Properties of Solutions of a Riccati Matrix Differential Equation

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1. Introduction. Recently the author [16] extended to Riccati matrix differential equations certain results of REDHEFFER [9; 10] on scalar Riccati equations. In a subsequent paper, REDHEFFER [11] pointed out that the conclusions of [16] did not provide a complete generalization of his earlier results for the scalar case, and proceeded to obtain a full generalization with the ingenious aid of algebraic results on “*-products” of matrices. Moreover, REDHEFFER answered a question raised by the author [16; p. 229] on a necessary and sufficient condition for solutions to possess a certain dissipative property.

In the present paper it is shown that the analytic methods of REID [16] yield properties of solutions of the type presented by REDHEFFER [11]. Section 2 is devoted to a comparison of the earlier results of the author [16] and REDHEFFER [11] and with statements of Theorems 2.1 and 2.2, which present the principal results of the paper. The result of Theorem 2.1 is essentially that of Theorem III of REDHEFFER [11]. Conclusion (i) of Theorem 2.2 corresponds to the first conclusion of Theorem XII of REDHEFFER [11], and in statement it is a direct extension and improvement of the result of Theorem 4.1 of REID [16]. The proof of this result involves an extension of a quadratic form problem which was considered a number of years ago by various authors (see ALBERT [1], DINES [3], FINSLER [4], HESTENES & McSHANE [6], and REID [13]); because of the intrinsic interest of the hermitian form problem, there are presented in Section 6 certain general results which include as a special instance the needed result of Theorem 5.1. Conclusion (ii) of Theorem 2.2 presents a corrected form of the final conclusion of Theorem XII of REDHEFFER [11], which is in error. It is to be noted that certain of the results appearing in the lemmas of Section 3 have been cast in a form to establish the existence of solutions, rather than to postulate their existence as in REID [16] and REDHEFFER [11]; in this respect

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also the results of Theorems 2.1 and 2.2 are direct improvements on the earlier results of [16] and [11].

Section 7 deals with a Riccati matrix differential equation which arises as the Legendre equation for a variational problem whose integrand function is an hermitian form in the dependent vector variable and its derivative. For such an equation there is a ready and simple alternative formulation of the considered definiteness properties for the corresponding hermitian form (2.2); moreover, in view of the existential nature of conclusion (i) of Theorem 2.2, this conclusion provides a result on the non-existence of focal points for certain conjoined families of solutions of the associated Euler linear second order vector differential equation. Finally, in Section 8 it is shown that the analytic methods of the author also yield the main theorem of a recent report by REDHEFFER [12] on the Mycielski-Paszowski diffusion problem.

Matrix notation is used throughout the present paper. Matrices of one column are termed vectors, and the norm $(|p_1|^2 + \cdots + |p_n|^2)^{\frac{1}{2}}$ of a vector $p = (p_\alpha)$ ($\alpha = 1, \cdots, n$) is denoted by $|p|$. If M is an $n \times n$ matrix, the symbol $|M|$ signifies the extremum of $|Mp|$ on the unit sphere $|p| = 1$. The symbols E and 0 are used for the identity and zero matrices, respectively, independent of the dimensions of these matrices. For a given matrix M its transpose is denoted by $\tilde{M}$, its conjugate by $\bar{M}$, and its conjugate transpose by M^* . The notations $M \geq N$ ($M > N$) are used to signify that M and N are hermitian matrices of the same dimensions and $M - N$ is a non-negative (positive) definite matrix. Finally, if all elements of a matrix possess a property of continuity or differentiability, for brevity we shall say that the matrix possesses this property.

2. Comparative statement of results. As in Section 4 of REID [16], it will be supposed that $A(y)$, $B(y)$, $C(y)$, $D(y)$ are $n \times n$ complex-valued matrix functions of the real variable y which are piecewise continuous on an open interval I of the real line. We shall be concerned with $n \times n$ matrix functions $U(x, y)$, $R(x, y)$, $S(x, y)$, $W(x, y)$ which are solutions of the differential system

$$\begin{aligned}
 (2.1) \quad & \begin{aligned}
 (a) \quad U_y &= A(y) + D(y)U + UB(y) + UC(y)U, & U(x, x) &= 0, \\
 (b) \quad R_y &= R[B(y) + C(y)U(x, y)], & R(x, x) &= E, \\
 (c) \quad S_y &= [D(y) + U(x, y)C(y)]S, & S(x, x) &= E, \\
 (d) \quad W_y &= R(x, y)C(y)S(x, y), & W(x, x) &= 0.
 \end{aligned}
 \end{aligned}$$

By a solution of (2.1) is meant a set of matrix functions U , R , S , W which satisfy the given initial conditions at $y = x$, are absolutely continuous on a subinterval $I_0 \subset I$ containing x , and are such that the differential equations of (2.1) hold a.e. (almost everywhere) on I_0 . Clearly an equivalent condition is that U , R , S , W are continuous in y on I_0 and satisfy corresponding integral equations; in particular, for a solution of (2.1) the differential equations hold at each point of continuity of $A(y)$, $B(y)$, $C(y)$, $D(y)$. From well-known existence

theorems (see, for example, [2; Chs. 1, 2] and [5; Ch. IX]) it follows that if $x \in I$, there is a maximal subinterval I_x of I on which a solution of (2.1) exists; moreover, the extent of this subinterval is the maximal subinterval containing x on which there exists a solution U of the differential equation (2.1a) satisfying $U(x, x) = 0$. In particular, in view of the linearity of (2.1b) and (2.1c) the matrices R and S are non-singular on I_x .

It is to be commented that whenever REDHEFFER [11] refers to a set U, R, S, W satisfying a system (2.1) in which the coefficient matrices are piecewise continuous, these matrix functions should be considered to have the properties described above, and in Theorems III and XII of [11] the conditions involving the coefficient matrices hold for values y at which these matrices are continuous; or equally well the conditions hold at each y with the arguments of all coefficient matrices the right-hand limit values or the left-hand limit values. If A, B, C, D are piecewise continuous on I , and by a solution of (2.1) is meant a set of differentiable matrix functions U, R, S, W satisfying the initial conditions and the differential equations for each $y \in I$, then an elementary argument shows that A, B, C, D must all be continuous on I whenever (2.1) admits a solution.

For brevity, we shall write I^c for the set of values $y \in I$ at which all the matrix functions $A(y), B(y), C(y), D(y)$ are continuous and denote by $\mathfrak{D}_0^+$ and $\mathfrak{D}^+$ the following conditions.

$\mathfrak{D}_0^+$: For $y \in I^c$ the hermitian form

$$(2.2) \quad \begin{aligned} \Omega(y; p, q) \equiv & q^*[D(y) + D^*(y)]q + q^*[A(y) + C^*(y)]p \\ & + p^*[A^*(y) + C(y)]q + p^*[B(y) + B^*(y)]p \end{aligned}$$

is non-positive for all n -dimensional vectors p, q .

$\mathfrak{D}^+$: For $y \in I^c$ the hermitian form $\Omega(y; p, q)$ is non-positive for all n -dimensional vectors p, q satisfying $|q|^2 - |p|^2 = 0$.

If in $\mathfrak{D}_0^+$ and $\mathfrak{D}^+$ we replace non-positive by non-negative, the resulting conditions are denoted by $\mathfrak{D}_0^-$ and $\mathfrak{D}^-$; if non-positive is replaced by zero, the respective conditions are designated $\mathfrak{D}_0^*$ and $\mathfrak{D}^*$.

Moreover, the symbols J^*, J^+, J^- are introduced for the following plane sets:

$$\begin{aligned} J^* &= I \times I, & J^+ &= \{(x, y) \mid (x, y) \in J^*, y \geq x\}, \\ J^- &= \{(x, y) \mid (x, y) \in J^*, y \leq x\}. \end{aligned}$$

If $x \in I$ and K is a given $n \times n$ matrix, $U(x, y; K)$ will denote the solution of the Riccati differential equation of (2.1a) satisfying the initial condition $U(x, x; K) = K$. The symbol $\mathfrak{O}^+$ will denote the following condition, which was designated as $\mathfrak{O}$ in REID [16].

$\mathfrak{O}^+$: If $|K| < 1$ then the solution $U(x, y; K)$ exists and satisfies $|U(x, y; K)| < 1$ on J^+ .

The results of Theorems 4.1 and 4.2 of REID [16] may be stated as follows.

Theorem A (REID). *If U, R, S, W is a solution of (2.1) on J^* , then*

- (i) $\mathfrak{D}^+$ is a necessary condition for $\mathfrak{O}^+$, and $\mathfrak{D}_0^+$ is a sufficient condition for $\mathfrak{O}^+$;
- (ii) if $D(y) \equiv \tilde{B}(y)$ and $A(y), C(y)$ are symmetric on I , then $\mathfrak{D}_0^+$ is a necessary and sufficient condition for $\mathfrak{O}^+$.

It is to be commented that since the proof of Theorem 4.2 of [16] establishes that $\mathfrak{D}^+$ implies $\mathfrak{D}_0^+$ whenever $D(y) \equiv \tilde{B}(y)$ and $A(y), C(y)$ are symmetric, in (ii) above one may equally well write $\mathfrak{D}^+$ in place of $\mathfrak{D}_0^+$.

