1+\beta_1}}{A_n^{\alpha_0, \alpha_1} A_n^{\beta_0, \beta_1}}\right) = o(1). \end{aligned}$$

Clearly, using (2),

$$\begin{aligned} Q''_n &= O\left(\frac{A_n^{\alpha_0-1, \alpha_1}}{(A_n^{\alpha_0, \alpha_1})^2}\right) \sum_{\nu=[\frac{1}{2}n]+1}^n A_{n-\nu}^{\beta_0-1, \beta_1} O(n-\nu) o(A_\nu^{\alpha_0, \alpha_1}) \\ &= O\left(\frac{A_n^{\alpha_0-1, \alpha_1}}{(A_n^{\alpha_0, \alpha_1})^2}\right) \sum_{\nu=0}^n |A_{n-\nu}^{\beta_0, \beta_1}| o(A_\nu^{\alpha_0, \alpha_1}) \\ &= o\left(\frac{A_n^{\alpha_0-1, \alpha_1} A_n^{\alpha_0+\beta_0+1, \alpha_1+\beta_1}}{(A_n^{\alpha_0, \alpha_1})^2}\right) = o(1). \end{aligned}$$

This proves (i) in the case $(\beta_0, \beta_1) \preceq (0, 0)$ and any $\alpha_0 > -1$.

The proof in the general case follows the same pattern provided we apply summation by parts; suppose, for example, that $(0, 0) \prec (\beta_0, \beta_1) \preceq (1, 0)$. The estimate for P_n remains unchanged. Write

$$Q_n = \frac{1}{A_n^{\beta_0, \beta_1}} \sum_{\nu=0}^n s_\nu^{\alpha_0, \alpha_1} \epsilon_\nu \quad \text{where} \quad \epsilon_\nu = \epsilon_{n, \nu} = A_{n-\nu}^{\beta_0-1, \beta_1} \left[\frac{1}{A_\nu^{\alpha_0, \alpha_1}} - \frac{1}{A_n^{\alpha_0, \alpha_1}} \right]$$

(hence $\epsilon_n = 0$). Writing $\Delta \epsilon_\nu = \epsilon_\nu - \epsilon_{\nu+1}$ and summing by parts, we have*

$$\begin{aligned} Q_n &= \frac{1}{A_n^{\beta_0, \beta_1}} \sum_{\nu=0}^{n-1} s_\nu^{\alpha_0+1, \alpha_1} \Delta \epsilon_\nu = \frac{1}{A_n^{\beta_0, \beta_1}} \sum_{\nu=0}^{n-1} s_\nu^{\alpha_0+1, \alpha_1} A_{n-\nu}^{\beta_0-2, \beta_1} \left[\frac{1}{A_\nu^{\alpha_0, \alpha_1}} - \frac{1}{A_n^{\alpha_0, \alpha_1}} \right] \\ &\quad + \frac{1}{A_n^{\beta_0, \beta_1}} \sum_{\nu=0}^{n-1} s_\nu^{\alpha_0+1, \alpha_1} A_{n-\nu-1}^{\beta_0-1, \beta_1} \left[\frac{1}{A_\nu^{\alpha_0, \alpha_1}} - \frac{1}{A_{\nu+1}^{\alpha_0, \alpha_1}} \right] = U_n + V_n. \end{aligned}$$

The hypothesis $(0, 0) \prec (\beta_0, \beta_1) \preceq (1, 0)$ implies that a series summable $(C, \alpha_0 + \beta_0, \alpha_1 + \beta_1)$ is also summable $(C, \alpha_0 + 1, \alpha_1)$. Hence $s_\nu^{\alpha_0+1, \alpha_1} = o(A_\nu^{\alpha_0+1, \alpha_1})$ and, applying to U_n the same decomposition as before into two sums extended

* In this context we systematically use the formula

$$a_\nu b_\nu - a_{\nu+1} b_{\nu+1} = (a_\nu - a_{\nu+1}) b_\nu + a_{\nu+1} (b_\nu - b_{\nu+1}).$$

respectively over $0 \leq \nu \leq [\frac{1}{2}n]$ and $[\frac{1}{2}n] + 1 \leq \nu \leq n - 1$, we find that $U_n = o(1)$. We also have

$$\begin{aligned} V_n &= \frac{1}{A_n^{\beta_0, \beta_1}} \sum_{\nu=0}^{n-1} A_{n-\nu}^{\beta_0-1, \beta_1} o(A_\nu^{\alpha_0+1, \alpha_1}) \frac{A_{\nu+1}^{\alpha_0-1, \alpha_1}}{A_\nu^{\alpha_0, \alpha_1} \cdot A_{\nu+1}^{\alpha_0, \alpha_1}} \\ &= \frac{1}{A_n^{\beta_0, \beta_1}} \sum_{\nu=0}^{n-1} A_{n-\nu}^{\beta_0-1, \beta_1} o(1) = \frac{1}{A_n^{\beta_0, \beta_1}} o(A_n^{\beta_0, \beta_1}) = o(1). \end{aligned}$$

This proves (i) in the case $(0, 0) \prec (\beta_0, \beta_1) \preceq (1, 0)$. Similarly, by further summation by parts we prove the result for $(\beta_0, \beta_1) \succ (1, 0)$, and so the result holds for any (β_0, β_1) with $\beta_0 > -1$ and any $\alpha_0 > -1$.

(ii) It is enough to show that $(C, \beta_0, \beta_1)(C, \alpha_0, \alpha_1)s_n \rightarrow 0$ implies that $(C, \alpha_0 + \beta_0, \alpha_1 + \beta_1)s_n \rightarrow 0$. We write

$$\begin{aligned} \frac{s_n^{\alpha_0+\beta_0, \alpha_1+\beta_1}}{A_n^{\alpha_0+\beta_0, \alpha_1+\beta_1}} &= \frac{\sum_{\nu=0}^n A_{n-\nu}^{\beta_0-1, \beta_1} s_\nu^{\alpha_0, \alpha_1}}{A_n^{\alpha_0+\beta_0, \alpha_1+\beta_1}} = \frac{\sum_{\nu=0}^n A_{n-\nu}^{\beta_0-1, \beta_1} \sigma_\nu^{\alpha_0, \alpha_1} A_\nu^{\alpha_0, \alpha_1}}{A_n^{\alpha_0+\beta_0, \alpha_1+\beta_1}} \\ &= \frac{A_n^{\alpha_0, \alpha_1} \sum_{\nu=0}^n A_{n-\nu}^{\beta_0-1, \beta_1} \sigma_\nu^{\alpha_0, \alpha_1}}{A_n^{\alpha_0+\beta_0, \alpha_1+\beta_1}} + \frac{\sum_{\nu=0}^n A_{n-\nu}^{\beta_0-1, \beta_1} \sigma_\nu^{\alpha_0, \alpha_1} (A_\nu^{\alpha_0, \alpha_1} - A_n^{\alpha_0, \alpha_1})}{A_n^{\alpha_0+\beta_0, \alpha_1+\beta_1}} \\ &= P_n + Q_n. \end{aligned}$$

By hypothesis, the sum $\sum$ in P_n is $o(A_n^{\beta_0, \beta_1})$, so that

$$P_n = o\left(\frac{A_n^{\alpha_0, \alpha_1} A_n^{\beta_0, \beta_1}}{A_n^{\alpha_0+\beta_0, \alpha_1+\beta_1}}\right) = o(1).$$

Suppose now that $(\beta_0, \beta_1) \preceq (0, 0)$; hence $\sigma_\nu^{\alpha_0, \alpha_1} \rightarrow 0$. We split the sum in Q_n into two parts, Q'_n and Q''_n , extended respectively over the ranges $0 \leq \nu \leq [\frac{1}{2}n]$, $[\frac{1}{2}n] + 1 \leq \nu \leq n$. Then

$$\begin{aligned} Q'_n &= \frac{\sum_{\nu=0}^{[\frac{1}{2}n]} A_{n-\nu}^{\beta_0-1, \beta_1} o(1)(|A_\nu^{\alpha_0, \alpha_1}| + |A_n^{\alpha_0, \alpha_1}|)}{A_n^{\alpha_0+\beta_0, \alpha_1+\beta_1}} \\ &= \frac{|A_n^{\beta_0-1, \beta_1}| \sum_{\nu=0}^{[\frac{1}{2}n]} o(A_\nu^{\alpha_0, \alpha_1})}{A_n^{\alpha_0+\beta_0, \alpha_1+\beta_1}} + \frac{o(|A_n^{\alpha_0, \alpha_1} A_n^{\beta_0-1, \beta_1}|) \sum_{\nu=0}^{[\frac{1}{2}n]} o(1)}{A_n^{\alpha_0+\beta_0, \alpha_1+\beta_1}} = o(1). \end{aligned}$$

