On Perturbations of Periodic Solutions

YASUTAKA SIBUYA¹

Communicated by A. ERDÉLYI

INTRODUCTION

Studying the problem of perturbation of a system of linear ordinary differential equations, E. A. CODDINGTON & N. LEVINSON gave criteria for the existence of periodic solutions in the case when the system of first variation has nontrivial periodic solutions [3]. Recently the author of the present paper obtained a method of constructing periodic solutions [9]. By using this method we shall give similar criteria in the case of perturbation of a nonlinear system under the situation similar to theirs.² We shall discuss this problem in Chapter I. In Chapter II, we shall give criteria for the stability of the periodic solutions which will be constructed in Chapter I. Our criteria for the stability are quite similar to those given by I. I. BLEKMAN [1, 2] and J. A. NOHEL [7].

I. EXISTENCE OF PERIODIC SOLUTIONS

1. Assumptions. We shall consider a system of nonlinear ordinary differential equations:

$$(1.1) \quad \frac{dy_i}{dt} = f_i(t, y, \mu) \quad (j = 1, 2, \dots, n),$$

where t is a real variable, μ a real parameter and the f_i are functions continuous in t, y, μ , sufficiently smooth with respect to y, μ , and periodic in t of period 2π .

Let us suppose that the unperturbed system

$$(1.2) \quad \frac{dy_i}{dt} = f_i(t, y, 0) \quad (j = 1, 2, \dots, n)$$

has a periodic solution of period 2π :

$$(1.3) \quad y_i = \phi_i(t) \quad (j = 1, 2, \dots, n).$$

¹ The author is a Fulbright grantee. This work is mainly based on discussions with Professor E. A. CODDINGTON at the University of California, Los Angeles, and the author's technical report at the Mathematics Research Center, U.S. Army, University of Wisconsin [10]. This paper was prepared at the Massachusetts Institute of Technology under the sponsorship of the Office of Naval Research.

² The study in this direction is not new. For example, see D. C. LEWIS [5], W. S. LOUD [6].

The main purpose of this chapter is to give a sufficient condition in order that the equations (1.1) have a periodic solution of period 2π which approaches the periodic solution (1.3) of the equations (1.2) as μ tends to 0.

By means of a change of variables

$$(1.4) \quad y_j = x_j + \phi_j(t) \quad (j = 1, 2, \dots, n),$$

we can reduce the system (1.1) to another system

$$(1.5) \quad \frac{dx_j}{dt} = g_j(t, x, \mu) \quad (j = 1, 2, \dots, n)$$

which satisfies the conditions

$$(1.6) \quad g_j(t, 0, 0) = 0 \quad (j = 1, 2, \dots, n).$$

The solution $x_j = 0$ ($j = 1, 2, \dots, n$) of the unperturbed system

$$\frac{dx_j}{dt} = g_j(t, x, 0) \quad (j = 1, 2, \dots, n)$$

corresponds to the solution (1.3) of the system (1.2). Therefore we shall investigate those periodic solutions of (1.5) which approach $x = 0$ as μ tends to zero.

Bearing these facts in mind, we shall suppose that the right-hand members of the system (1.1) satisfy the conditions

$$(1.7) \quad f_j(t, 0, 0) = 0 \quad (j = 1, 2, \dots, n).$$

Furthermore the f_j are assumed to be given in the form of expansion in powers of $y_1, y_2, \dots, y_n$, and μ with coefficients continuous and periodic in t of period 2π . The series representing the f_j are uniformly convergent for $|y_j| \leq \delta$, ($j = 1, 2, \dots, n$), $|\mu| \leq \rho$, and $-\infty < t < +\infty$, where δ and ρ are positive constants. Under our assumptions, the f_j are given in the following form:

$$(1.8) \quad f_j(t, y, \mu) = F_j(t, y) + \mu G_j(t, y, \mu)$$

with

$$(1.9) \quad F_j(t, y) = \sum_{k=1}^n a_{jk} y_k + \sum_{h=2}^{\infty} b_{jh}(t, y)$$

and

$$(1.10) \quad G_j(t, y, \mu) = \sum_{h=0}^{\infty} \mu^h G_{jh}(t, y),$$

where b_{jh} is a homogeneous polynomial of degree h in y with coefficients periodic in t of period 2π , whereas the G_{jh} are power series in y with periodic coefficients. The matrix $A = (a_{jk})$ is assumed to be constant and in the Jordan canonical form. Then we can write the matrix A in the following form:

$$A = \begin{bmatrix} \lambda_1 & & & & \\ \delta_2 & \lambda_2 & & & \\ & \delta_3 & \lambda_3 & & \\ & & \ddots & \ddots & \\ & & & \delta_n & \lambda_n \end{bmatrix},$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the characteristic values of the matrix A , while $\delta_2, \delta_3, \dots, \delta_n$ are equal to 1 or 0, and $\lambda_{i-1} = \lambda_i$ if $\delta_i = 1$. Hence

$$(1.9') \quad F_i(t, y) = \lambda_i y_i + \delta_i y_{i-1} + \sum_{h=2}^{\infty} b_{ih}(t, y).$$

Let us suppose that

$$(1.11) \quad \begin{cases} \lambda_i \equiv 0 \pmod{i} & (j = 1, 2, \dots, r), \\ \lambda_i \not\equiv 0 \pmod{i} & (j \neq 1, 2, \dots, r), \end{cases}$$

where $i = \sqrt{-1}$. We assume $r > 0$; in other words, we suppose that there is at least one index j such that $\lambda_j \equiv 0 \pmod{i}$. The system of linear equations

$$(1.12) \quad \frac{dy_i}{dt} = \lambda_i y_i + \delta_i y_{i-1} \quad (j = 1, 2, \dots, n)$$

is the system of first variation of (1.2) with respect to $y \equiv 0$. This system has nontrivial periodic solutions under our assumptions.

Our main tool is the following lemma:

Lemma 1. *Let $C_1, C_2, \dots, C_r$ be arbitrary constants. For any set $p = (p_1, \dots, p_r)$, where the p_i are nonnegative integers, let C^p denote $C_1^{p_1} C_2^{p_2} \dots C_r^{p_r}$, and $|p| = p_1 + p_2 + \dots + p_r$. Then there exist power series in C and μ with periodic coefficients:*

$$(1.13) \quad V_i(t, C, \mu) = \sum_{|p|+h \geq 1} V_{hp}^i(t) C^p \mu^h \quad (j = 1, 2, \dots, n)$$

and power series with constant coefficients:

$$(1.14) \quad \Gamma_i(C, \mu) = \sum_{|p|+h \geq 1} \Gamma_{hp}^i C^p \mu^h \quad (j = 1, 2, \dots, r)$$

which satisfy the equations

$$(1.15) \quad \begin{cases} dV_i/dt = f_i(t, V, \mu) + \Gamma_i e^{\lambda_i t} & (j = 1, 2, \dots, r), \\ dV_i/dt = f_i(t, V, \mu) & (j \neq 1, 2, \dots, r), \end{cases}$$

and the conditions

$$(1.16) \quad C_i = \frac{1}{2\pi} \int_0^{2\pi} V_i(t, C, \mu) e^{-\lambda_i t} dt \quad (j = 1, 2, \dots, r).$$

For the proof, see, Y. SIBUYA [9].

Periodic solutions of the system (1.1) are constructed by substituting in (1.13) the roots of the equations

$$(1.17) \quad \Gamma_i(C, \mu) = 0 \quad (j = 1, 2, \dots, r).$$

The relations between the Γ_i and the V_i are clearly given by the identities

$$(1.18) \quad \Gamma_i(C, \mu) = -\frac{1}{2\pi} \int_0^{2\pi} [f_i(t, V, \mu) - \lambda_i V_i] e^{-\lambda_i t} dt \quad (j = 1, 2, \dots, r).$$

2. Coefficients V_{hp}^i . In this section, we shall calculate the coefficients V_{hp}^i for $|p| + h = 1$.

Lemma 2. Let $e_k = (\delta_{1k}, \delta_{2k}, \dots, \delta_{rk})$ ($k = 1, 2, \dots, r$), where $\delta_{jk} = 1$ if $j = k$, and $\delta_{jk} = 0$ otherwise. Then we get

$$(2.1) \quad V_{0e_k}^i = \delta_{ik} e^{\lambda_i t} \quad (j = 1, 2, \dots, n; k = 1, 2, \dots, r).$$