Now let $Z(x, y)$ denote the $2n \times 2n$ matrix

$$(2.3) \quad Z(x, y) = \begin{vmatrix} S(x, y) & U(x, y) \\ W(x, y) & R(x, y) \end{vmatrix},$$

where U, R, S, W is the solution of (2.1), and let $\mathfrak{O}_0^+, \mathfrak{U}_0^+$ denote the following properties.

$\mathfrak{O}_0^+ : Z(x, y)$ exists and satisfies $|Z(x, y)| \leq 1$ on J^+ .

$\mathfrak{U}_0^+ : Z(x, y)$ exists and is unitary on J^+ .

If in $\mathfrak{O}_0^+$ and $\mathfrak{U}_0^+$ we replace J^+ by J^- , the resulting conditions are denoted by $\mathfrak{O}_0^-$ and $\mathfrak{U}_0^-$; if J^+ is replaced by J^* , the respective conditions are designated $\mathfrak{O}_0^*$ and $\mathfrak{U}_0^*$.

As pointed out by REDHEFFER [11; p. 350], for a matrix differential system (2.1) the result corresponding to the conclusion of Theorem IV of REDHEFFER [10] consists of both properties $\mathfrak{O}^+$ and $\mathfrak{O}_0^+$, and not merely the property $\mathfrak{O}^+$ considered by REID [16]. Moreover, in Theorem III of [11] REDHEFFER has established the following result.

Theorem B (REDHEFFER). *If U, R, S, W is a solution of (2.1) on J^* , then*

- (i) $\mathfrak{D}_0^+$ is a necessary and sufficient condition for $\mathfrak{O}_0^+$;
- (ii) $\mathfrak{D}_0^*$ is a necessary and sufficient condition for $\mathfrak{U}_0^*$.

Let $\mathfrak{D}_\rho^+$ and $\mathfrak{U}^+$ denote the following conditions:

$\mathfrak{D}_\rho^+ : \text{For } y \in I^c \text{ there exists a real scalar } \rho(y) \text{ such that the hermitian form}$

$$(2.4) \quad \Omega_\rho(y; p, q) \equiv \Omega(y; p, q) - \rho(y)(|q|^2 - |p|^2)$$

is non-positive for all n -dimensional vectors p, q .

$\mathfrak{U}^+ : \text{If } K \text{ is unitary then } U(x, y; K) \text{ exists and is unitary on } J^+.$

If in $\mathfrak{D}_\rho^+$ we replace non-positive by non-negative $\{\text{zero}\}$, the resulting condition is denoted by $\mathfrak{D}_\rho^- \{D_\rho^*\}$; if in $\mathfrak{U}^+$ we replace J^+ by $J^- \{J^*\}$, the resulting condition is denoted by $\mathfrak{U}^- \{\mathfrak{U}^*\}$.

With this notation Theorem XII of REDHEFFER [9] may be stated as follows; in particular, conclusion (i) answers a question raised by REID [16; p. 229].

Theorem C (REDHEFFER). *If U, R, S, W is a solution of (2.1) on J^* , then*

- (i) $\mathfrak{D}_\rho^+$ is a necessary and sufficient condition for $\mathfrak{O}^+$;
- (ii) $\mathfrak{D}_0^*$ is a necessary and sufficient condition for $\mathfrak{U}^+$.

Actually, the statement in (ii) of Theorem C that $\mathfrak{D}_0^*$ is necessary for $\mathfrak{U}^+$ is in error. The discussion of REDHEFFER on p. 367, when properly interpreted, implies merely that $\mathfrak{D}^*$ is necessary for $\mathfrak{U}^+$, and indeed the example of a system (2.1) with

$$A(y) \equiv C(y) \equiv 0, \quad B(y) \equiv -D(y) \equiv E$$

shows that $\mathfrak{D}_0^*$ is not necessary for $\mathfrak{U}^+$. In conclusion (i) of Theorem 2.2 below it will be established that $\mathfrak{D}^*$ is a necessary and sufficient condition for $\mathfrak{U}^+$.

In each of Theorems A, B, C the specific hypothesis that U, R, S, W is a solution of (2.1) on J^* has been inserted since this assumption was made in the earlier discussions of REDHEFFER [9; 10; 11] and REID [16]. Actually this assumption is unnecessary, for in the proof of each of the sufficiency statements of the above theorems the existence of the solution on J^+ may be deduced from extension theorems for solutions of the involved differential equations, as is done in Lemmas 3.2 and 3.3 of the next section.

We shall proceed to show that the methods used by REID to prove Theorem 4.1 of [16] yield the following results, corresponding to the result of REDHEFFER cited above as Theorem B.

Theorem 2.1. (i) $\mathfrak{D}_0^+$, $\mathfrak{D}_0^-$, or $\mathfrak{D}_0^*$ is a necessary and sufficient condition for the respective property $\mathfrak{O}_0^+$, $\mathfrak{O}_0^-$, or $\mathfrak{O}_0^*$; (ii) $\mathfrak{D}_0^*$ is a necessary condition for either $\mathfrak{U}_0^+$ or $\mathfrak{U}_0^-$ and a sufficient condition for $\mathfrak{U}_0^*$.

In Section 5 we shall establish the following result, again using the methods of REID [16], now supplemented by an algebraic theorem that shows the equivalence of $\mathfrak{D}_p^+$ $\{\mathfrak{D}_p^-$ or $\mathfrak{D}_p^*\}$ and $\mathfrak{D}^+$ $\{\mathfrak{D}^-$ or $\mathfrak{D}^*\}$.

Theorem 2.2. (i) $\mathfrak{D}^+$, $\mathfrak{D}^-$, or $\mathfrak{D}^*$ is a necessary and sufficient condition for the respective property $\mathfrak{O}^+$, $\mathfrak{O}^-$, or $\mathfrak{O}^*$; (ii) $\mathfrak{D}^*$ is a necessary condition for either $\mathfrak{U}^+$ or $\mathfrak{U}^-$ and is a sufficient condition for $\mathfrak{U}^*$.

3. Lemmas. Preliminary to the proofs of Theorems 2.1, 2.2 we shall establish the following results, which are actually slight extensions of the component parts of the proof of Theorem 4.1 of [16], together with an application of an extension theorem for solutions of differential systems.

In general, if K is a given $n \times n$ matrix, $P = P(x, y; K)$, $Q = Q(x, y; K)$ denotes the solution of the linear differential system

$$(3.1) \quad \begin{aligned} P_y &= -B(y)P - C(y)Q, & P(x, x) &= E, \\ Q_y &= A(y)P + D(y)Q, & Q(x, x) &= K. \end{aligned}$$

If I_{1x} is a subinterval of I containing $y = x$, and throughout which $P(x, y; K)$ is non-singular, then on this subinterval $U(x, y; K)$ exists and

$$Q(x, y; K)P^{-1}(x, y; K) = U(x, y; K).$$

Conversely, if I_x is the maximal subinterval of I containing $y = x$, and through-

out which $U(x, y; K)$ exists, the matrix function $P = P(x, y; K)$ defined on I_x as the solution of

$$P_y = -[B(y) + C(y)U(x, y; K)]P, \quad P(x, x; K) = E,$$

is non-singular and

$$P(x, y; K), \quad Q(x, y; K) = U(x, y; K)P(x, y; K)$$

is the solution of (3.1) on I_x . Moreover,

$$(3.2) \quad V(x, y; K) = Q^*(x, y; K)Q(x, y; K) - P^*(x, y; K)P(x, y; K)$$

is such that for ξ an arbitrary n -dimensional vector

$$(3.3) \quad [\xi^*V(x, y; K)\xi]_y = \Omega(y; P(x, y; K)\xi, Q(x, y; K)\xi) \quad \text{for } y \in I_x \cap I^c.$$

Lemma 3.1. *If K is a given $n \times n$ unitary matrix and $U(x, y; K)$ exists and satisfies $|U(x, y; K)| \leq 1$ {is unitary} on J^+ , then for $y \in I^c$ the form $\Omega(y; \xi, K\xi)$ is non-positive {zero} for arbitrary vectors ξ .*

If $U(x, y; K)$ exists and satisfies $|U(x, y; K)| \leq 1$ on J^* , then for arbitrary vectors ξ we have $\xi^*V(x, y; K)\xi \leq 0$ on J^+ . As $V(x, x; K) = 0$ whenever K is unitary, the conclusion of the lemma is an immediate consequence of (3.3). Alternatively, if $U(x, y; K)$ is unitary on J^+ then $V(x, y; K) \equiv 0$ on J^+ , so that for $y \in I^c$ and ξ an arbitrary vector we have $\Omega(y; \xi, K\xi) = 0$ by (3.3).

Lemma 3.2. *$\mathfrak{D}^*$ is a necessary condition for $\mathfrak{U}^+$, and $\mathfrak{D}_0^*$ is a sufficient condition for $\mathfrak{U}^*$.*

If condition $\mathfrak{U}^+$ holds, then for arbitrary unitary K the solution $U(x, y; K)$ exists and is unitary on J^+ , so that Lemma 3.1 implies $\Omega(y; \xi, K\xi) \equiv 0$ for arbitrary unitary K , vectors ξ , and $y \in I^c$; this latter condition is clearly equivalent to $\mathfrak{D}^*$. On the other hand, if $\mathfrak{D}_0^*$ holds, K is a unitary matrix, and I_x is the maximal subinterval of I containing $y = x$ throughout which $U(x, y; K)$ exists, then the initial condition $V(x, x; K) = 0$ and (3.3) for arbitrary ξ imply that $V(x, y; K) \equiv 0$, so that $U(x, y; K)$ is unitary on I_x . By a well-known type of extension argument (see, for example, GRAVES [5; Ch. IX, Th. 4]) it then follows that $I_x = I$ for each x ; that is, $\mathfrak{D}_0^*$ implies $\mathfrak{U}^*$.