Using the fact that

$$A_\nu^{\alpha_0, \alpha_1} - A_n^{\alpha_0, \alpha_1} = - \sum_{i=\nu+1}^n A_i^{\alpha_0-1, \alpha_1} = O[(n-\nu)A_n^{\alpha_0-1, \alpha_1}]$$

if $[\frac{1}{2}n] + 1 \leq \nu \leq n$, we see that

$$Q''_n = \frac{o(A_n^{\alpha_0-1, \alpha_1}) \sum_{\nu=[\frac{1}{2}n]+1}^n |A_{n-\nu}^{\beta_0, \beta_1}|}{A_n^{\alpha_0+\beta_0, \alpha_1+\beta_1}} = o\left(\frac{A_n^{\alpha_0-1, \alpha_1} A_n^{\beta_0+1, \beta_1}}{A_n^{\alpha_0+\beta_0, \alpha_1+\beta_1}}\right) = o(1).$$

This proves (ii) in the case when $(\beta_0, \beta_1) \preceq (0, 0)$ and $\alpha_0 > -1$. Using, as in (i), summation by parts we prove (ii) successively for $(\beta_0, \beta_1) \preceq (1, 0)$, $(\beta_0, \beta_1) \preceq (2, 0)$, etc. and so in the general case.

We shall now consider theorems giving summability of series $\sum u_n \mu_n$ in terms of summability of $\sum u_n$. In the theorem which follows, and which in the case $k = 0$ was proved by ZYGMUND [14], we denote by $\langle \gamma \rangle$ the least integer not less than γ (hence $\langle \gamma \rangle = \gamma$ if γ is an integer, $\langle \gamma \rangle = [\gamma] + 1$ otherwise).

Theorem 2. Suppose $\alpha_0 > -1$ and that the $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$ means of $\sum u_n$ satisfy

$$\sigma_n^{\alpha_0, \alpha_1, \dots, \alpha_k} = o(\mu_n),$$

where $\{\mu_n\}$ is monotonically increasing to $+\infty$ and such that

$$(3) \quad \Delta^j \frac{1}{\mu_n} = O\left(\frac{1}{(n+1)^j \mu_n}\right), \quad j = 1, 2, \dots, \langle \alpha_0 \rangle + 2.$$

Then, if the series $\sum u_n/\eta_n$ is summable A , it is also summable $(C, \alpha_0, \dots, \alpha_k)$.

We write here $\Delta \epsilon_\nu = \epsilon_\nu - \epsilon_{\nu+1}$, $\Delta^s \epsilon_\nu = \Delta(\Delta^{s-1} \epsilon_\nu)$.

It is again enough to consider the case $k = 1$.

Lemma. If $\sigma_n^{\alpha_0, \alpha_1} = o(\mu_n)$, $(\alpha_0, \alpha_1) \succ (-1, 0)$, and $\{\mu_n\}$ is monotonically increasing to $+\infty$, then $\sigma_n^{\alpha_0+\epsilon, \alpha_1+\delta} = o(\mu_n)$ provided $(\epsilon, \delta) \succ (0, 0)$.

The lemma holds if “ o ” is replaced throughout by “ O ”.

The proof is immediate if one expresses $\sigma_n^{\alpha_0+\epsilon, \alpha_1+\delta}$ in terms of the $\sigma_n^{\alpha_0, \alpha_1}$ and uses Toeplitz conditions.

Return to Theorem 2 and suppose first that $(\alpha_0, \alpha_1) \preceq (0, 0)$. Write $u_\nu/\mu_\nu = b_\nu$ and denote the (C, α_0, α_1) means of $\sum b_\nu$ by $\bar{\sigma}_n^{\alpha_0, \alpha_1}$. Then

$$\begin{aligned} \bar{\sigma}_n^{\alpha_0, \alpha_1} &= \frac{1}{A_n^{\alpha_0, \alpha_1}} \sum_{\nu=0}^n A_{n-\nu}^{\alpha_0, \alpha_1} b_\nu \\ &= \frac{1}{\mu_n} \frac{1}{A_n^{\alpha_0, \alpha_1}} \sum_{\nu=0}^n A_{n-\nu}^{\alpha_0, \alpha_1} u_\nu + \frac{1}{A_n^{\alpha_0, \alpha_1}} \sum_{\nu=0}^n A_{n-\nu}^{\alpha_0, \alpha_1} u_\nu \left[\frac{1}{\mu_\nu} - \frac{1}{\mu_n} \right] = P_n + Q_n. \end{aligned}$$

The hypothesis of Theorem 2 implies that $P_n = o(1)$. Summation by parts shows that

$$\begin{aligned} Q_n &= \frac{1}{A_n^{\alpha_0, \alpha_1}} \sum_{\nu=0}^n s_\nu A_{n-\nu}^{\alpha_0-1, \alpha_1} \left[\frac{1}{\mu_\nu} - \frac{1}{\mu_n} \right] \\ &\quad + \frac{1}{A_n^{\alpha_0, \alpha_1}} \sum_{\nu=0}^{n-1} s_\nu A_{n-\nu-1}^{\alpha_0, \alpha_1} \left[\frac{1}{\mu_\nu} - \frac{1}{\mu_{\nu+1}} \right] = \frac{1}{A_n^{\alpha_0, \alpha_1}} \sum_{\nu=0}^n s_\nu A_{n-\nu}^{\alpha_0-1, \alpha_1} \left[\frac{1}{\mu_\nu} - \frac{1}{\mu_n} \right] \\ &\quad + \frac{1}{A_n^{\alpha_0, \alpha_1}} \sum_{\nu=1}^n A_{n-\nu}^{\alpha_0, \alpha_1} s_{\nu-1} \left[\frac{1}{\mu_{\nu-1}} - \frac{1}{\mu_\nu} \right] = Q'_n + Q''_n, \end{aligned}$$

say. We split the sum in Q'_n into two parts extended over the ranges $0 \leq \nu \leq [\frac{1}{2}n]$

and $[\frac{1}{2}n] + 1 \leq \nu \leq n$. The corresponding parts of Q'_n are, respectively, majorized by

$$\frac{1}{A_n^{\alpha_0, \alpha_1}} \sum_{\nu=0}^{[\frac{1}{2}n]} \left| \frac{s_\nu}{\mu_\nu} \right| A_{n-\nu}^{\alpha_0-1, \alpha_1} = O\left(\frac{A_n^{\alpha_0-1, \alpha_1}}{A_n^{\alpha_0, \alpha_1}}\right) \sum_{\nu=1}^{[\frac{1}{2}n]} o(1) = O\left(\frac{A_n^{\alpha_0-1, \alpha_1}}{A_n^{\alpha_0, \alpha_1}}\right) o(n) = o(1)$$

(since $(\alpha_0, \alpha_1) \preceq (0, 0)$, we have $s_\nu = o(\mu_\nu)$) and by

$$\begin{aligned} \frac{1}{A_n^{\alpha_0, \alpha_1}} \sum_{\nu=[\frac{1}{2}n]+1}^n |s_\nu| A_{n-\nu}^{\alpha_0-1, \alpha_1} \sum_{j=\nu}^{n-1} \left| \Delta \frac{1}{\mu_j} \right| \\ = \frac{1}{A_n^{\alpha_0, \alpha_1}} \sum_{\nu=[\frac{1}{2}n]+1}^n o(\mu_\nu) A_{n-\nu}^{\alpha_0-1, \alpha_1} (n-\nu) O\left(\frac{1}{\nu \mu_\nu}\right) \\ = O\left(\frac{1}{A_n^{\alpha_0, \alpha_1} n}\right) \sum_{\nu=[\frac{1}{2}n]+1}^n A_{n-\nu}^{\alpha_0, \alpha_1} = O\left(\frac{A_n^{\alpha_0+1, \alpha_1}}{A_n^{\alpha_0, \alpha_1} n}\right) = o(1). \end{aligned}$$

Write $c_0 = 0$, $c_\nu = s_{\nu-1} \Delta(1/\mu_{\nu-1})$, $\nu = 1, 2, \dots$, and denote the (C, α_0, α_1) means of $\sum c_\nu$ by $\bar{\sigma}_n^{\alpha_0, \alpha_1}$. Clearly $Q'_n = \bar{\sigma}_n^{\alpha_0, \alpha_1}$. Collecting results we have

$$\bar{\sigma}_n^{\alpha_0, \alpha_1} = \bar{\sigma}_n^{\alpha_0, \alpha_1} + o(1).$$

This means that the series $\sum (b_\nu - c_\nu)$ is summable (C, α_0, α_1) to the sum 0. Hence it is also summable A to 0. Since by hypothesis $\sum b_\nu$ is summable A , the same holds for $\sum c_\nu$ (and the sums of both series are the same). But (by hypothesis (3) for $j = 1$) $c_\nu = o(1/\nu)$; hence, by Tauber's theorem, $\sum c_\nu$ converges, and, by the theorem of CHAPMAN stated in section 4, is summable (C, α_0, α_1) , $\alpha_0 > -1$, to the sum

$$0 + \sum_{\nu=1}^{\infty} s_{\nu-1} \Delta(1/\mu_{\nu-1}) = \sum_{\nu=0}^{\infty} s_\nu \Delta(1/\mu_\nu).$$

This completes the proof of the theorem if $(\alpha_0, \alpha_1) \preceq (0, 0)$.