Proof. Since the V_i satisfy the equation (1.15), we have

$$(2.2) \quad dV_{0e_k}^i/dt = \lambda_i V_{0e_k}^i + \delta_i V_{0e_k}^{i-1} + \Gamma_{0e_k}^i e^{\lambda_i t} \quad (j = 1, 2, \dots, r),$$

and

$$(2.3) \quad dV_{0e_k}^i/dt = \lambda_i V_{0e_k}^i + \delta_i V_{0e_k}^{i-1} \quad (j \neq 1, 2, \dots, r).$$

On the other hand, the conditions (1.16) imply the conditions

$$(2.4) \quad \frac{1}{2\pi} \int_0^{2\pi} V_{0e_k}^i(t) e^{-\lambda_i t} dt = \delta_{ik} \quad (j = 1, 2, \dots, r).$$

The identities (1.18) imply

$$(2.5) \quad \Gamma_{0e_k}^i = -\frac{1}{2\pi} \int_0^{2\pi} \int_i \delta_i V_{0e_k}^{i-1}(t) e^{-\lambda_i t} dt \quad (j = 1, 2, \dots, r).$$

Hence we get (2.1).

It should be remarked that (2.1) and (2.5) give us formulas

$$(2.6) \quad \Gamma_{0e_k}^i = -\delta_{j-1,k} \delta_j \quad (j, k = 1, 2, \dots, r).$$

Let us put

$$(2.7) \quad B_j(t) = V_{i_0}^j(t) \quad (j = 1, 2, \dots, n).$$

Then by similar arguments we can prove the following lemma:

Lemma 3. The periodic functions $B_1, B_2, \dots, B_n$ satisfy the equations

$$(2.8) \quad dB_i/dt = \lambda_i B_i + \delta_i B_{i-1} + G_i(t, 0, 0) + \beta_i e^{\lambda_i t} \quad (j = 1, 2, \dots, r),$$

and

$$(2.9) \quad dB_i/dt = \lambda_i B_i + \delta_i B_{i-1} \quad (j \neq 1, 2, \dots, r),$$

with conditions

$$(2.10) \quad \int_0^{2\pi} B_i(t) e^{-\lambda_i t} dt = 0 \quad (j = 1, 2, \dots, r),$$

where

$$(2.11) \quad \beta_i = -\frac{1}{2\pi} \int_0^{2\pi} G_i(t, 0, 0) e^{-\lambda_i t} dt \quad (j = 1, 2, \dots, r).$$

3. Main theorem. Let

$$(3.1) \quad G_{i0}(t, y) = H_i(t) + L_i(t, y) + \dots,$$

where the dots stand for the terms of degree not less than two with respect to y , the L_i are linear homogeneous in y , and

$$(3.2) \quad H_i(t) = G_i(t, 0, 0).$$

Letting $K_i(t) = G_{i1}(t, 0)$, we shall put

$$(3.3) \quad D_i = b_{i2}(t, y) + L_i(t, y) + K_i(t) \quad (j = 1, 2, \dots, r).$$

On the other hand, we shall put

$$(3.4) \quad U_i(t, C) = \begin{cases} C_i e^{\lambda_i t} + B_i(t) & (j = 1, 2, \dots, r), \\ B_i(t) & (j \neq 1, 2, \dots, r), \end{cases}$$

and

$$(3.5) \quad E_i(C) = \int_0^{2\pi} D_i(t, U(t, C)) e^{-\lambda_i t} dt \quad (j = 1, 2, \dots, r).$$

Finally letting

$$(3.6) \quad \delta_{i+1} = \begin{cases} 0 & (j = j_1, j_2, \dots, j_v), \\ 1 & (j \neq j_1, j_2, \dots, j_v), \end{cases}$$

where $j \leq r$, $1 \leq j_1 < j_2 < \dots < j_v = r$, we shall suppose that

$$(3.7) \quad \beta_i = -\frac{1}{2\pi} \int_0^{2\pi} G_i(t, 0, 0) e^{-\lambda_i t} dt = 0 \quad (j = 1, 1 + j_1, \dots, 1 + j_{v-1}).$$

D. C. LEWIS [5] and W. S. LOUD assumed similar conditions. In the case of CODDINGTON & LEVINSON [3], the periodic solution (1.3) of the unperturbed system (1.2) contained a finite number of arbitrary constants. Since we have made the change of variables (1.4), the right-hand members of (1.5) depend on those arbitrary constants and μ . Therefore in their case the left-hand members of (3.7) contained the arbitrary constants, and these constants were determined in such a manner that the conditions (3.7) were satisfied. In general we may put $\mu = \sigma^2$ and regard σ as a new parameter. Then (3.7) are satisfied.