Lemma 3.3. *$\mathfrak{D}^+$ is a necessary condition for $\mathfrak{O}^+$, and $\mathfrak{D}_0^+$ is a sufficient condition for $\mathfrak{O}^+$.*

If $\mathfrak{O}^+$ holds and K is an arbitrary unitary matrix, then for $0 < r < 1$ the solution $U(x, y; rK)$ exists and $|U(x, y; rK)| < 1$ on J^+ . By the continuity of $U(x, y; rK)$ as a function of r it then follows that $U(x, y; K)$ exists and satisfies $|U(x, y; K)| \leq 1$ on J^+ . Consequently, by Lemma 3.1 we have that for $y \in I^c$ the form $\Omega(y; \xi, K\xi)$ is non-positive for arbitrary unitary K and vectors ξ , and this latter condition is clearly equivalent to $\mathfrak{D}^+$. Conversely, if $\mathfrak{D}_0^+$ holds and $|K| < 1$, let I_x denote the maximal subinterval of I with left-hand end-point

$y = x$ throughout which $U(x, y; K)$ exists. It follows from (3.2), (3.3) that if ξ is an arbitrary non-null vector, then $\xi^*V(x, x; K)\xi = |K\xi|^2 - |\xi|^2 < 0$, and $[\xi^*V(x, y; K)\xi]_v \leq 0$ for $y \in I_x \cap I^c$. Consequently, $\xi^*V(x, y; K)\xi < 0$ and $|U(x, y; K)| < 1$ for all $y \in I_x$. By an extension argument similar to that cited in the above proof of Lemma 3.2 it then follows that I_x contains all $y \in I, y \geq x$; that is, $\mathfrak{D}_0^+$ implies $\mathfrak{P}^+$.

4. Proof of Theorem 2.1. Let $A_0(y)$ denote the $2n \times 2n$ direct sum matrix

$$A_0(y) = \begin{vmatrix} A(y) & 0 \\ 0 & 0 \end{vmatrix},$$

where 0 is the $n \times n$ zero matrix, with similar definitions of $B_0(y), C_0(y), D_0(y)$ in terms of the corresponding $B(y), C(y), D(y)$. If $U_0(x, y)$ is the $2n \times 2n$ matrix

$$(4.1) \quad U_0(x, y) = \begin{vmatrix} U(x, y) & S(x, y) \\ R(x, y) & W(x, y) \end{vmatrix},$$

then it may be verified readily that U, R, S, W is a solution of (2.1) on a subinterval I_x containing $y = x$ if and only if on this subinterval $U_0(x, y)$ is a solution of the Riccati system

$$(4.2) \quad U_{0v} = A_0(y)U_0 + U_0B_0(y) + U_0C_0(y)U_0, \quad U_0(x, x) = F,$$

where F is the $2n \times 2n$ unitary matrix

$$(4.3) \quad F = \begin{vmatrix} 0 & E \\ E & 0 \end{vmatrix}.$$

Corresponding to (2.2), for the Riccati differential equation of (4.2) we have the hermitian form

$$(4.4) \quad \begin{aligned} \Omega_0(y; \pi, \sigma) \equiv & \sigma^*[D_0(y) + D_0^*(y)]\sigma + \sigma^*[A_0(y) + C_0^*(y)]\pi \\ & + \pi^*[A_0^*(y) + C_0(y)]\sigma + \pi^*[B_0(y) + B_0^*(y)]\pi. \end{aligned}$$

If ξ, ξ', η, η' are n -dimensional vectors and $\pi_\alpha = \xi_\alpha, \pi_{n+\alpha} = \xi'_\alpha, \sigma_\alpha = \eta_\alpha, \sigma_{n+\alpha} = \eta'_\alpha$ ($\alpha = 1, \dots, n$), from (4.4) it follows readily that

$$(4.5) \quad \Omega_0(y; \pi, \sigma) = \Omega(y; \xi, \eta).$$

If condition $\mathfrak{P}_0^+$ holds, then application of Lemma 3.1 to the solution U_0 of (4.2) yields the result that for $y \in I^c$ the form $\Omega_0(y; \pi, F\pi)$ is non-positive for arbitrary $2n$ -dimensional vectors π . If $\pi_\alpha = p_\alpha, \pi_{n+\alpha} = q_\alpha$ ($\alpha = 1, \dots, n$), then $\sigma = F\pi$ has $\sigma_\alpha = q_\alpha$ ($\alpha = 1, \dots, n$), and from (4.5) it follows $\Omega(y; p, q) \leq 0$ for $y \in I^c$ and p, q arbitrary n -dimensional vectors. That is, $\mathfrak{D}_0^+$ is a necessary condition for $\mathfrak{P}_0^+$. Conversely, if $\mathfrak{D}_0^+$ holds then (4.5) implies that for $y \in I^c$ the form $\Omega_0(y; \pi, \sigma)$ is non-positive for arbitrary $2n$ -dimensional vectors π, σ ; that is, for the Riccati differential equation of (4.2) the corresponding

condition $\mathfrak{D}_0^+$ holds. The sufficiency conclusion of Lemma 3.3 applied to the differential equation of (4.2) shows that this differential equation has the corresponding property $\mathfrak{O}^+$, that is, if $U_0(x, y; K_0)$ is the solution of this equation with $U_0(x, x; K_0) = K_0$ and $|K_0| < 1$, then $U_0(x, y; K_0)$ exists and satisfies $|U_0(x, y; K_0)| < 1$ on J^+ . This result for $K_0 = rF$, $0 < r < 1$, and a continuity argument show that $U_0(x, y) = U_0(x, y; F)$ exists and satisfies $|U_0(x, y)| \leq 1$ on J^+ , thus proving that $\mathfrak{D}_0^+$ is a sufficient condition for $\mathfrak{O}_0^+$.

The conclusion of (i) of Theorem 2.1 involving $\mathfrak{D}_0^-$ and $\mathfrak{O}_0^-$ is a corollary of the above established results involving $\mathfrak{D}_0^+$ and $\mathfrak{O}_0^+$, under the change of independent variable $y = -y'$; the conclusion of (i) involving $\mathfrak{D}_0^*$ and $\mathfrak{O}_0^*$ follows from the fact that $\mathfrak{D}_0^*$ is equivalent to the two conditions $\mathfrak{D}_0^+$ and $\mathfrak{D}_0^-$, while $\mathfrak{O}_0^*$ is equivalent to the two properties $\mathfrak{O}_0^+$, $\mathfrak{O}_0^-$.

Similarly, if $\mathfrak{U}_0^+$ holds then Lemma 3.1 applied to the solution U_0 of (4.2) yields the result that $\Omega_0(y; \pi, F\pi) = 0$ for $y \in I^c$ and arbitrary $2n$ -dimensional vectors π , and for $\pi_\alpha = p_\alpha$, $\pi_{n+\alpha} = q_\alpha$ ($\alpha = 1, \dots, n$) this latter condition becomes $\mathfrak{D}_0^*$. In turn, this result and the change of independent variable $y = -y'$ leads to the conclusion that $\mathfrak{D}_0^*$ is also a necessary condition for $\mathfrak{U}_0^-$. Conversely, if $\mathfrak{D}_0^*$ holds then in view of (4.5) we have that the corresponding condition $\mathfrak{D}_0^*$ holds for the Riccati differential equation of (4.2), and from Lemma 3.2 the solutions of this differential equation have property $\mathfrak{U}^*$. In particular, $U_0(x, y)$ is unitary on J^* , thus establishing that $\mathfrak{D}_0^*$ is a sufficient condition for $\mathfrak{U}_0^*$.

In Section 2 of [11] REDHEFFER notes that the functional equations determined by REID [16] are not equivalent to the semi-group property for the Riccati differential equation of (2.1a). It is to be commented, however, that one may verify readily that these functional equations are equivalent to the semi-group property for the associated Riccati equation (4.2).

5. Proof of Theorem 2.2. From Lemmas 3.3 and 3.2 it follows that $\mathfrak{D}^+$ is a necessary condition for $\mathfrak{O}^+$ and $\mathfrak{D}^*$ is a necessary condition for $\mathfrak{U}^+$. In turn, these results and the change of independent variable $y = -y'$ imply that $\mathfrak{D}^-$ is a necessary condition for $\mathfrak{O}^-$ and $\mathfrak{D}^*$ is a necessary condition for $\mathfrak{U}^-$; moreover, the fact that $\mathfrak{D}^*$ is a necessary condition for $\mathfrak{O}^*$ follows from the fact that $\mathfrak{D}^*$ is equivalent to the two conditions $\mathfrak{D}^+$ and $\mathfrak{D}^-$, while $\mathfrak{O}^*$ is equivalent to the two properties $\mathfrak{O}^+$, $\mathfrak{O}^-$. Consequently, there remains to prove only the sufficiency statements of Theorem 2.2.