If $(0, 0) \prec (\alpha_0, \alpha_1) \preceq (1, 0)$ we proceed similarly. As before we have $P_n = o(1)$. Summing by parts twice, so as to introduce $s_\nu^{(1)}$, and using the formula

$$\Delta^2(u_\nu v_\nu) = (\Delta^2 u_\nu) v_\nu + 2(\Delta u_{\nu+1})(\Delta v_\nu) + u_{\nu+2} \Delta^2 v_\nu,$$

we decompose Q_n into three sums Q'_n , Q''_n , Q'''_n , where

$$Q'''_n = \frac{1}{A_n^{\alpha_0, \alpha_1}} \sum_{\nu=0}^n s_\nu^{(1)} A_{n-\nu}^{\alpha_0, \alpha_1} \Delta^2 \frac{1}{\mu_\nu}$$

is the (C, α_0, α_1) mean of the series $\sum s_\nu^{(1)} \Delta^2(1/\mu_\nu)$ and the remaining two sums Q'_n and Q''_n , as well as the "integrated" terms, give contributions $o(1)$. Arguing as before we show that if the series $\sum u_\nu/\mu_\nu$ is summable A , then it is summable (C, α_0, α_1) to sum $\sum s_\nu^{(1)} \Delta^2(1/\mu_\nu)$. The pattern of the proof in the general case is now clear and we may consider the theorem proved.

We assumed in Theorem 2 that $\alpha_0 > -1$. For $\alpha_0 = -1$ we have the following theorem, where we suppose for simplicity that $k = 1$.

Theorem 2'. Suppose that the means σ_n^{-1, α_1} of the series $\sum u_\nu$ satisfy the condition $\sigma_n^{-1, \alpha_1} = o(\mu_n)$, $\alpha_1 > 0$, where μ_n increases monotonically to $+\infty$ and $\Delta(1/\mu_n) = O(\mu_n^{-1} n^{-1} (\log n)^{-1})$; then, if $\sum u_\nu/\mu_\nu$ is summable A , it is also summable $(C, -1, \alpha_1)$.

The proof is the same as that of Theorem 2 (for $\alpha_0 < 0$) but instead of using CHAPMAN's theorem we apply Theorem 1 of section 4.

We now give an extension of a theorem of ANDERSEN [1].

Let s be a nonnegative integer and $\eta_0, \eta_1, \dots$ a sequence of numbers. We shall say that $\{\eta_\nu\}$ satisfies the condition B_s if

$$(i) \quad \eta_\nu = O(1), \quad (ii) \quad \sum_{n=1}^{\infty} n^s |\Delta^{s+1} \eta_n| < \infty.$$

It is very well known (and not difficult to prove) that (i) and (ii) imply

$$(i') \quad \Delta^t \eta_\nu = o(\nu^{-t}), \quad (ii') \quad \sum_{n=1}^{\infty} n^t |\Delta^{t+1} \eta_n| < \infty, \quad t = 1, 2, \dots, s.$$

Theorem 3. Suppose that $\sum u_\nu$ is summable $(C, \alpha_0, \dots, \alpha_k)$, where $(\alpha_0, \alpha_1, \dots, \alpha_k) \geq (0, 0, \dots, 0)$, and that $\{\eta_\nu\}$ satisfies condition B_s , where s is the least integer such that $(\alpha_0, \alpha_1, \dots, \alpha_k) \leq (s, 0, \dots, 0)$. Then $\sum u_\nu \eta_\nu$ is summable $(C, \alpha_0, \dots, \alpha_k)$. The theorem holds for

$$(-1, 0, \dots, 0) \prec (\alpha_0, \alpha_1, \dots, \alpha_k) \prec (0, 0, \dots, 0)$$

if in addition to conditions (i) and (ii) for $s = 0$ one requires that (iii) $\Delta \eta_\nu = O(1/\nu \log \nu \dots \log_\nu \nu)$.

Proof. Suppose for simplicity that $k = 1$ and that the (C, α_0, α_1) sum of $\sum u_\nu$ is 0. We consider first the case when $s = 0$, i.e. $(\alpha_0, \alpha_1) \leq (0, 0)$. We may assume that we have the sign $\prec$ here, the case when $(\alpha_0, \alpha_1) = (0, 0)$ being both very well known and immediate. The (C, α_0, α_1) mean of $\sum u_\nu \eta_\nu$ is

$$\begin{aligned} & (A_n^{\alpha_0, \alpha_1})^{-1} \sum_{\nu=0}^n A_{n-\nu}^{\alpha_0, \alpha_1} u_\nu (\eta_\nu - \eta_n) + o(1) \\ (4) \quad &= (A_n^{\alpha_0, \alpha_1})^{-1} \sum_{\nu=0}^{n-1} s_\nu \Delta \{A_{n-\nu}^{\alpha_0, \alpha_1} (\eta_\nu - \eta_n)\} + o(1) \\ &= (A_n^{\alpha_0, \alpha_1})^{-1} \sum_{\nu=0}^{n-1} A_{n-\nu}^{\alpha_0-1, \alpha_1} (\eta_\nu - \eta_n) s_\nu \\ &\quad + (A_n^{\alpha_0, \alpha_1})^{-1} \sum_{\nu=0}^{n-1} A_{n-\nu-1}^{\alpha_0, \alpha_1} s_\nu \Delta \eta_\nu + o(1) = P_n + Q_n + o(1), \end{aligned}$$

say. Now Q_n , when multiplied by $A_n^{\alpha_0, \alpha_1}/A_{n-1}^{\alpha_0, \alpha_1}$ is a (C, α_0, α_1) mean of the series $\sum s_\nu \Delta \eta_\nu$, which is convergent (by (ii) with $s = 0$), and by (iii) has terms $o(1/\nu \log \nu)$. By Theorem 1 of section 4, this series Q_n converges to the limit $\sum s_\nu \Delta \eta_\nu$. The hypothesis $\sum |\Delta \eta_\nu| < \infty$ implies that η_ν tends to a limit. If

we now split the sum in P_n into two extended over the ranges $0 \leq \nu \leq [\frac{1}{2}n]$ and $[\frac{1}{2}n] < \nu < n$, and use for the first range the fact that $\eta_\nu - \eta_n \rightarrow 0$ as $\nu \rightarrow \infty$ ($n \geq \nu$) and for the second the estimate

$$|\eta_\nu - \eta_n| \leq |\Delta\eta_\nu + \Delta\eta_{\nu+1} + \cdots + \Delta\eta_{n-1}| \leq (n - \nu)O(1/n \log n)$$

and the fact that $\sum |A_i^{\alpha_0-1, \alpha_1}| < \infty$, we obtain

$$\begin{aligned} |P_n| &= (A_n^{\alpha_0, \alpha_1})^{-1} O(A_{[\frac{1}{2}n]}^{\alpha_0-1, \alpha_1}) \sum_{\nu=0}^{[\frac{1}{2}n]} o(1) + (A_n^{\alpha_0, \alpha_1})^{-1} O\left(\frac{1}{n \log n}\right) \sum_{\nu=[\frac{1}{2}n]+1}^{n-1} |A_{n-\nu}^{\alpha_0-1, \alpha_1}| \\ &= o\left(\frac{n A_n^{\alpha_0-1, \alpha_1}}{A_n^{\alpha_0, \alpha_1}}\right) + o\left(\frac{1}{n \log n A_n^{\alpha_0, \alpha_1}}\right) = o(1). \end{aligned}$$

This proves the theorem in the case when $(-1, 0) \prec (\alpha_0, \alpha_1) \preceq (0, 0)$. If $(0, 0) \prec (\alpha_0, \alpha_1) \preceq (1, 0)$, we apply summation by parts once more to the sum in (4) so as to introduce the $(C, 1)$ sums $s_\nu^{(1)}$, and the argument proceeds as before; we omit the details (cf. [16], pp. 322–324).