Then we can prove the following theorem:

Theorem 1. *If the equations*

$$(3.8) \quad E_i(C) = 0 \quad (j = 1, 1 + j_1, \dots, 1 + j_{r-1})$$

have a solution

$$(3.9) \quad C_i = \begin{cases} \gamma_i & (j = j_1, j_2, \dots, j_r), \\ \beta_{i+1} & (j \neq j_1, j_2, \dots, j_r), \end{cases}$$

and the functional determinant of (3.8) with respect to $C_{i_1}, C_{i_2}, \dots, C_{i_r}$ is different from zero at (3.9), then there exists a periodic solution of (1.1) which can be represented by power series in μ and tends to $y = 0$ as μ tends to zero.

Proof. Letting $C_i = \mu\gamma_i$ ($j = 1, 2, \dots, r$), we shall put

$$(3.10) \quad \Gamma_i(\mu\gamma, \mu) = \sum_{h=1}^{\infty} \mu^h \hat{\Gamma}_{ih}(\gamma).$$

Since

$$V_i(t, \mu\gamma, \mu) = \mu U_i(t, \gamma) + O(\mu^2),$$

(1.18) and (2.6) imply that

$$(3.11) \quad \hat{\Gamma}_{i1}(\gamma) = -\delta_i \gamma_{i-1} + \beta_i,$$

and

$$(3.12) \quad \hat{\Gamma}_{i2}(\gamma) = -\frac{1}{2\pi} E_i(\gamma).$$

Since we assume the conditions (3.7), we can derive our theorem immediately.

4. Remarks. The criterion of CODDINGTON & LEVINSON is derived in the following manner:

Suppose that the unperturbed system (1.2) is linear. Then

$$(4.1) \quad f_i(t, y, \mu) = \lambda_i y_i + \delta_i y_{i-1} + \mu G_{i0}(t, y) + O(\mu^2).$$

On the other hand it is easily shown that

$$(4.2) \quad V_i(t, C, \mu) = \begin{cases} C_i e^{\lambda_i t} + O(\mu) & (j = 1, 2, \dots, r), \\ O(\mu) & (j \neq 1, 2, \dots, r). \end{cases}$$

Put

$$(4.3) \quad \hat{U}_i(t, C) = \begin{cases} C_i e^{\lambda_i t} & (j = 1, 2, \dots, r), \\ 0 & (j \neq 1, 2, \dots, r). \end{cases}$$

We shall retain the assumptions (3.7). The identities (1.18) imply that

$$(4.4) \quad \Gamma_i(C, \mu) = -\delta_i C_{i-1} + \mu \left\{ \beta_i - \frac{1}{2\pi} \int_0^{2\pi} L_i(t, \hat{U}(t, C)) e^{-\lambda_i t} dt + \dots \right\},$$

where the dots stands for terms of degree not less than two with respect to C , or not less than one with respect to μ . By the use of (3.7), we can easily see that $\delta_i = 0$ implies $\beta_i = 0$. Put

$$(4.5) \quad M_i(C) = \int_0^{2\pi} L_i(t, \hat{U}(t, C)) e^{-\lambda_i t} dt.$$

The M_i are linear and homogeneous in C . Let

$$(4.6) \quad \begin{bmatrix} M_1(C) \\ M_2(C) \\ \vdots \\ M_r(C) \end{bmatrix} = Q \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_r \end{bmatrix},$$

where Q is an r by r matrix:

$$(4.7) \quad Q = \begin{bmatrix} q_{11} & \cdots & q_{1r} \\ \cdots & \cdots & \cdots \\ q_{r1} & \cdots & q_{rr} \end{bmatrix}.$$

Let $\hat{Q}$ be a v by v submatrix of Q whose elements are those q_{ik} 's such that $j = 1, 1 + j_1, \dots, 1 + j_{v-1}$ and $k = j_1, j_2, \dots, j_v$. Then if $\hat{Q}$ is nonsingular, we can obtain the same conclusion as given in our Theorem 1. This is exactly the same criterion as given by CODDINGTON & LEVINSON [3].