In regard to the sufficiency statement involving $\mathfrak{D}^+$ and $\mathfrak{O}^+$, if one employs the result of REDHEFFER as stated in conclusion (i) of Theorem C in Section 2 one need prove only that $\mathfrak{D}^+$ implies $\mathfrak{D}_\rho^+$. Actually, one may establish the following result.

Theorem 5.1. *If $A(y)$, $B(y)$, $C(y)$, $D(y)$ are piecewise continuous on I and condition $\mathfrak{D}^+ \{ \mathfrak{D}^* \}$ holds, then there exists a real scalar function $\rho(y)$ which is piecewise continuous on I and such that for $y \in I^c$ the hermitian form $\Omega_\rho(y; p, q)$ is non-positive {zero} for all n -dimensional vectors p, q .*

Once Theorem 5.1 is established, however, the sufficiency statements of Theorem 2.2 are immediate consequences of Lemmas 3.3 and 3.2. Indeed, the solution $U(x, y; K)$ of the Riccati differential equation of (2.1a) satisfying $U(x, x; K) = K$ is also a solution of the equation

$$(5.1) \quad U_y = A(y) + [D(y) - \frac{1}{2}\rho(y)E]U + U[B(y) + \frac{1}{2}\rho(y)E] + UC(y)U.$$

If condition $\mathfrak{D}^+$ holds and $\rho(y)$ is chosen as in Theorem 5.1, so that for $y \in I^c$ the form $\Omega_\rho(y; p, q)$ is non-positive for arbitrary p, q , then from the sufficiency statement of Lemma 3.3 applied to (5.1) we have that property $\mathfrak{O}^+$ holds. This result and the change of independent variable $y = -y'$ then lead to the sufficiency statement in (i) of Theorem 2.2 involving $\mathfrak{D}^-$, $\mathfrak{O}^-$, and in turn these results imply the sufficiency statement involving $\mathfrak{D}^*$, $\mathfrak{O}^*$. Correspondingly, if $\mathfrak{D}^*$ holds and $\rho(y)$ is chosen as in Theorem 5.1, so that for $y \in I^c$ the form $\Omega_\rho(y; p, q)$ is zero for arbitrary p, q , then the sufficiency statement of Lemma 3.2 applied to (5.1) yields the conclusion that $\mathfrak{U}^*$ holds.

Now in the case of matrices A, B, C, D independent of y the existence of a real scalar ρ such that the corresponding form Ω_ρ is non-positive is a slight extension of a quadratic form problem considered a number of years ago by a variety of authors. For literature the reader is referred to ALBERT [1], DINES [3], FINSLER [4], HESTENES & MCSHANE [6], and REID [13] of the bibliography. It is to be commented, however, that for this case of constant coefficients the stated result is not contained explicitly in any of these references. Because of the intrinsic interest of the hermitian form problem, in the following section there will be established some general results which include as a special case the needed result of Theorem 5.1; in particular, Theorem 5.1 is a direct consequence of Theorem 6.1 and its Corollary.

6. Results on pencils of hermitian forms. We shall consider first two hermitian forms

$$(6.1) \quad F(z) \equiv z^*Fz, \quad G(z) \equiv z^*Gz,$$

where F, G are $n \times n$ hermitian matrices and $z \equiv (z_\alpha)$ ($\alpha = 1, \dots, n$) is an n -dimensional vector.

Lemma 6.1. *If $F(z) \leq 0$ for all z satisfying $G(z) = 0$ and $G(z)$ is an indefinite form, there exists a real constant ρ such that*

$$F_\rho(z) \equiv F(z) - \rho G(z) \leq 0$$

for all vectors z .

Independent of the assumption that $G(z)$ is indefinite, for the case in which $F(z) < 0$ for all non-null vectors z satisfying $G(z) = 0$ the cited papers of ALBERT, DINES, FINSLER, HESTENES & MCSHANE, and REID on real quadratic forms afford separate proofs of the existence of a real ρ such that $F_\rho(z)$ is a negative definite hermitian form. Consequently, under the hypotheses of the lemma we

have for each $\epsilon > 0$ the existence of a $\rho[\epsilon]$ such that the form

$$(6.2) \quad F(z) - \epsilon |z|^2 - \rho[\epsilon]G(z)$$

is negative definite. Now under the assumption that $G(z)$ is indefinite there exist vectors z^1, z^2 such that $|z^1| = |z^2| = 1$, $G(z^1) > 0$, $G(z^2) < 0$, and consequently for $\rho[\epsilon]$ such that (6.2) is negative definite we have

$$(6.3) \quad -\frac{|F|}{G(z^1)} + \epsilon \leq \rho[\epsilon] \leq -\frac{|F|}{G(z^2)} + \epsilon.$$

In particular, $\rho[\epsilon]$ is bounded for $0 < \epsilon \leq 1$, and hence there exists a value ρ and a sequence $\epsilon_i \rightarrow 0$ such that $\rho[\epsilon_i] \rightarrow \rho$. By continuity it then follows that such a ρ satisfies the conclusion of Lemma 6.1.

It is to be commented that the conclusion of Lemma 6.1 is invalid if one omits the hypothesis that $G(z)$ is indefinite. Indeed, for $n = 2$, $F(z) = z_1 z_2$, $G(z) = z_1^2$, one has $F(z) \leq 0$ for all z satisfying $G(z) = 0$, but there is no ρ such that $F_\rho(z)$ is non-positive for all z .

Lemma 6.2. *Suppose that $F(z) \leq 0$ for all z satisfying $G(z) = 0$, and that $G(z)$ is an indefinite form. Then the set Λ of real ρ defined as*

$$(6.4) \quad \Lambda = \{ \rho \mid F_\rho(z) \leq 0 \text{ for all } z \}$$

is a compact interval Λ : $\rho_m \leq \rho \leq \rho_M$, and $\rho_m < \rho_M$ whenever $F(z) < 0$ for all non-null z satisfying $G(z) = 0$. In general, if $\rho_m < \rho_M$ then for $\rho_m < \tau < \rho_M$ a vector z^0 is such that $F_\tau(z^0) = 0$ if and only if $Fz^0 = 0$, $Gz^0 = 0$; in particular, if there is a member of the pencil $F - \rho G$ which is non-singular then $F_\rho(z)$ is negative definite for $\rho_m < \rho < \rho_M$.

Clearly the set Λ is closed; moreover, Λ is an interval since for $\sigma \in \Lambda$, $\tau \in \Lambda$, and $\rho = (1 - \theta)\sigma + \theta\tau$, $0 \leq \theta \leq 1$, we have

$$F_\rho(z) = (1 - \theta)F_\sigma(z) + \theta F_\tau(z).$$

In addition, Λ is bounded since an argument similar to that used in the proof of Lemma 6.1 shows that

$$(6.5) \quad -\frac{|F|}{G(z^1)} \leq \rho \leq -\frac{|F|}{G(z^2)} \quad \text{if } \rho \in \Lambda,$$

where z^1, z^2 are vectors such that $|z^1| = 1 = |z^2|$, $G(z^1) > 0$, $G(z^2) < 0$. In case $F(z) < 0$ for all non-null z satisfying $G(z) = 0$, the above cited result insures the existence of a value τ such that $F_\tau(z)$ is a negative definite form; by continuity $F_\rho(z)$ is negative definite for ρ in a suitable neighborhood of τ , so that Λ is non-degenerate and $\rho_m < \rho_M$.

In general, if $\rho_m < \rho_M$ then $F_\rho(z) \leq 0$ for $\rho_m \leq \rho \leq \rho_M$ and arbitrary vectors z , so that if $\rho_m < \tau < \rho_M$ and z^0 is a vector such that $F_\tau(z^0) = 0$, then the linearity of $F_\rho(z^0)$ in ρ implies that $F_\rho(z^0) = 0$ for arbitrary ρ . The non-positiveness of the form $F_\rho(z)$ for $\rho_m \leq \rho \leq \rho_M$ then implies $Fz^0 - \rho Gz^0 = 0$ on $\rho_m \leq \rho \leq \rho_M$,

and hence $Fz^0 = 0 = Gz^0$. Finally, if there is a member of the pencil $F - \rho G$ which is non-singular then $z^0 = 0$, so that $F_\tau(z)$ is a negative definite form for $\rho_m < \rho < \rho_M$.

We shall consider now hermitian forms

$$(6.6) \quad F(y; z) \equiv z^* F(y) z, \quad G(y; z) \equiv z^* G(y) z,$$

where $F(y)$, $G(y)$ are $n \times n$ hermitian matrices for y on a given interval I_0 of the real line.

Theorem 6.1. *Suppose that the hermitian matrices $F(y)$, $G(y)$ are continuous on I_0 , and that for each $y \in I_0$:*

(a) $F(y; z) \leq 0$ for all z satisfying $G(y; z) = 0$;

(b) $G(y; z)$ is an indefinite form;

(c) *there is a member of the pencil $F(y) - \rho G(y)$ which is non-singular.*

Then there is a real-valued scalar function $\rho(y)$ continuous and such that for each $y \in I_0$ the form $F(y; z) - \rho(y)G(y; z)$ is non-positive for arbitrary vectors z ; moreover, $\rho(y)$ may be chosen so that if $y_0 \in I_0$ and $F(y_0; z) < 0$ for all non-null vectors z satisfying $G(y_0; z) = 0$, then $F(y_0; z) - \rho(y_0)G(y_0; z)$ is a negative definite form.