The theorem which follows reduces when $k = 0$ to the very well known result of M. RIESZ ([10], [3], [6]).

Theorem 4. Suppose that the series $\sum u_n$ is summable $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$, $(\alpha_0, \alpha_1, \dots, \alpha_k) \succ (-1, 0, \dots, 0)$, and that $(\beta_0, \beta_1, \dots, \beta_k) \succ (0, 0, \dots, 0)$; then the series

$$(5) \quad \sum' \frac{u_n}{n^{\beta_0} (\log n)^{\beta_1} \cdots (\log_k n)^{\beta_k}}$$

is summable $(C, \alpha_0 - \beta_0, \alpha_1 - \beta_1, \dots, \alpha_k - \beta_k)$ except possibly in the case when $\alpha_0 > -1$, $\alpha_0 - \beta_0 = -1$. The prime ' in (5) indicates that the early terms of the series for which $\log_k n$ is not defined are replaced by 0.

The theorem is an easy consequence of Theorems 2 and 2' (and the extensions of Theorem 2' to general k). Suppose, for simplicity, that $k = 1$. It is easy to see that if the means $\sigma_n^{\alpha_0, \alpha_1}$ of $\sum u_\nu$ tend, say, to limit 0, that is if $s_n^{\alpha_0, \alpha_1} = o(A_n^{\alpha_0, \alpha_1})$, then $s_n^{\alpha_0 - \beta_0, \alpha_1 - \beta_1} = o(A_n^{\alpha_0, \alpha_1})$, so that

$$\sigma_n^{\alpha_0 - \beta_0, \alpha_1 - \beta_1} = o(A_n^{\alpha_0, \alpha_1} / A_n^{\alpha_0 - \beta_0, \alpha_1 - \beta_1}) = o(n^{\beta_0} (\log n)^{\beta_1})$$

provided α_0 and $\alpha_0 - \beta_0$ are either both greater than -1 or both equal to -1 (in the latter case $\beta_0 = 0$). Since the series $\sum' u_n n^{-\beta_0} (\log n)^{-\beta_1}$ is clearly summable A , Theorem 2 (in the case $\alpha_0 - \beta_0 > -1$) and Theorem 2' (in the case $\alpha_0 = -1$ so that $\beta_0 = 0$) imply that $\sum' u_n n^{-\beta_0} (\log n)^{-\beta_1}$ is summable $(C, \alpha_0 - \beta_0, \alpha_1 - \beta_1)$. Thus the only case we excluded was $\alpha_0 > -1$ and $\alpha_0 - \beta_0 = -1$. We shall show by example that in this exceptional case the theorem is false. It is enough to consider the convergent series $\sum' (-1)^\nu / \log \nu$. Since

$$\sum' \frac{u_\nu}{\nu (\log \nu)^{-1}} \sim \sum' \frac{(-1)^\nu}{\nu},$$

the series $\sum' u_\nu / \nu (\log \nu)^{-1}$ cannot be summable $(C, -1, 1)$, for otherwise the terms would be $o(A_\nu^{-1,1}) = o(1/\nu)$.

The following theorem will be needed in section 5 (for $k = 0$ it is given in ZYGMUND [14]).

Theorem 5. Suppose that

$$(\beta_0, \beta_1, \dots, \beta_k) \succ (0, 0, \dots, 0),$$

$$(\alpha_0 - \beta_0, \alpha_1 - \beta_1, \dots, \alpha_k - \beta_k) \succ (-1, 0, \dots, 0)$$

(so that also $(\alpha_0, \alpha_1, \dots, \alpha_k) \succ (-1, 0, \dots, 0)$) and that the means $\sigma_n^{\alpha_0, \dots, \alpha_k}$ of $\sum_{\nu=0}^\infty u_\nu$ satisfy

$$\sigma_n^{\alpha_0, \dots, \alpha_k} = s + o\left(\frac{A_n^{\alpha_0 - \beta_0, \dots, \alpha_k - \beta_k}}{A_n^{\alpha_0, \dots, \alpha_k}}\right);$$

then $\sum u_\nu$ is summable $(C, \alpha_0 - \beta_0, \dots, \alpha_k - \beta_k)$ to s .

Proof. We may suppose that $s = 0$. We also suppose for simplicity that $k = 1$. The hypothesis is then that $s_n^{\alpha_0, \alpha_1} = o(A_n^{\alpha_0 - \beta_0, \alpha_1 - \beta_1})$. We have

$$s_n^{\alpha_0 - \beta_0, \alpha_1 - \beta_1} = \sum_{\nu=0}^n A_{n-\nu}^{-\beta_0-1, -\beta_1} s_\nu^{\alpha_0, \alpha_1} = \sum_{\nu=0}^{\lfloor \frac{1}{2}n \rfloor} + \sum_{\nu=\lfloor \frac{1}{2}n \rfloor+1}^n = P_n + Q_n,$$

say. Hence

$$|P_n| \leq O(A_n^{-\beta_0-1, -\beta_1}) \sum_{\nu=0}^{\lfloor \frac{1}{2}n \rfloor} o(A_\nu^{\alpha_0 - \beta_0, \alpha_1 - \beta_1}) = O(A_n^{-\beta_0-1, -\beta_1}) o(A_n^{\alpha_0 - \beta_0 + 1, \alpha_1 - \beta_1})$$

and, observing that the hypothesis $(\beta_0, \beta_1) \succ (0, 0)$ implies that $A_n^{-\beta_0-1, -\beta_1} = O(1/n \log n)$, we see that $P_n = o(A_n^{\alpha_0 - \beta_0, \alpha_1 - \beta_1})$. Also

$$|Q_n| \leq o(A_n^{\alpha_0 - \beta_0, \alpha_1 - \beta_1}) \sum_{\nu=\lfloor \frac{1}{2}n \rfloor+1}^n |A_{n-\nu}^{-\beta_0-1, -\beta_1}| = o(A_n^{\alpha_0 - \beta_0, \alpha_1 - \beta_1}),$$

since

$$\sum_{\nu=0}^\infty |A_\nu^{-\beta_0-1, -\beta_1}| < \infty.$$

This proves the theorem.

We conclude this section with remarks of a different nature.

Let $\epsilon_0, \epsilon_1, \epsilon_2, \dots$ be an arbitrary number-sequence and $\Delta^1 \epsilon_\nu = \epsilon_\nu - \epsilon_{\nu+1}$, $\nu = 0, 1, 2, \dots$. These differences can be represented as the coefficients of $x^{-\nu}$ in the series expansion

$$(1-x) \left(\epsilon_0 + \frac{\epsilon_1}{x} + \frac{\epsilon_2}{x^2} + \dots \right) = -\epsilon_0 x + \left(\Delta^1 \epsilon_0 + \frac{\Delta^1 \epsilon_1}{x} + \frac{\Delta^1 \epsilon_2}{x^2} + \dots \right).$$

Denoting in the usual way the iterated differences

$$\Delta^2 \epsilon_\nu = \Delta^1(\Delta^1 \epsilon_\nu), \dots, \Delta^m \epsilon_\nu = \Delta^1(\Delta^{m-1} \epsilon_\nu), \dots$$

we find immediately that the difference $\Delta^m \epsilon_\nu$, $\nu = 0, 1, 2, \dots$, for an arbitrary natural number m is the coefficient of $x^{-\nu}$ in the series expansion

$$(1-x)^m \left(\epsilon_0 + \frac{\epsilon_1}{x} + \frac{\epsilon_2}{x^2} + \dots \right) = P(x) + \Delta^m \epsilon_0 + \frac{\Delta^m \epsilon_1}{x} + \frac{\Delta^m \epsilon_2}{x^2} + \dots,$$

where $P(x)$ is a finite polynomial with positive powers of x , in whose coefficients we are not interested.

H. BOHR [2] defines the difference $\Delta^r \epsilon_\nu$ for any real number r by the corresponding formula

$$(1-x)^r \left(\epsilon_0 + \frac{\epsilon_1}{x} + \frac{\epsilon_2}{x^2} + \dots \right) = \dots + \Delta^r \epsilon_0 + \frac{\Delta^r \epsilon_1}{x} + \frac{\Delta^r \epsilon_2}{x^2} + \dots,$$

where the dots before the constant term on the right of the equation denote a power series in x .