II. STABILITY OF PERIODIC SOLUTIONS

5. Preliminaries. Usually, in order to derive criteria for the stability of perturbed periodic solutions, we have to study linear systems with coefficients periodic in the independent variable and containing a small parameter. Several years ago the author of the present paper obtained a reduction theorem for such linear systems [8]. This result will now be stated.

Consider a system of linear ordinary differential equations of the following type:

$$(5.1) \quad \frac{dy}{dt} = A(t, \mu)y,$$

where y is an n -dimensional vector and A an n by n matrix of the form

$$(5.2) \quad A(t, \mu) = A_0 + \sum_{k=1}^{\infty} \mu^k A^{(k)}(t).$$

The series (5.2) is uniformly convergent for $|\mu| \leq \delta$, $-\infty < t < +\infty$, where δ is a positive constant. The coefficients $A^{(k)}$ are assumed to be continuous and periodic of period 2π in t , whereas the matrix A_0 is constant and in the Jordan canonical form. Let $\lambda_1, \lambda_2, \dots, \lambda_s$ be distinct characteristic values of the

matrix A_0 and $n_1, n_2, \dots, n_s$ be their respective multiplicities. Then A_0 can be written in the form

$$(5.3) \quad A_0 = \begin{bmatrix} \mathring{A}_1 & & & \\ & \mathring{A}_2 & & \\ & & \ddots & \\ & & & \mathring{A}_s \end{bmatrix},$$

where

$$(5.4) \quad \mathring{A}_j = \lambda_j E_{n_j} + D_j \quad (j = 1, 2, \dots, s),$$

E_{n_j} being the unit-matrix of order n_j and

$$(5.5) \quad D_j = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ d_{j2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & d_{j3} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & d_{j4} & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & d_{jn_j} & 0 \end{bmatrix}.$$

Let $P(t, \mu)$ be an n by n matrix of the form

$$(5.6) \quad P(t, \mu) = E + \sum_{k=1}^{\infty} \mu^k P^{(k)}(t),$$

where E is the unit-matrix of order n and $P^{(k)}$ periodic in t of period 2π .

Lemma 4. *If $\lambda_1, \lambda_2, \dots, \lambda_s$ satisfy the conditions*

$$(5.7) \quad \lambda_j \not\equiv \lambda_k \pmod{i} \quad (j \neq k),$$

then there exists a matrix P of the form (5.6) such that by the transformation

$$(5.8) \quad y = P(t, \mu)z$$

the system (5.1) is reduced to a system

$$(5.9) \quad \frac{dz}{dt} = A(\mu)z$$

whose matrix $A(\mu)$ is independent of t and has the following form

$$(5.10) \quad A(\mu) = \begin{bmatrix} A_1(\mu) & & & \\ & A_2(\mu) & & \\ & & \ddots & \\ & & & A_s(\mu) \end{bmatrix},$$

where $A_i(\mu)$ is an n_i by n_i matrix of the type

$$(5.11) \quad A_i(\mu) = \mathring{A}_i + \sum_{k=1}^{\infty} \mu^k B_i^{(k)},$$

the $B_i^{(k)}$ being constant matrices.

For the proof, see, Y. SIBUYA [8]. For further reductions, see, M. HUKUHARA [4].

In order to derive criteria for the stability from this result, we shall show how to determine the matrices $P^{(k)}$ and $B^{(k)}$, where

$$(5.12) \quad A(\mu) = A_0 + \sum_{k=1}^{\infty} \mu^k B^{(k)}.$$

The matrices P and $A(\mu)$ must satisfy the equation

$$(5.13) \quad \frac{dP}{dt} = A(t, \mu)P - PA(\mu).$$

From this equation we can derive the sequence of equations

$$(5.14) \quad \frac{dP^{(k)}}{dt} = A_0 P^{(k)} - P^{(k)} A_0 - B^{(k)} + H^{(k)}(t),$$

where

$$(5.15) \quad H^{(k)} = \sum_{p+q=k} \{A^{(p)}(t)P^{(q)}(t) - P^{(q)}(t)B^{(p)}\} + A^{(k)}(t).$$