In view of conditions (a), (b), (c), for each $y \in I_0$ the hermitian forms $F(y; z)$, $G(y; z)$ satisfy the hypotheses of Lemma 6.2, and hence the set $\Lambda(y)$ of real values ρ such that the form $F(y; z) - \rho G(y; z)$ is non-positive is a compact interval $\Lambda(y): \rho_m(y) \leq \rho \leq \rho_M(y)$. In view of Lemma 6.2, for a given $y_0 \in I_0$ we have $\rho_m(y_0) < \rho_M(y_0)$ if and only if $F(y_0; z) < 0$ for all non-null vectors z satisfying $G(y_0; z) = 0$, and the function $\rho(y) = \frac{1}{2}[\rho_m(y) + \rho_M(y)]$ satisfies the conditions of Theorem 6.1 provided $\rho_m(y)$ and $\rho_M(y)$ are continuous on I_0 .

If $y_0 \in I_0$, then from an inequality corresponding to (6.5) there exist constants k_1 , k_2 and a neighborhood $N(y_0)$ of y_0 such that

$$k_1 < \rho_m(y), \quad \rho_M(y) < k_2 \quad \text{for } y \in I_0 \cap N(y_0).$$

If

$$0 < \epsilon < \min \{ \rho_m(y_0) - k_1, k_2 - \rho_M(y_0) \}$$

and

$$\Lambda_1 = \{ \rho \mid k_1 \leq \rho \leq \rho_m(y_0) - \epsilon \},$$

$$\Lambda_2 = \{ \rho \mid \rho_M(y_0) + \epsilon \leq \rho \leq k_2 \},$$

then for each $\tau \in \Lambda_1 \cup \Lambda_2$ there is a vector z^τ such that $F_\tau(y_0; z^\tau) > 0$, and from the continuity of $F_\rho(y; z^\tau)$ as a function of ρ , y it follows that there is a $\delta(\tau) > 0$ such that $F_\rho(y; z^\tau) > 0$ for all ρ, y with $|\rho - \tau| < \delta(\tau)$, $|y - y_0| < \delta(\tau)$, $y \in I_0 \cap N(y_0)$. Consequently, an elementary application of the Heine-Borel theorem yields the semicontinuity conclusion that there is a $\delta_\epsilon > 0$ such that

$$(6.7) \quad \rho_m(y_0) - \epsilon < \rho_m(y), \quad \rho_M(y) < \rho_M(y_0) + \epsilon \quad \text{if } |y - y_0| < \delta_\epsilon, \quad y \in I_0.$$

Now if y_0 is such that $\rho_m(y_0) = \rho_M(y_0)$, relations (6.7) imply the continuity of $\rho_m(y)$ and $\rho_M(y)$ at $y = y_0$. On the other hand, if $\rho_m(y_0) < \rho_M(y_0)$ then Lemma 6.2 implies that the form $F_\rho(y_0; z)$ is negative definite for $\rho_m(y_0) < \rho < \rho_M(y_0)$. If $0 < \epsilon < \frac{1}{2}(\rho_M(y_0) - \rho_m(y_0))$ then a simple continuity argument shows that there is a $\delta'_\epsilon > 0$ such that the form $F_\rho(y; z)$ is negative definite for

$$\rho_m(y_0) + \epsilon \leq \rho \leq \rho_M(y_0) - \epsilon, \quad |y - y_0| < \delta'_\epsilon, \quad y \in I_0,$$

and hence

$$(6.8) \quad \rho_m(y) < \rho_m(y_0) + \epsilon, \quad \rho_M(y_0) - \epsilon < \rho_M(y) \quad \text{if} \quad |y - y_0| < \delta'_\epsilon, \quad y \in I_0.$$

Combination of (6.7) and (6.8) then shows that $\rho_m(y)$ and $\rho_M(y)$ are continuous at $y = y_0$.

Corollary. *Suppose that the hermitian matrices $F(y)$, $G(y)$ are continuous on I_0 , and for each $y \in I_0$ conditions (b), (c), of Theorem 6.1 hold, while condition (a) is replaced by*

$$(a') \quad F(y; z) = 0 \text{ for all } z \text{ satisfying } G(y; z) = 0.$$

Then there is a (unique) real-valued scalar function $\rho(y)$ continuous on I_0 and such that for $y \in I_0$ the form $F(y; z) - \rho(y)G(y; z)$ is zero for arbitrary vectors z .

Under the conditions of the corollary the hypotheses of Theorem 6.1 are satisfied by the pair of forms $F(y; z)$, $G(y; z)$ and also the pair $-F(y; z)$, $G(y; z)$, and thus there exist real-valued continuous scalar functions $\rho_1(y)$, $\rho_2(y)$ such that for $y \in I_0$ each of the forms $F(y; z) - \rho_1(y)G(y; z)$, $-F(y; z) - \rho_2(y)G(y; z)$ is non-positive for arbitrary vectors z . If $\rho_1(y)$, $\rho_2(y)$ are any two such functions, then for $y \in I_0$ the form $(\rho_1(y) + \rho_2(y))G(y; z)$ is non-negative for arbitrary vectors z , and since $G(y; z)$ is indefinite we have $\rho_2(y) \equiv -\rho_1(y)$. Thus the functions $\rho_1(y)$, $\rho_2(y)$ are individually unique, the conclusion of the corollary holds for $\rho(y) = \rho_1(y)$, and as any $\rho(y)$ satisfying the conclusion is a permissible choice for $\rho_1(y)$ we have that $\rho(y)$ is unique.

It is to be remarked that the conclusion of Theorem 6.1 is invalid whenever hypothesis (c) is deleted. For example, let $n = 4$ and

$$\begin{aligned} G(y; z) = G(z) &= |z_3 - z_4|^2 - |z_1 - z_2|^2, \\ F(y; z) &= |z_3 - z_4|^2 + 2y(|z_3|^2 - |z_4|^2) - \frac{y^2}{\theta(y)} |z_3 + z_4|^2 - 2|z_1 - z_2|^2 \\ &\quad + 2y(|z_1|^2 - |z_2|^2) - \frac{y^2}{\varphi(y)} |z_1 + z_2|^2, \end{aligned}$$

where

$$\begin{aligned} \theta(y) &= \frac{1}{4} + \frac{1}{8} \sin(1/y), & \varphi(y) &= \frac{1}{2} - \frac{1}{4} \sin(1/y) \quad \text{for } y \neq 0, \\ \theta(0) &= \frac{1}{4}, & \varphi(0) &= \frac{1}{2}. \end{aligned}$$

It follows readily that the coefficients of the forms F, G are continuous functions of y on $I_0 : -\infty < y < \infty$. Also

$$F(y; z) = (1 + \theta(y)) |z_3 - z_4|^2 - \theta(y) \left| (z_3 - z_4) - \frac{y}{\theta(y)} (z_3 + z_4) \right|^2 \\ + (\varphi(y) - 2) |z_1 - z_2|^2 - \varphi(y) \left| (z_1 - z_2) - \frac{y}{\varphi(y)} (z_1 + z_2) \right|^2,$$

from which it follows readily that $F(y; z) \leq 0$ for all z satisfying $G(z) = 0$. In particular,

$$F_\rho(0; z) = (\rho - 2) |z_1 - z_2|^2 + (1 - \rho) |z_3 - z_4|^2,$$

and hence $\rho_m(0) = 1$, $\rho_M(0) = 2$. Moreover,

$$F(y; z) - \rho G(y; z) = (\rho + \varphi(y) - 2) |z_1 - z_2|^2 \\ - \varphi(y) \left| (z_1 - z_2) - \frac{y}{\varphi(y)} (z_1 + z_2) \right|^2 + (1 + \theta(y) - \rho) |z_3 - z_4|^2 \\ - \theta(y) \left| (z_3 - z_4) - \frac{y}{\theta(y)} (z_3 + z_4) \right|^2,$$

and for $y \neq 0$ the form $F(y; z) - \rho G(y; z)$ is non-positive for arbitrary vectors z if and only if $1 + \theta(y) \leq \rho \leq 2 - \phi(y)$. That is, for $y \neq 0$ we have

$$\rho_m(y) = 1 + \theta(y) = \frac{5}{4} + \frac{1}{8} \sin(1/y), \quad \rho_M(y) = 2 - \varphi(y) = \frac{3}{2} + \frac{1}{4} \sin(1/y),$$

and $\rho_m(y)$, $\rho_M(y)$ are both discontinuous at $y = 0$. Moreover, since

$$\limsup_{y \rightarrow 0} \rho_m(y) = \frac{11}{8} > \frac{5}{4} = \liminf_{y \rightarrow 0} \rho_M(y),$$

for this example there is no scalar function $\rho(y)$ which is continuous at $y = 0$ and such that for all values y the form $F(y; z) - \rho(y)G(z)$ is non-positive for arbitrary vectors z .

Finally, in line with the comment on p. 230 of REID [16] for a system (2.1) with coefficient matrices merely (Lebesgue) integrable on compact subintervals of I , results corresponding to those of Theorem 2.2 may be obtained by the same general methods as before, but with Theorem 5.1 replaced by a corresponding result that is an immediate application of the following theorem.