We define correspondingly the difference $\Delta^{\alpha_0, \alpha_1, \dots, \alpha_k} \epsilon_\nu$ for any real numbers $\alpha_0, \alpha_1, \dots, \alpha_k$ by

$$\begin{aligned} \Psi(\alpha_0, \dots, \alpha_k; x) \left(\epsilon_0 + \frac{\epsilon_1}{x} + \frac{\epsilon_2}{x^2} + \dots \right) \\ = \dots + \Delta^{\alpha_0, \dots, \alpha_k} \epsilon_0 + \frac{\Delta^{\alpha_0, \dots, \alpha_k} \epsilon_1}{x} + \frac{\Delta^{\alpha_0, \dots, \alpha_k} \epsilon_2}{x^2} + \dots, \end{aligned}$$

where

$$\Psi(\alpha_0, \alpha_1, \dots, \alpha_k; x) = \frac{(1-x)^{\alpha_0}}{\left(\log_1 \frac{e_1}{1-x} \right)^{\alpha_1} \left(\log_2 \frac{e_2}{1-x} \right)^{\alpha_2} \dots \left(\log_k \frac{e_k}{1-x} \right)^{\alpha_k}}$$

and, as above, the dots before $\Delta^{\alpha_0, \dots, \alpha_k} \epsilon_0$ denote a power series in x which does not interest us.

Since

$$\Psi(\alpha_0, \alpha_1, \dots, \alpha_k; x) = \sum_{n=0}^{\infty} A_n^{-\alpha_0-1, -\alpha_1, \dots, -\alpha_k} x^n,$$

the above definition is equivalent to

$$(6) \quad \Delta^{\alpha_0, \alpha_1, \dots, \alpha_k} \epsilon_\nu = \sum_{p=\nu}^{\infty} \epsilon_p A_{p-\nu}^{-\alpha_0-1, -\alpha_1, \dots, -\alpha_k},$$

and we attach meaning to the symbol $\Delta^{\alpha_0, \dots, \alpha_k} \epsilon_\nu$ if and only if the series on the right side of (6) is convergent. Clearly the differences $\Delta^{\alpha_0, \dots, \alpha_k} \epsilon_\nu$, $\nu = 0, 1, 2, \dots$, exist if the ϵ_ν are bounded and $(\alpha_0, \dots, \alpha_k) \succ (0, \dots, 0)$.

Theorem 6. *If the ϵ_ν are bounded,*

$$(\alpha_0, \dots, \alpha_k) \succ (0, \dots, 0), \quad (\beta_0, \beta_1, \dots, \beta_k) \succ (-1, 0, \dots, 0),$$

and

$$(\alpha_0 + \beta_0, \dots, \alpha_k + \beta_k) \succ (0, \dots, 0),$$

then

$$\Delta^{\beta_0, \dots, \beta_k} \Delta^{\alpha_0, \dots, \alpha_k} \epsilon_\nu = \Delta^{\alpha_0 + \beta_0, \dots, \alpha_k + \beta_k} \epsilon_\nu.$$

For $k = 0$ this is the well known theorem of ANDERSEN [1], for which H. BOHR [*loc. cit.*] gave a very simple proof. This proof applies equally well to the general case and we omit it here.

4. Tauberian theorems. Tauberian theorems usually assert that if B_1 and B_2 are two methods of summation, B_1 weaker than B_2 , and if $\sum u_n$ is summable B_2 and its terms satisfy certain additional conditions, then $\sum u_n$ is summable B_1 . For example, a theorem of CHAPMAN [4] asserts that if $\sum u_n$ is convergent and $u_n = o(1/n)$, then it is summable $(C, -1 + \delta)$ for each $\delta > 0$. HARDY & LITTLEWOOD [6] have shown that the conclusion holds if the condition $u_n = o(1/n)$ is replaced by $u_n = O(1/n)$. CHAPMAN's theorem implies, of course, that if $\sum u_n$ is convergent and $u_n = o(1/n)$, then it is also summable $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$ for each $\alpha_0 > -1$. We now consider the case $\alpha_0 = -1$.

Theorem 1. *If the series $\sum u_n$ converges to s and*

$$u_n = o(1/n \log n \log_2 n \cdots \log_k n),$$

then $\sum u_n$ is summable $(C, \alpha_0, \alpha_1, \dots, \alpha_k) = (C, -1, 0, \dots, 0, \delta)$ to s for every positive δ .

It is a curious fact that, unlike in the theorem of CHAPMAN, we cannot replace here the o in the condition for u_n by O . We return to this in a moment.

Proof. It is enough to consider the case $k = 1$, and we may suppose that $s = 0$, i.e., $s_\nu = \sum_{i=0}^\nu u_i \rightarrow 0$. We have to show that

$$s_n^{-1, \delta} = \sum_{\nu=0}^n u_\nu A_{n-\nu}^{-1, \delta} = \sum_{\nu=0}^{\lfloor \frac{1}{2}n \rfloor} + \sum_{\nu=\lfloor \frac{1}{2}n \rfloor+1}^n = o(A_n^{-1, \delta}).$$

Now

$$\begin{aligned} \left| \sum_{\nu=0}^{\lfloor \frac{1}{2}n \rfloor} u_\nu A_{n-\nu}^{-1, \delta} \right| &= \left| \sum_{\nu=0}^{\lfloor \frac{1}{2}n \rfloor} (s_\nu - s_{\nu-1}) A_{n-\nu}^{-1, \delta} \right| \\ &= \left| \sum_{\nu=0}^{\lfloor \frac{1}{2}n \rfloor} s_\nu A_{n-\nu}^{-2, \delta} + o(1) O(n^{-1}(\log n)^{\delta-1}) \right| \leq O(n^{-2}(\log n)^{\delta-1}) \sum_{\nu=0}^{\lfloor \frac{1}{2}n \rfloor} |s_\nu| \\ &\quad + o(n^{-1}(\log n)^{\delta-1}) = O(n^{-2}(\log n)^{\delta-1}) o(n) + o(n^{-1}(\log n)^{\delta-1}) = o(A_n^{-1, \delta}). \end{aligned}$$

Also

$$\left| \sum_{\nu=\lfloor \frac{1}{2}n \rfloor+1}^n u_\nu A_{n-\nu}^{-1, \delta} \right| \leq o\left(\frac{1}{n \log n}\right) \sum_{\nu=\lfloor \frac{1}{2}n \rfloor+1}^n |A_{n-\nu}^{-1, \delta}| = o\left(\frac{1}{n \log n}\right) O(A_n^{0, \delta}) = o(A_n^{-1, \delta}).$$

Let us take a sequence $n_1 < n_2 < \dots < n_i < \dots$ of even natural numbers such that

$$1^\circ \quad n_i > 2n_{i-1}, \quad 2^\circ \quad \sum \frac{1}{\log n_i} < \infty.$$

Let us define the series $\sum b_\nu$ by the condition that for

$$\frac{1}{2}n_i < \nu \leq n_i \quad (i = 1, 2, \dots)$$

we have $b_\nu = 1/n_i \log n_i$, and for all other $\nu > 0$ we set $b_\nu = 0$. Clearly $b_\nu \leq 1/\nu \log \nu$ for all $\nu \geq 2$. The series is convergent by condition 2^0 , since the sum of the terms b_ν with indices between $\frac{1}{2}n_i$ and n_i is $1/2 \log n_i$. By giving a suitable value to b_0 , we may assume that the series converges to 0.

We shall now show that the means $\sigma_n^{-1, \delta}$ do not tend to 0 as $n \rightarrow \infty$ through the sequence $\{n_i\}$. Write

$$s_{n_i}^{-1, \delta} = \sum_{\nu=0}^{n_i} b_\nu A_{n_i-\nu}^{-1, \delta} = \sum_{\nu=0}^{\frac{1}{2}n_i} + \sum_{\nu=\frac{1}{2}n_i+1}^{n_i} = I_1 + I_2.$$

Following the proof of Theorem 1, we have $I_1 = o(A_{n_i}^{-1, \delta})$, while

$$\begin{aligned} I_2 &= \sum_{\nu=\frac{1}{2}n_i+1}^{n_i} b_\nu A_{n_i-\nu}^{-1, \delta} = \frac{1}{n_i \log n_i} \sum_{\nu=0}^{\frac{1}{2}n_i-1} A_\nu^{-1, \delta} \\ &= \frac{1}{n_i \log n_i} A_{\frac{1}{2}n_i-1}^{0, \delta} \geq C_1 n_i^{-1} (\log n_i)^{\delta-1} \geq C_2 A_{n_i}^{-1, \delta}. \end{aligned}$$

Hence $\sigma_{n_i}^{-1, \delta} > C_3 > 0$, so that the series is not summable $(C, -1, \delta)$.