In particular,

$$(5.16) \quad \frac{dP^{(1)}}{dt} = A_0 P^{(1)} - P^{(1)} A_0 - B^{(1)} + A^{(1)}(t).$$

According to the form (5.3) of the matrix A_0 , we can write the $P^{(k)}$, $A^{(k)}$, and $H^{(k)}$ in the following form:

$$\begin{aligned} P^{(k)} &= \begin{bmatrix} P_{11}^{(k)} & \cdots & P_{1s}^{(k)} \\ \vdots & \ddots & \vdots \\ P_{s1}^{(k)} & \cdots & P_{ss}^{(k)} \end{bmatrix}, \\ A^{(k)} &= \begin{bmatrix} A_{11}^{(k)} & \cdots & A_{1s}^{(k)} \\ \vdots & \ddots & \vdots \\ A_{s1}^{(k)} & \cdots & A_{ss}^{(k)} \end{bmatrix}, \\ H^{(k)} &= \begin{bmatrix} H_{11}^{(k)} & \cdots & H_{1s}^{(k)} \\ \vdots & \ddots & \vdots \\ H_{s1}^{(k)} & \cdots & H_{ss}^{(k)} \end{bmatrix}, \end{aligned}$$

where the $P_{ih}^{(k)}$, $A_{ih}^{(k)}$, and $H_{ih}^{(k)}$ are n_i by n_h matrices. The equations (5.14) can be written in the form:

$$(5.17) \quad \frac{dP_{ih}^{(k)}}{dt} = \mathring{A}_i P_{ih}^{(k)} - P_{ih}^{(k)} \mathring{A}_h + H_{ih}^{(k)} \quad (j \neq h)$$

and

$$(5.17') \quad \frac{dP_{ii}^{(k)}}{dt} = \mathring{A}_i P_{ii}^{(k)} - P_{ii}^{(k)} \mathring{A}_i - B_i^{(k)} + H_{ii}^{(k)} \quad (j = 1, 2, \dots, s).$$

The $P_{ih}^{(k)}$ ($j \neq h$) can be determined uniquely under the assumption (5.7), whereas the $P_{ii}^{(k)}$ can be given by the formulas

$$(5.18) \quad P_{ii}^{(k)}(t) = \exp(\mathring{A}_i t) \int_0^t \exp(-\mathring{A}_i \tau) \{H_{ii}^{(k)}(\tau) - B_i^{(k)}\} \\ \cdot \exp(\mathring{A}_i \tau) d\tau \exp(-\mathring{A}_i t) \quad (j = 1, 2, \dots, s),$$

with

$$(5.19) \quad \int_0^{2\pi} \exp(-\mathring{A}_i \tau) B_i^{(k)} \exp(\mathring{A}_i \tau) d\tau = \int_0^{2\pi} \exp(-\mathring{A}_i \tau) H_{ii}^{(k)}(\tau) \\ \cdot \exp(\mathring{A}_i \tau) d\tau \quad (j = 1, 2, \dots, s).$$

The $B_i^{(k)}$ are determined by (5.19). In particular,

$$(5.20) \quad \int_0^{2\pi} \exp(-\mathring{A}_i \tau) B_i^{(1)} \exp(\mathring{A}_i \tau) d\tau = \int_0^{2\pi} \exp(-\mathring{A}_i \tau) A_{ii}^{(1)}(\tau) \\ \cdot \exp(\mathring{A}_i \tau) d\tau \quad (j = 1, 2, \dots, s).$$

The characteristic exponents of the system (5.1) are given by the equation

$$(5.21) \quad J_1(X, \mu) J_2(X, \mu) \cdots J_s(X, \mu) = 0,$$

where

$$(5.22) \quad J_h(X, \mu) = \det(XE_{n_h} - A_h(\mu)) \quad (h = 1, 2, \dots, s),$$

X being unknown.

Suppose that

$$(5.23) \quad \operatorname{Re} \lambda_i = 0 \quad (j = 1, 2, \dots, N)$$

and

$$(5.24) \quad \operatorname{Re} \lambda_i < 0 \quad (j \neq 1, 2, \dots, N).$$

Lemma 5. Suppose that the $\mathring{A}_i$ ($j = 1, 2, \dots, N$) have the diagonal form. If all the characteristic values of the matrices

$$(5.25) \quad \int_0^{2\pi} A_{ii}^{(1)}(t) dt \quad (j = 1, 2, \dots, N)$$

have negative real parts, then all the characteristic exponents of the system (5.1) have negative real parts for $\mu > 0$.