Theorem 6.2. *If the hermitian matrix $F(y)$ is integrable on the compact interval I_0 , G is a constant hermitian matrix such that the form $G(z) \equiv z^*Gz$ is indefinite, and for y a.e. on I_0 the form $F(y; z)$ is non-positive {zero} for all vectors z satisfying $G(z) = 0$, then there is a real-valued scalar function $\rho(y)$ integrable on I_0 and such that for y a.e. on I_0 the form $F(y; z) - \rho(y)G(z)$ is non-positive {zero} for arbitrary vectors z .*

Consideration will be limited to the proof of the stated conclusion in the

case that for y a.e. on I_0 we have $F(y; z) \leq 0$ for all vectors z satisfying $G(z) = 0$, since this result implies the result for the parenthetical alternative by an argument entirely analogous to that used above in deducing the corollary from Theorem 6.1. Moreover, throughout the following argument it will be assumed that for *each* $y \in I_0$ the form $F(y; z)$ is non-positive for all z satisfying $G(z) = 0$, since on a possible exceptional set of measure zero the redefinition of $F(y)$ as the zero matrix provides this property and does not affect the other conditions in the statement of the result. Then from Lemma 6.2 it follows that for each $y \in I_0$ the set $\Lambda(y)$ of real values ρ such that the form $F(y; z) - \rho G(z)$ is non-positive is a compact interval $\Lambda(y): \rho_m(y) \leq \rho \leq \rho_M(y)$. Now if $\{z^i\}$ ($i = 1, 2, \dots$) is a sequence of vectors dense in the unit ball $|z| \leq 1$, we have

$$|F(y)| = \sup_i |F(y)z^i| \leq \sum_{\alpha, \beta=1}^n |F_{\alpha\beta}(y)|,$$

and the integrability of $|F(y)|$ on I_0 follows from the integrability of $F(y)$ on this interval. Moreover, analogous to (6.5) we have

$$-\frac{|F(y)|}{G(z^1)} \leq \rho_m(y) \leq \rho_M(y) \leq -\frac{|F(y)|}{G(z^2)},$$

where z^1, z^2 are vectors with $|z^1| = |z^2| = 1$, $G(z^1) > 0$, $G(z^2) < 0$, and consequently the stated conclusion is immediate if it is established that $\rho_m(y)$ and $\rho_M(y)$ are measurable. In turn, the measurability of $\rho_m(y)$ and $\rho_M(y)$ under the stated hypothesis is a ready consequence of the measurability of $\rho_m(y)$, $\rho_M(y)$ under the following more stringent hypothesis:

(6.9) $F(y; z) < 0$ for $y \in I_0$ and non-null vectors z satisfying $G(z) = 0$.

Indeed, if $F(y; z) \leq 0$ for $y \in I_0$ and all vectors z satisfying $G(z) = 0$, then for each $h > 0$ the form $F_h(y; z) \equiv F(y; z) - h|z|^2$ is negative for $y \in I_0$ and all non-null vectors z satisfying $G(z) = 0$, while for the corresponding intervals $\Lambda_h(y): \rho_{m,h}(y) \leq \rho \leq \rho_{M,h}(y)$ we have $\rho_{m,h}(y) < \rho_m(y)$, $\rho_M(y) < \rho_{M,h}(y)$, and $\rho_{m,h}(y) \rightarrow \rho_m(y)$, $\rho_{M,h}(y) \rightarrow \rho_M(y)$ as $h \rightarrow 0$; in particular, if $\rho_{m,h}(y)$ and $\rho_{M,h}(y)$ are measurable for $h > 0$ then $\rho_m(y)$ and $\rho_M(y)$ are measurable.

Now suppose that the stronger hypothesis (6.9) is satisfied; from Lemma 6.2 it follows that $\rho_m(y) < \rho_M(y)$ and the form $F(y; z) - \rho G(z)$ is negative definite for $\rho_m(y) < \rho < \rho_M(y)$. If I_0 is the interval $a \leq y \leq b$, let the domain of definition of $F(y)$ be extended to the real line, with $F(y) \equiv F(a)$ on $y \leq a$, $F(y) \equiv F(b)$ on $y \geq b$. If

$$F(y | j) = \frac{j}{2} \int_{-1/j}^{1/j} F(y + s) ds \quad (j = 1, 2, \dots),$$

then each $F(y|j)$ is a continuous hermitian matrix, for y a.e. on I_0 we have $F(y|j) \rightarrow F(y)$ as $j \rightarrow \infty$, and (6.9) implies that $F(y|j; z) \equiv z^* F(y|j) z < 0$ for each y and arbitrary non-null vectors z satisfying $G(z) = 0$. In view of Lemma 6.2, for each $j = 1, 2, \dots$ the forms $F(y|j; z)$, $G(z)$ satisfy the hypotheses of Theorem

6.1, and hence there exist real-valued continuous scalar functions $\rho_m(y|j)$, $\rho_M(y|j)$ ($j = 1, 2, \dots$) such that the corresponding interval $\Delta(y|j)$ of values ρ for which the form $F(y|j; z) - \rho G(z)$ is non-positive definite is given by $\rho_m(y|j) \leq \rho \leq \rho_M(y|j)$. By an elementary argument, similar to that employed to establish (6.7) and (6.8) in the proof of Theorem 6.1, one may prove that if y is such that $F(y|j) \rightarrow F(y)$ as $j \rightarrow \infty$, then $\rho_m(y|j) \rightarrow \rho_m(y)$ and $\rho_M(y|j) \rightarrow \rho_M(y)$. In particular, these limit relations imply the measurability of $\rho_m(y)$ and $\rho_M(y)$, thus completing the proof of Theorem 6.2.

7. Riccati equations associated with variational problems. Suppose that on an interval I of the real line $F(y)$, $G(y)$, $H(y)$ are complex-valued $n \times n$ matrix functions with $F(y)$ non-singular and $F(y)$, $G(y)$ hermitian, while $F(y)$, $F^{-1}(y)$, $G(y)$, and $H(y)$ are piecewise continuous on this interval. For the variational integral

$$(7.1) \quad \mathcal{J}(\eta; y_1, y_2) = \int_{y_1}^{y_2} \{ \eta^* [F(y)\eta' + H(y)\eta] + \eta^* [H^*(y)\eta' + G(y)\eta] \} dy$$

the vector Euler differential equation is

$$(7.2) \quad [F(y)\eta' + H(y)\eta]' - [H^*(y)\eta' + G(y)\eta] = 0.$$

In terms of the canonical variables η , $\zeta = F(y)\eta' + H(y)\eta$ the equation (7.2) becomes

$$(7.3) \quad \eta' = -B(y)\eta - C(y)\zeta, \quad \zeta' = A(y)\eta + D(y)\zeta,$$

where on I the matrices $A(y)$, $B(y)$, $C(y)$, $D(y)$ are

$$(7.4) \quad A = G - H^*F^{-1}H, \quad B = F^{-1}H, \quad C = -F^{-1}, \quad D = H^*F^{-1} = B^*.$$

For the variational integral (7.1) the corresponding Legendre matrix differential equation is then the Riccati differential equation of (2.1a); it follows readily that this equation may be written also in the "Legendre" form

$$(7.5) \quad K[U] \equiv U' + (H^* - U)F^{-1}(H - U) - G = 0.$$

For a system (2.1) associated in this manner with the variational integral one has the following simple interpretation of the conditions $\mathfrak{D}^+$ and $\mathfrak{D}_0^+$.

Theorem 7.1. *For the coefficient matrices of (2.1) defined by (7.4) the condition $\mathfrak{D}^+$ is equivalent to $\mathfrak{D}_0^+$, and this condition is equivalent to the statement that $K[-E] \leq 0$ and $K[E] \geq 0$ on I^c ; alternatively, in terms of the matrices F , G , H this condition may be written*

$$(7.6) \quad (H^* + E)F^{-1}(H + E) \leq G \leq (H^* - E)F^{-1}(H - E) \quad \text{for } y \in I^c.$$

For this particular problem one may verify readily that the form $\Omega(y; p, q)$ of (2.2) is

$$(7.7) \quad \Omega(y; p, q) = q^*M(y)q + q^*N(y)p + p^*N(y)q + p^*M(y)p,$$

where $M(y)$ and $N(y)$ are the hermitian matrices

$$(7.8) \quad M = F^{-1}H + H^*F^{-1}, \quad N = G - H^*F^{-1}H - F^{-1}.$$

Whenever condition $\mathfrak{D}^+$ holds, the choices $q = -p$ and $q = p$ lead to the respective conditions $M - N \leq 0$ and $M + N \leq 0$ for $y \in I^c$, and the identity

$$2\Omega(y; p, q) = (q^* + p^*)(M + N)(q + p) + (q^* - p^*)(M - N)(q - p)$$

shows that condition $\mathfrak{D}^+$ implies $\mathfrak{D}_0^+$. The remaining statements of the theorem are immediate consequences of the relations $M - N \equiv K[-E]$, $M + N \equiv -K[E]$, and the form (7.5) of the differential equation of (2.1a).