Another Tauberian theorem is the familiar result of HARDY [6], namely that if $\sum u_n$ is summable (C, α) , $\alpha > 0$, and has terms $O(1/n)$, then $\sum u_n$ converges. (HARDY & LITTLEWOOD have also shown that if $w(n)$ is any sequence tending monotonically to $+\infty$, then there are divergent series $\sum u_n$ which are summable (C, α) and have terms $u_n = O(w(n)/n)$; their argument is reproduced in ZYGMUND [16], Chapter VIII, Theorem (2.5).) We shall prove the following Tauberian theorem:

Theorem 2. Suppose that the series $\sum u_n$ is summable $(C, 0, \dots, 0, \beta)$, where we have k zeros and $\beta > 0$. Then if

$$u_n = o(n^{-1} \log n \log_2 n \cdots \log_k n),$$

the series $\sum u_n$ converges.

Proof. We again suppose that $k = 1$. It is enough to prove that $s_n - \sigma_n^{0, \beta} \rightarrow 0$. We have

$$\begin{aligned} s_n - \sigma_n^{0, \beta} &= \sum_{\nu=0}^n u_\nu - \frac{1}{A_n^{0, \beta}} \sum_{\nu=0}^n A_{n-\nu}^{0, \beta} u_\nu = \frac{1}{A_n^{0, \beta}} \sum_{\nu=0}^n (A_n^{0, \beta} - A_{n-\nu}^{0, \beta}) u_\nu \\ &= \frac{1}{A_n^{0, \beta}} \sum_{\nu=1}^{\lfloor \frac{1}{2}n \rfloor} + \frac{1}{A_n^{0, \beta}} \sum_{\nu=\lfloor \frac{1}{2}n \rfloor+1}^n = I_1 + I_2. \end{aligned}$$

Since

$$A_n^{0, \beta} - A_{n-\nu}^{0, \beta} = A_n^{-1, \beta} + A_{n-1}^{-1, \beta} + \cdots + A_{n-\nu+1}^{-1, \beta},$$

we have

$$\begin{aligned}
 |I_1| &\leq \frac{1}{A_n^{0,\beta}} \sum_{\nu=1}^{\lfloor \frac{1}{2}n \rfloor} (|A_n^{-1,\beta}| + |A_{n-1}^{-1,\beta}| + \cdots + |A_{n-\nu+1}^{-1,\beta}|) |u_\nu| \\
 &\leq \frac{1}{A_n^{0,\beta}} \sum_{\nu=1}^{\lfloor \frac{1}{2}n \rfloor} \nu \cdot \max_{1 \leq \nu \leq \lfloor \frac{1}{2}n \rfloor} |A_{n-\nu}^{-1,\beta}| \cdot o\left(\frac{\log \nu}{\nu}\right) \\
 &\leq O\left(\frac{n^{-1}(\log n)^{\beta-1}}{(\log n)^\beta}\right) \sum_{\nu=1}^{\lfloor \frac{1}{2}n \rfloor} o(\log \nu) = O(n^{-1}(\log n)^{-1})o(n \log n) = o(1).
 \end{aligned}$$

Next,

$$\begin{aligned}
 |I_2| &= \left| \frac{1}{A_n^{0,\beta}} \sum_{\nu=\lfloor \frac{1}{2}n \rfloor+1}^n (A_n^{0,\beta} - A_{n-\nu}^{0,\beta}) o\left(\frac{\log \nu}{\nu}\right) \right| \\
 &\leq \frac{1}{(\log n)^\beta} o\left(\frac{\log n}{n}\right) \sum_{\nu=\lfloor \frac{1}{2}n \rfloor+1}^n |A_n^{0,\beta} - A_{n-\nu}^{0,\beta}| \\
 &= o\left(\frac{1}{n(\log n)^{\beta-1}}\right) \sum_{\nu=0}^{n-1} |A_n^{0,\beta} - A_\nu^{0,\beta}| = o\left(\frac{1}{n(\log n)^{\beta-1}}\right) \left| \sum_{\nu=0}^{n-1} (A_n^{0,\beta} - A_\nu^{0,\beta}) \right|
 \end{aligned}$$

since $\beta > 0$, and the $A_n^{0,\beta}$ are increasing from some place on. Write

$$\sum_{\nu=0}^{n-1} (A_n^{0,\beta} - A_\nu^{0,\beta}) = C_n.$$

Thus to prove $I_2 = o(1)$, it is enough to show that $C_n = O(n(\log n)^{\beta-1})$. Clearly, for n large enough,

$$0 \leq C_n < \sum_{\nu=0}^n A_n^{0,\beta} - \sum_{\nu=0}^{n-1} A_\nu^{0,\beta} = nA_n^{0,\beta} - A_{n-1}^{1,\beta}.$$

Also

$$(7) \quad \sum_{n=0}^{\infty} A_n^{0,\beta} x^n = \frac{1}{1-x} \left(\log \frac{e}{1-x} \right)^\beta, \quad \sum_{n=0}^{\infty} A_n^{1,\beta} x^n = \frac{1}{(1-x)^2} \left(\log \frac{e}{1-x} \right)^\beta.$$

Differentiating both sides of the first equation,

$$\sum_{n=0}^{\infty} n A_n^{0,\beta} x^{n-1} = \frac{1}{(1-x)^2} \left(\log \frac{e}{1-x} \right)^\beta + \frac{\beta}{(1-x)^2} \left(\log \frac{e}{1-x} \right)^{\beta-1},$$

so that

$$\begin{aligned}
 \sum_{n=0}^{\infty} n A_n^{0,\beta} x^{n-1} - \sum_{n=0}^{\infty} A_n^{1,\beta} x^n &= \frac{\beta}{(1-x)^2} \left(\log \frac{e}{1-x} \right)^{\beta-1}, \\
 \sum_{n=0}^{\infty} (n A_n^{0,\beta} - A_{n-1}^{1,\beta}) x^{n-1} &= \beta \sum_{n=0}^{\infty} A_n^{1,\beta-1} x^n,
 \end{aligned}$$

and therefore $C_n = \beta A_{n-1}^{1,\beta-1} = O(n (\log n)^{\beta-1})$.

5. Trigonometric and power series. In this section we give some applications of the methods $(C, \alpha_0, \dots, \alpha_k)$ to trigonometric and power series. They generalize familiar results established for the method (C, α) . Since by now the pattern of generalizations from (C, α) to $(C, \alpha_0, \dots, \alpha_k)$ ought to be clear to the reader, in many cases we merely state generalizations, referring the reader to the places where proofs for the case $k = 0$ can be found.

Given two trigonometric series

$$S = \sum_{n=-\infty}^{+\infty} c_n e^{inx}, \quad T = \sum_{n=-\infty}^{+\infty} \gamma_n e^{inx},$$

we call the series

$$U = \sum_{n=-\infty}^{+\infty} C_n e^{inx}, \quad C_n = \sum_{p=-\infty}^{+\infty} c_p \gamma_{n-p}$$

the *formal product* of S and T , and we write $U = ST$. It is assumed that the series defining the C_n converge absolutely; then $ST = TS$.

Let $b(u)$ be defined and strictly positive for $u > u_0$. We say that $b(u)$ is a *slowly varying* function if, for every $\epsilon > 0$, $b(u)u^\epsilon$ is ultimately increasing and $b(u)u^{-\epsilon}$ ultimately decreasing as $u \rightarrow +\infty$. The definition implies that $b(du) \simeq b(u)$ as $u \rightarrow \infty$ for any fixed $d > 0$, and even uniformly if d is situated in a bounded closed interval not containing the point 0. We shall consider functions of the form $u^r \varphi(u)$ where r is a real number and $\varphi(u)$ is slowly varying. In this section by $\sum$ we shall mean $\sum_{-\infty}^{+\infty}$.

Lemma 1. Suppose that $r > 0$ and $\varphi(u)$ is slowly varying. Then if

$$1^\circ \quad c_n = o\{|n|^r \varphi(|n|)\} \text{ (as } n \rightarrow \pm\infty), \quad 2^\circ \quad \sum |\gamma_n| |n|^r \varphi(|n|)$$

converges, we have $C_n = o\{|n|^r \varphi(|n|)\}$. The result holds for $r = 0$ if $\varphi(|n|)$ is increasing.