Proof. Under the assumptions of the lemma we have

$$(5.26) \quad B_i^{(1)} = \frac{1}{2\pi} \int_0^{2\pi} A_{ii}^{(1)}(t) dt \quad (j = 1, 2, \dots, N).$$

Therefore the conclusion of the lemma is evident.

Lemma 6. Suppose that those components of the matrices

$$(5.27) \quad \int_0^{2\pi} \exp(-\mathring{A}_i t) A_{ij}^{(1)}(t) \exp(\mathring{A}_i t) dt \quad (j = 1, 2, \dots, N)$$

which are above the main diagonal are equal to zero, and that all the characteristic values of the matrices (5.27) have negative real parts. Then all the characteristic exponents of the system (5.1) have negative real parts for $\mu > 0$.

Proof. Because of the formulas (5.20), the $B_i^{(1)}$ ($j = 1, 2, \dots, N$) have the same properties as those of the matrices (5.27). Therefore the conclusion of the lemma is evident.

6. Criteria for the stability. Let us revert to the nonlinear system (1.1). We shall write the system in the form

$$(1.1') \quad \frac{dy}{dt} = Ay + \sum_{h=2}^{\infty} b_h(t, y) + \mu \sum_{h=0}^{\infty} \mu^h G_h(t, y),$$

where A is the matrix (a_{ik}) , and, the b_h and G_h are n -dimensional vectors whose components are the b_{ih} and the G_{ih} respectively. We shall retain the assumption that the matrix A has the Jordan canonical form. Let

$$(6.1) \quad y = q(t, \mu) = \sum_{h=1}^{\infty} q_h(t) \mu^h$$

be the periodic solution of (1.1) which was constructed in section 3. It should be remarked that the components of the vector $q_1(t)$ are the $U_i(t, \mathring{\gamma})$ defined by (3.4), where the $\mathring{\gamma}_i$ are given in Theorem 1. The system of first variation of (1.1) with respect to q is

$$(6.2) \quad \frac{dz}{dt} = \left[A + \sum_{h=2}^{\infty} b_{hv}(t, q(t, \mu)) + \mu \sum_{h=0}^{\infty} \mu^h G_{hv}(t, q(t, \mu)) \right] z.$$

The matrix A may not satisfy the conditions (5.7). In such a case, there exists a constant diagonal matrix T such that (i) $AT = TA$, (ii) $\exp(itT)$ is periodic of period 2π in t , and (iii) $A_0 = A - T$ has the Jordan canonical form and satisfies the conditions (5.7). By the transformation

$$(6.3) \quad z = \exp(itT)u$$

the system (6.2) is reduced to

$$(6.4) \quad \frac{du}{dt} = A(t, \mu)u,$$

where

$$(6.5) \quad A(t, \mu) = A_0 + \mu A^{(1)}(t) + O(\mu^2)$$

with

$$(6.6) \quad A^{(1)}(t) = \exp(-tT) \{b_{2\nu}(t, U(t, \hat{y})) + L(t)\} \exp(tT),$$

$L(t)$ being the matrix of the linear forms $L_1(t, y), L_2(t, y), \dots, L_n(t, y)$ in the formulas (3.1).

Then by applying lemmas 5 and 6, we can prove the following theorem:

Theorem 2. *If the matrix $A^{(1)}(t)$ defined by (6.6) satisfies the conditions given in lemma 5 or 6, the periodic solution (6.1) of the system (1.1) is stable for $\mu > 0$.*

Remarks. (i) $\exp(-tT) L(t) \exp(tT)$ is the matrix which was relevant to the criteria of BLEKMAN and NOHEL. (ii) Let the $E_i(C)$ be the functions defined by (3.5), and the E_{ik} be the value of $\partial E_i / \partial C_k$ at (3.9). Then the matrix (E_{ik}) , $j, k = 1, 2, \dots, r$, is a submatrix of the matrix

$$\int_0^{2\pi} A^{(1)}(t) dt.$$

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Massachusetts Institute of Technology
Cambridge, Massachusetts
and
Ochanomizu University
Tokyo