By a similar argument it follows that condition $\mathfrak{D}^-$ is equivalent to $\mathfrak{D}_0^-$, and that this condition is equivalent to the statement that $K[-E] \geq 0$ and $K[E] \leq 0$, which also may be written in the form

$$(7.6') \quad (H^* - E)F^{-1}(H - E) \leq G \leq (H^* + E)F^{-1}(H + E) \quad \text{for } y \in I^c.$$

If $\Phi(y)$ is a non-singular continuous $n \times n$ matrix function which has piecewise continuous derivatives on I , then under the transformation

$$(7.9) \quad \eta = \Phi(x)\eta_1$$

the functional (7.1) becomes

$$(7.10) \quad \int_{y_1}^{y_2} \{ \eta_1'^* [F_1(y)\eta_1' + H_1(y)\eta_1] + \eta_1^* [H_1^*(y)\eta_1' + G_1(y)\eta_1] \} dy,$$

with coefficient matrices $F_1(y)$, $G_1(y)$, $H_1(y)$ given by

$$(7.11) \quad \begin{aligned} F_1 &= \Phi^*F\Phi, & H_1 &= \Phi^*(F\Phi' + H\Phi), \\ G_1 &= \Phi^{*'}(F\Phi' + H\Phi) + \Phi^*(H^*\Phi' + G\Phi). \end{aligned}$$

Clearly the matrices F_1 , G_1 , H_1 satisfy the same conditions as specified above for the respective matrices F , G , H . Moreover, if $K_1[U]$ denotes the corresponding Legendre differential expression

$$K_1[U] = U' + (H_1^* - U)F_1^{-1}(H_1 - U) - G_1,$$

it may be verified readily that $\Phi^*K[U]\Phi \equiv K_1[\Phi^*U\Phi]$, and consequently we have the following result.

Corollary. *For the coefficient matrices of (2.1) defined by*

$$(7.12) \quad A = G_1 - H^*F_1^{-1}H_1, \quad B = F_1^{-1}H_1, \quad C = -F_1^{-1}, \quad D = H_1^*F_1^{-1} = B^*$$

and F_1 , G_1 , H_1 given by (7.11), the condition $\mathfrak{D}^+$ is equivalent to $\mathfrak{D}_0^+$, and this condition is equivalent to the condition that $K[-\Phi^{*-1}\Phi^{-1}] \leq 0$ and $K[\Phi^{*-1}\Phi^{-1}] \geq 0$ on I^c .

In connection with the result of this corollary, it is to be emphasized that if $W(y)$ is any $n \times n$ hermitian positive definite matrix function which is con-

tinuous and has piecewise continuous derivatives on this interval, then there exists a non-singular matrix function $\Phi(y)$ with similar continuity and differentiability properties and such that $\Phi(y)\Phi^*(y) \equiv W^{-1}(y)$ on I (see REID [15; Section 4]); indeed, the most general such $\Phi(y)$ is of the form $V(y)\Theta(y)$, where $V(y)$ is the positive definite hermitian square root matrix of $W^{-1}(y)$ and $\Theta(y)$ is a unitary matrix which is continuous and has piecewise continuous derivatives on I . Consequently, whenever $W(y)$ is positive definite and satisfies $K[-W] \leq 0$, $K[W] \geq 0$ on I° , for any such $\Phi(y)$ we have $\Phi^*W\Phi \equiv E$ on I , so that for F_1, G_1, H_1 defined by (7.11) and A, B, C, D by (7.12) the condition $\mathfrak{D}_0^+$ is satisfied.

Finally, if $P = P(x, y; K)$, $Q = Q(x, y; K)$ is the solution of (3.1) satisfying $P(x, x; K) = E$, $Q(x, x; K) = K$, the property $\mathcal{O}^+ \{ \mathcal{O}^- \}$ implies that $P(x, y; K)$ is non-singular on $J^+ \{ J^- \}$ whenever $|K| < 1$. In particular, if the hermitian matrix $F(y)$ is positive definite on I° and K is hermitian, then the non-singularity of $P(x, y; K)$ on $J^+ \{ J^- \}$ may be phrased as follows (see, for example, REID [14; 17] for terminology): *The conjoined family of solutions of (3.1) with base $(P(x, y; K); Q(x, y; K))$ has no focal point for $y > x, y \in I \{ y < x, y \in I \}$.* Consequently, for such systems (3.1) the result of conclusion (i) of Theorem 2.2 provides a sufficient condition for the non-existence of focal points for conjoined families of solutions of (3.1) with K an hermitian matrix having $|K| < 1$. Moreover, in view of the comparison theorems of the Morse generalization of the Sturmian theory (see, for example, MORSE [7; Sec. 11], MORSE [8; Ch. IV], REID [14], and REID [17; Sec. 4]), only one of the inequalities in each of the relations (7.6), (7.6') is significant for the non-existence of focal points, and we have the following result.

Theorem 7.2. *If $F(y)$ is positive definite on I° , K is an hermitian matrix with $|K| < 1$, and*

$$G \geq (H^* + E)F^{-1}(H + E) \quad \{ G \geq (H^* - E)F^{-1}(H - E) \}$$

holds for $y \in I^\circ$, then for each $x \in I$ the conjoined family of solutions of (3.1) with base $(P(x, y; K); Q(x, y; K))$ has no focal point for $y > x, y \in I \{ y < x, y \in I \}$.

Although the author is not aware of any previous mention of this specific criterion for the non-existence of focal points, it is to be remarked that quite independent of the discussion of the present paper this criterion may be established by methods of the Morse generalization of Sturmian methods. For example, if

$$G \geq (H^* + E)F^{-1}(H + E)$$

on I° then the integrand function of (7.1) is not less than

$$[F\eta' + (H + E)\eta]^*F^{-1}[F\eta' + (H + E)\eta] - [\eta^*\eta]',$$

and for $x \in I$ the conclusion that for K an hermitian matrix with $|K| < 1$ the conjoined family of solutions of (3.1) with base $(P(x, y; K); Q(x, y; K))$ has no focal point on $y > x, y \in I$, follows from the fact that if $y_2 > x, y_2 \in I$, and $\eta(y)$

is a piecewise continuously differentiable vector function on $x \leq y \leq y_2$ with $\eta(y_2) = 0$, then

$$\eta^*(x)K\eta(x) + \mathcal{G}(\eta; x, y_2) \geq \eta^*(x)[K + E]\eta(x) \geq 0,$$

and the equality signs hold only if $\eta(y) \equiv 0$ on $x \leq y \leq y_2$.

8. A theorem of Redheffer for the Mycielski-Paszowski diffusion problem.

Recently REDHEFFER [12] has employed the methods of REID [16] to show that the "scattering matrix" of a generalized Mycielski-Paszowski diffusion problem satisfies a system of matrix differential equations, and with the aid of algebraic methods analogous to those of his earlier paper [11] he has obtained necessary and sufficient conditions for this scattering matrix to be dissipative in a certain sense. In this section it will be shown that analytic methods analogous to those of Sections 2-5 may be employed to obtain the main theorem of [12].

For a general real $m \times n$ matrix $||M_{\alpha\beta}||$ we shall use the symbol $M \cdot \geq \cdot 0$ to signify that $M_{\alpha\beta} \geq 0$ ($\alpha = 1, \dots, m; \beta = 1, \dots, n$). If M is a real square matrix $||M_{\alpha\beta}||$ ($\alpha, \beta = 1, \dots, n$), the symbol $M^* \geq \cdot 0$ will signify that $M_{\alpha\beta} \geq 0$ for $\alpha \neq \beta$, $\alpha, \beta = 1, \dots, n$. The symbols $M \cdot \leq \cdot 0$ and $M^* \leq \cdot 0$ will be used to denote the respective conditions $(-M) \cdot \geq \cdot 0$ and $(-M)^* \geq \cdot 0$. If M is an $m \times n$ matrix, θM will denote the corresponding $1 \times n$ matrix $||N_{1\beta}||$ with $N_{1\beta} = \sum_{\alpha=1}^m M_{\alpha\beta}$ ($\beta = 1, \dots, n$). In the following Γ will denote the convex set of n -dimensional real vectors $z \equiv (z_\alpha)$ with $z_\alpha \geq 0$ ($\alpha = 1, \dots, n$) and $z_1 + \dots + z_n \leq 1$. Finally, a real $n \times n$ matrix M will be said to have property Δ (dissipative in the sense of REDHEFFER [12]) if $Mz \in \Gamma$ for arbitrary $z \in \Gamma$. It is seen readily that M has property Δ if and only if $M \cdot \geq \cdot 0$ and $\theta(M - E) \cdot \leq \cdot 0$.

The following preliminary results will be concerned with solutions of the Riccati matrix equation

$$(8.1) \quad V_y = A(y) + D(y)V + VB(y) + VC(y)V,$$

where $A(y)$, $B(y)$, $C(y)$, $D(y)$ are real-valued piecewise continuous $n \times n$ matrix functions of the real variable y on an interval I on the real line. As in the preceding sections, I^c will denote the subset of I at which all the coefficient matrices A, B, C, D are continuous, and J^+ will signify the subset of $J = I \times I$ on which $y \geq x$.

Lemma 8.1. *If $V(y)$ is a solution of (8.1) on a subinterval $I_0 : y_1 \leq y \leq y_2$ of I such that $V(y_1) \cdot \geq \cdot 0$ and*

$$(8.2) \quad A(y) \cdot \geq \cdot 0, \quad C(y) \cdot \geq \cdot 0, \quad B(y)^* \geq \cdot 0, \quad D(y)^* \geq \cdot 0 \quad \text{for } y \in I_0 \cap I^c,$$

then $V(y) \cdot \geq \cdot 0$ on I_0 .