The proof in the case $\varphi(u) \equiv 1$ is given in [16], p. 363, and the extension to the general case is not difficult. A corollary of the lemma is that if $c_n = o(A_{|n|}^{\alpha_0, \dots, \alpha_k})$, $\sum |\gamma_n| |A_n^{\alpha_0, \dots, \alpha_k}| < \infty$, then $C_n = o(A_{|n|}^{\alpha_0, \dots, \alpha_k})$, provided $(\alpha_0, \dots, \alpha_k) \succ (0, \dots, 0)$.

Lemma 2. (i) Suppose that $-1 < r \leq 0$, that $\varphi(u)$ is slowly varying and $c_n = o\{|n|^r \varphi(|n|)\}$; if $r = 0$, we assume in addition that $\varphi(u)$ is monotonically decreasing as $u \rightarrow +\infty$. Then, if $\gamma_n = O(1/|n|)$, $\sum |\gamma_n| < \infty$, we have $C_n = o\{|n|^r \varphi(|n|)\}$.

(ii) Suppose that $c_n = o\{|n|^{-1} \varphi(|n|)\}$, where $\varphi(u)$ is slowly varying and

$$\Phi(u) = \sum_{1 \leq j \leq u} \frac{\varphi(j)}{j} \quad (u \geq 1)$$

tends to $+\infty$. Then, if $\sum |\gamma_n| < \infty$ and

$$\gamma_n = O\left(\frac{\varphi(|n|)}{n\Phi(|n|)}\right),$$

we have $C_n = o\{|n|^{-1} \varphi(|n|)\}$.

Proof. Suppose, for example, that $n \rightarrow +\infty$. In the formula $C_n = \sum c_p \gamma_{n-p}$ we split the sum into two parts extended over $|p| \leq \frac{1}{2}n$ and $|p| > \frac{1}{2}n$, and denote these sums by $\sum_1$ and $\sum_2$ respectively. Clearly,

$$|\sum_2| \leq o(n^r \varphi(n)) \sum_{|p| > \frac{1}{2}n} |\gamma_{n-p}| = o(n^r \varphi(n)) \cdot O(1) = o(n^r \varphi(n)),$$

and this result holds for $-1 \leq r \leq 0$. Now, if $-1 < r \leq 0$,

$$|\sum_1| \leq \text{Max}_{|p| \geq \frac{1}{2}n} |\gamma_p| \sum_{|p| \leq \frac{1}{2}n} |c_p| = O\left(\frac{1}{n}\right) o(n^{r+1} \varphi(n))^\dagger = o(n^r \varphi(n)).$$

To handle the case of $r = -1$, we observe that $\Phi(2n) = O\{\Phi(n)\}$. For

$$\Phi(2n) = \sum_{j=1}^n \frac{\varphi(j)}{j} + \sum_{j=n+1}^{2n} \frac{\varphi(j)}{j},$$

and it is enough to note that since $\varphi(j)$ is slowly varying, the second sum on the right is $O\{\varphi(n)\}$ and the first sum exceeds

$$\sum_{\frac{1}{2}n \leq j \leq n} \frac{\varphi(j)}{j} \geq \text{const.} \cdot \varphi(n).$$

Now, in the case of $r = -1$,

$$|\sum_1| \leq O(1) \cdot \text{Max}_{|p| \geq \frac{1}{2}n} \frac{\varphi(p)}{p\Phi(p)} \cdot \sum_{|p| \leq \frac{1}{2}n} O\left(\frac{\varphi(p)}{p}\right) = O(1) \cdot \frac{\varphi(\frac{1}{2}n)}{\frac{1}{2}n\Phi(\frac{1}{2}n)} \cdot o(\Phi(n))^\dagger = o(n^{-1} \varphi(n)).$$

Corollary. Suppose that $c_n = o(A_{|n|}^{\alpha_0, \alpha_1, \dots, \alpha_k})$, where $(-1, 0, \dots, 0) \prec (\alpha_0, \alpha_1, \dots, \alpha_k) \prec (0, 0, \dots, 0)$. Then (i) if $\alpha_0 > -1$, $\gamma_n = O(1/|n|)$, $\sum |\gamma_n| < \infty$, we have $C_n = o(A_{|n|}^{\alpha_0, \alpha_1, \dots, \alpha_k})$. (ii) If $\alpha_0 = -1$,

$$\gamma_n = O(1/|n| \log |n| \cdots \log_k |n|),$$

$\sum |\gamma_n| < \infty$, we have $C_n = o(A_{|n|}^{-1, \alpha_1, \dots, \alpha_k})$.

The result follows from Lemma 2 if $\alpha_0 > -1$. If $\alpha_0 = -1$ and (for simplicity) $k = 1$, the hypothesis is $c_n = o(|n|^{-1} (\log |n|)^{\alpha_1-1})$, we may set $\varphi(u) = (\log u)^{\alpha_1-1}$, $\Phi(u)$ becomes strictly of the order $\alpha_1^{-1} (\log u)^{\alpha_1}$, and the condition $\gamma_n = O\{\varphi(|n|)/n\Phi(|n|)\}$ takes the form $\gamma_n = O(1/|n| \log |n|)$.

We shall now consider the summability of formal products of trigonometric series. Together with series $S = \sum c_n e^{inx}$ we shall also consider the conjugate series $\tilde{S} = \sum (-i \text{sign } n) c_n e^{inx}$.

Theorem 1. Suppose that $(-1, 0, \dots, 0) \prec (\alpha_0, \alpha_1, \dots, \alpha_k) \prec (0, 0, \dots, 0)$, that $c_n = o(A_{|n|}^{\alpha_0, \dots, \alpha_k})$ and that $\gamma_{|n|} = O(A_{|n|}^{\alpha_0-2, \alpha_1, \dots, \alpha_k})$, so that the series $T =$

* It is not difficult to see that if $\varphi(u)$ is slowly varying and $r > -1$, then

$$\sum_{j=1}^n j^r \varphi(j) \simeq \frac{n^{r+1}}{r+1} \varphi(n).$$

† Observe that $\varphi(p)/p$ decreases for p large enough.

$\sum \gamma_n e^{inx}$ converges absolutely to a function $\lambda(x)$. Then the formal product $U = \sum C_n e^{inx}$ of $S = \sum c_n e^{inx}$ and $T = \sum \gamma_n e^{inx}$ is uniformly equisummable $(C, \alpha_0, \dots, \alpha_k)$ with $\lambda(x) \sum c_n e^{inx}$ and $\tilde{U}$ is uniformly equisummable $(C, \alpha_0, \dots, \alpha_k)$ with $\lambda(x) \sum (-i \operatorname{sign} n) c_n e^{inx}$ in the wider sense.*

It is enough to consider the summability of U . We prove the result for a fixed point x , and note that the hypotheses are satisfied uniformly in x . Without loss in generality we may assume that $x = 0$ and that $\lambda(0) = 0$. Suppose, for example, that $k = 1$ (compare the corresponding argument in [16], p. 364, for $k = 0$). Then the partial sums S_n of $\sum C_\nu$ are

$$(1) \quad \begin{aligned} S_n &= \sum_{\nu=-n}^{+n} C_\nu = \sum_{\nu=-n}^{+n} \sum_{p=-\infty}^{+\infty} c_p \gamma_{\nu-p} = \sum_{p=-\infty}^{+\infty} c_p \sum_{\nu=-n}^{n-p} \gamma_\nu \\ &= \sum_{p=-\infty}^{+\infty} c_p (R_{-n-p} - R_{n-p+1}) = \sum_{p=-\infty}^{+\infty} c_p R_{-n-p} - \sum_{p=-\infty}^{+\infty} c_p R_{n-p+1}, \end{aligned}$$

where $R_s = \gamma_s + \gamma_{s+1} + \dots$. Since $\gamma_s = O(A_{|s|}^{\alpha_0-2, \alpha_1})$, we have $R_s = O(A_{|s|}^{\alpha_0-1, \alpha_1})$ as $s \rightarrow +\infty$, and the hypothesis $\lambda(0) = 0$ implies that $R_{-s} = -\gamma_{-s} - \gamma_{-s-1} - \dots$, so that $R_s = O(A_{|s|}^{\alpha_0-1, \alpha_1})$ as $s \rightarrow -\infty$. It follows that if $(-1, 0) \prec (\alpha_0, \alpha_1) \prec (0, 0)$, then $\sum |R_s| < \infty$, $R_s = O(1/s \log |s|)$. Hence, using the estimate $c_n = o(A_{|n|}^{\alpha_0, \alpha_1})$ and the corollary of Lemma 2, we see that $S_n = o(A_n^{\alpha_0, \alpha_1})$. By Theorem 5 of section 3, where now $k = 1$, (α_0, α_1) is replaced by $(0, 0)$ and (β_0, β_1) by $(-\alpha_0, -\alpha_1)$, we see that $\sum C_\nu$ is summable (C, α_0, α_1) to 0. This completes the proof of Theorem 1.