Whenever (8.2) holds there exists a piecewise continuous real scalar function $\rho(y)$ such that

$$B(y) + \frac{1}{2}\rho(y)E \cdot \geq \cdot 0, \quad D(y) + \frac{1}{2}\rho(y)E \cdot \geq \cdot 0;$$

for instance, $\rho(y)$ may be chosen as $2 \sum_{\alpha=1}^n (|D_{\alpha\alpha}(y)| + |B_{\alpha\alpha}(y)|)$. Then

$$W(y) = \sigma(y) V(y) \quad \text{with} \quad \sigma(y) = \exp \left\{ \int_{y_1}^y \rho(s) ds \right\}$$

is a solution of

$$W_v = A_1(y) + D_1(y)W + WB_1(y) + WC_1(y)W$$

with

$$\begin{aligned} A_1(y) &= \sigma(y)A(y), & B_1(y) &= B(y) + \tfrac{1}{2}\rho(y)E, \\ C_1(y) &= \frac{1}{\sigma(y)}C(y), & D_1(y) &= D(y) + \tfrac{1}{2}\rho(y)E, \end{aligned}$$

and $W(y) \cdot \geq \cdot 0$ whenever $V(y) \cdot \geq \cdot 0$. Moreover, in view of (8.2) we have

$$A_1(y) \cdot \geq \cdot 0, \quad B_1(y) \cdot \geq \cdot 0, \quad C_1(y) \cdot \geq \cdot 0, \quad D_1(y) \cdot \geq \cdot 0$$

on $I_0 \cap I^c$. If $y_0 \notin I_0$ with $W(y_0) \cdot \geq \cdot 0$ and $W^{(j)}(y)$ ($j = 0, 1, \dots$) denotes the sequence of Picard iterates for (8.1) with $W^{(0)}(y) \equiv W(y_0)$, it follows readily that $W^{(j)}(y) \cdot \geq \cdot 0$ for $y \geq y_0$, $y \notin I_0$. The local convergence of $\{W^{(j)}(y)\}$ to $W(y)$ then implies that there is a subinterval $y_0 \leq y \leq y_0 + \delta$ of I_0 throughout which $W(y) \cdot \geq \cdot 0$. Moreover, if $W(y) \cdot \geq \cdot 0$ for $y_1 \leq y < y_3$ and $y_3 \notin I_0$, then $W(y_3) \cdot \geq \cdot 0$ by continuity. Hence the set of values $y_3 \notin I_0$ such that $W(y) \cdot \geq \cdot 0$ on $y_1 \leq y \leq y_3$ is both open and closed relative to I_0 , and consequently throughout I_0 we have $W(y) \cdot \geq \cdot 0$ and $V(y) \cdot \geq \cdot 0$.

The following lemma may be established by an argument similar to that used in the above proof of Lemma 8.1, but simpler in detail in view of the linearity of the involved differential equation.

Lemma 8.2. Suppose that $A_2(y)$, $B_2(y)$ are real-valued piecewise continuous $n \times n$ matrix functions on an interval $I_0 : y_1 \leq y \leq y_2$ on the real line, and $Z(y)$ is the $n \times n$ matrix solution of

$$(8.3) \quad Z_v = A_2(y) + ZB_2(y), \quad Z(y_1) = 0.$$

If $\theta A_2(y) \cdot \leq \cdot 0$ and $B_2(y) * \geq * 0$ for all values $y \in I_0$ at which $A_2(y)$ and $B_2(y)$ are continuous, then $\theta Z(y) \cdot \leq \cdot 0$ on I_0 .

Theorem 8.1. For $x \in I$, let $V(x, y)$ be the solution of (8.1) satisfying $V(x, x) = E$. If $V(x, y)$ exists and has property Δ on J^+ , then the matrix $T(y) = A(y) + B(y) + C(y) + D(y)$ satisfies

$$(8.4) \quad T(y) * \geq * 0, \quad \theta T(y) \cdot \leq \cdot 0 \quad \text{for} \quad y \in I^c.$$

Moreover, if for $y \in I^c$ we have

$$(8.5) \quad A \cdot \geq \cdot 0, \quad C \cdot \geq \cdot 0, \quad B * \geq * 0, \quad D * \geq * 0, \quad \theta(D + C) \cdot \leq \cdot 0, \quad \theta(A + B) \cdot \leq \cdot 0,$$

then $V(x, y)$ exists and has property Δ on J^+ .

Since the condition that $V(x, y)$ has property Δ on J^+ is equivalent to $V(x, y) \cdot \geq \cdot 0$ and $\theta[V(x, y) - E] \cdot \leq \cdot 0$ on J^+ , the necessity of (8.4) is an immediate consequence of the fact that $V(x, x) = E$ and $V_x(x, x) = T(x)$ for $x \in I^c$. Now suppose that conditions (8.5) hold on I^c , and for a given $x \in I$ let I_x denote the maximal subinterval of I with left-hand end-point $y = x$ on which the solution $V(x, y)$ exists. In view of the first four relations of (8.5), it follows from Lemma 8.1 that $V(x, y) \cdot \geq \cdot 0$ on I_x . Moreover, from (8.1) it follows that on I_x the matrix function $Z = V(x, y) - E$ is the solution of (8.3) with

$$A_2(y) = A(y) + B(y) + [D(y) + C(y)]V(x, y), \quad B_2(y) = B(y) + C(y)V(x, y).$$

Since $V(x, y) \cdot \geq \cdot 0$ on I_x , the conditions

$$C \cdot \geq \cdot 0, \quad B^* \geq \cdot 0, \quad \theta(D + C) \cdot \leq \cdot 0, \quad \theta(A + B) \cdot \leq \cdot 0$$

of (8.5) imply $\theta A_2 \cdot \leq \cdot 0$ and $B_2^* \geq \cdot 0$ on I_x , and from Lemma 8.2 it follows that $\theta[V(x, y) - E] \cdot \leq \cdot 0$ on I_x . That is, $V(x, y)$ has property Δ on I_x . Finally, use of the extension theorem as in Lemmas 3.2 and 3.3 yields the result that I_x contains all $y \in I$ with $y \geq x$, thus completing the proof that $V(x, y)$ exists and has property Δ on J^+ .

The main theorem of REDHEFFER [12] is concerned with a system of matrix differential equations

$$\begin{aligned} (8.6) \quad P_v &= P(K + \lambda Q), & p_v &= (k + Q\lambda)p, \\ Q_v &= \Lambda + kQ + QK + Q\lambda Q, & q_v &= P\lambda p, \\ R_v &= M + \mu Q + RK + R\lambda Q, & r_v &= (R\lambda + \mu)p, \end{aligned}$$

where the coefficient matrices $\lambda(y)$, $k(y)$, $\mu(y)$, $\Lambda(y)$, $K(y)$, $M(y)$ are $n \times n$ real-valued matrix functions which are piecewise continuous on an interval I on the real line, and the solution matrices P , Q , R , p , q , r are $n \times n$ matrix functions such that the $3n \times 3n$ matrix

$$(8.7) \quad S(x, y) = \begin{bmatrix} p & Q & 0 \\ q & P & 0 \\ r & R & E \end{bmatrix}$$

satisfies the initial condition $S(x, x) = E$. Corresponding to preceding terminology, in reference to (8.6), (8.7) the symbol I^c will denote the subset of I on which all the coefficient matrices λ , k , μ , Λ , K , M are continuous. As an immediate consequence of Theorem 8.1 we have for (8.6), (8.7) the following result, which is the main theorem of [12] amplified by the existential property of $S(x, y)$ on J^+ .

Theorem 8.2. *A necessary and sufficient condition for $S(x, y)$ to exist and have property Δ on J^+ is that the $3n \times 3n$ matrix*

$$(8.8) \quad \mathfrak{C}(y) = \begin{vmatrix} k & \Lambda & 0 \\ \lambda & K & 0 \\ \mu & M & 0 \end{vmatrix}$$

be such that

$$(8.9) \quad \mathfrak{C}(y) * \geq * 0, \quad \theta \mathfrak{C}(y) \cdot \leq \cdot 0 \quad \text{for } y \in I^c.$$

Indeed, to deduce the result of Theorem 8.2 from Theorem 8.1 it suffices to note that $\mathfrak{s} = \mathfrak{s}(x, y)$ is the solution of

$$\mathfrak{s}_y = A(y) + D(y)\mathfrak{s} + \mathfrak{s}B(y) + \mathfrak{s}C(y)\mathfrak{s}, \quad \mathfrak{s}(x, x) = E,$$

where A, B, C, D are now the $3n \times 3n$ matrix functions

$$A = \begin{vmatrix} 0 & \Lambda & 0 \\ 0 & 0 & 0 \\ 0 & M & 0 \end{vmatrix}, \quad B = \begin{vmatrix} 0 & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad C = \begin{vmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad D = \begin{vmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ \mu & 0 & 0 \end{vmatrix}.$$

Moreover, the corresponding matrix $T = A + B + C + D$ is $\mathfrak{C}$, the condition $\mathfrak{C}(y)* \geq * 0$ is equivalent to the set of conditions

$$A \cdot \geq \cdot 0, \quad C \cdot \geq \cdot 0, \quad B * \geq * 0, \quad D * \geq * 0,$$

and the condition $\theta \mathfrak{C}(y) \cdot \leq \cdot 0$ is equivalent to the set of conditions

$$\theta(A + B) \cdot \leq \cdot 0, \quad \theta(D + C) \cdot \leq \cdot 0.$$

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