Theorem 2. Suppose that $c_n = o(A_{|n|}^{\alpha_0, \alpha_1, \dots, \alpha_k})$, where $(\alpha_0, \alpha_1, \dots, \alpha_k) \succ (0, 0, \dots, 0)$. Let m be the least integer such that $(\alpha_0, \alpha_1, \dots, \alpha_k) \preceq (m, 0, \dots, 0)$. Suppose that $\sum |\gamma_n| A_{|n|}^{\alpha_0+m+1, \alpha_1, \dots, \alpha_k} < \infty$, so that λ has at least $m+1$ continuous derivatives. Suppose also that at each point of a set E we have $\lambda' = \lambda'' = \dots = \lambda^{(m)} = 0$. Then in the set E the series $U = \sum C_n e^{inx}$ and $\lambda(x) \sum c_n e^{inx}$ (where U is the formal product of $S = \sum c_n e^{inx}$ and $T = \sum \gamma_n e^{inx}$) are uniformly equisummable $(C, \alpha_0, \dots, \alpha_k)$, and the series $\tilde{U}$ and $\lambda(x)\tilde{S}$ are uniformly equisummable in the wider sense.

The result is known (see [16], p. 364) when $k = 0$, and we shorten the proof by taking this special case for granted. Take any $x_0 \in E$, and suppose for simplicity that $x_0 = 0$, $\lambda(x_0) = 0$. Consider first the case $\lambda(x) = (1 - e^{ix})^{m+1}$. If $s_n^m = s_n^{m, 0, \dots, 0}$ are the (C, m) sums of U at $x = 0$, then (cf. [16], p. 365, formula (9.9))

$$s_n^m = c_n + (-1)^{m-1} c_{-n-m-1} + O(n^{m-1}), \quad \sigma_n^m = \frac{s_n^m}{A_n^m} = o\left(\frac{A_n^{\alpha_0, \dots, \alpha_k}}{A_n^m}\right) + O\left(\frac{1}{m}\right).$$

If $(\alpha_0, \alpha_1, \dots, \alpha_k) = (0, 0, \dots, 0)$, at $x = 0$ U is clearly summable $(C, \alpha_0, \dots, \alpha_k)$ to sum 0, and an application of Theorem 5 of section 3 shows the same if $(\alpha_0, \alpha_1, \dots, \alpha_k) \prec (m, 0, \dots, 0)$.

Consider now the case of general λ . Since, by hypothesis, $\lambda(0) = \lambda'(0) = \dots =$

* Two series are said to be equisummable if their difference (i) is summable, (ii) has sum 0; if condition (ii) is omitted, the two series are called equisummable in the wider sense.

$\lambda^{(m)}(0)$, each of the functions $\lambda_h(x) = \lambda(x)(1 - e^{ix})^{-h}$ completed by continuity at $x = 0$, is continuous, provided $h = 0, 1, \dots, m+1$. If we set $\lambda_h(x) \sim \sum \gamma_h^h e^{i\nu x}$, we have (compare the corresponding argument in [16], pp. 365–366) that

$$(2) \quad \sum_{n \neq 0} |\gamma_n^{m+1}| |A_{|n|}^{\alpha_0, \alpha_1, \dots, \alpha_k}| \leq \text{const.} \sum_{n=1}^{\infty} (|\gamma_n| + |\gamma_{-n}|) |A_{|n|}^{\alpha_0+m+1, \alpha_1, \dots, \alpha_k}|.$$

Now

$$\lambda(x) = \sum \gamma_n e^{inx} = (1 - e^{ix})^{m+1} \sum \gamma_n^{m+1} e^{inx}.$$

The formal product ST is obtained by first multiplying S by $T_1 = \sum \gamma_n^{m+1} e^{inx}$ and then the result by $T_2 = (1 - e^{ix})^{m+1}$. In view of (2) and Lemma 1, ST_1 has coefficients $o(A_{|n|}^{\alpha_0, \alpha_1, \dots, \alpha_k})$, and so, by the special case dealt with, $(ST_1)T_2$ is summable $(C, \alpha_0, \alpha_1, \dots, \alpha_k)$ to 0 at $x = 0$. Hence $ST = (ST_1)T_2$ is summable $(C, \alpha_0, \dots, \alpha_k)$ to 0 at $x = 0$. This completes the proof for the series $U = ST$, and a similar argument is valid for $\tilde{U}$.

Consider a trigonometric series $S = \sum c_n e^{inx}$. Suppose that there is an integer $m > 0$ such that the series when integrated termwise m times converges uniformly to a continuous function $F(x)$:

$$(3) \quad F(x) = c_0 \frac{x^m}{m!} + A_{m-1} x^{m-1} + \dots + A_1 x + A_0 + \sum_{-\infty}^{\infty} \frac{c_n e^{inx}}{(in)^m}$$

(in what follows it would be enough to assume that the periodic part on the right is a Fourier series). Let $D_n(x)$ and $\tilde{D}_n(x)$ be respectively the Dirichlet kernel and the conjugate Dirichlet kernel.

Theorem 3. Suppose that $S = \sum c_n e^{inx}$ has coefficients $o(A_{|n|}^{\alpha_0, \dots, \alpha_k})$, where $(\alpha_0, \alpha_1, \dots, \alpha_k) \succ (-1, 0, \dots, 0)$. Let m be so large that the series (3) converges uniformly. If $\lambda(x)$ is a periodic function with sufficiently many derivatives, equal to 1 for $a' \leq x \leq b'$, equal to 0 outside (a, b) , where $a < a' < b' < b$, $b - a < 2\pi$, then the two sequences

$$\begin{aligned} \sum_{j=-n}^n c_j e^{ijx} - \frac{(-1)^m}{\pi} \int_a^b F(t) \lambda(t) \frac{d^m}{dt^m} D_n(x-t) dt, \\ \sum_{j=-n}^n (-i \operatorname{sign} j) c_j e^{ijx} - \frac{(-1)^m}{\pi} \int_a^b F(t) \lambda(t) \frac{d^m}{dt^m} \tilde{D}_n(x-t) dt \end{aligned}$$

are uniformly limitable $(C, \alpha_0, \dots, \alpha_k)$ in (a', b') , the first limit being 0.

The result is well known for $k = 0$. If we use Theorems 1 and 2, the proof follows the same pattern for $k > 0$ and we omit it.

Theorem 4. Suppose that a power series $\sum_0^{\infty} c_n z^n$ converges for $|z| < 1$ and that its sum $F(z)$ is continuable across an arc $a < \arg z < b$ of the unit circle. Then, if $c_n = o(A_n^{\alpha_0, \dots, \alpha_k})$, $(\alpha_0, \alpha_1, \dots, \alpha_k) \succ (-1, 0, \dots, 0)$, the series is uniformly summable $(C, \alpha_0, \dots, \alpha_k)$ on each arc $a + \epsilon \leq \arg z \leq b - \epsilon$ of the unit circle.

This is a corollary of Theorem 3.

6. Bibliographical remarks. § 1. Historical comments concerning the Nörlund-Voronoi methods are given in HARDY [6], p. 91. The book also contains

an exposition of the theory (see Chapter IV, §1), but since it only considers the case $\alpha_0 > 0$, $\alpha_n \geq 0$ ($n = 1, 2, \dots$), which roughly corresponds to the method (C, α) , $\alpha \geq 0$, we reproduce a slightly more general treatment by ZYGMUND [13] which suffices for our purposes. The results about the multiplication of series are also taken from the latter paper. The Inclusion Theorems I, II and Theorems 6 and 7 are essentially in M. RIESZ [11]. We give proofs here since the results are needed under somewhat more general conditions than RIESZ's.

§2. The asymptotic estimates of the numbers $A_n^{\alpha_0, \alpha_1, \dots, \alpha_k}$ were given by FABER [5]; see also ZYGMUND [16], p. 192.

§3. The proofs given in this section apply a certain device which for the first time seems to have been used systematically by ZYGMUND [14].

§5. Theorems about summability (C, α) of formal products were first proved by ZYGMUND [15] and are reproduced in ZYGMUND [16] with simpler proofs. Our arguments here follow the presentation of the latter book. Theorem 4 is stated without proof in ZYGMUND [12].

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