*Supplementary Note on Matrix Riccati Equations*¹

RAYMOND REDHEFFER

Communicated by E. HILLE

This supplements the author's *Inequalities for a Matrix Riccati Equation* cited as Reference 2. To avoid dilution by repetition, the present paper is written as though the reader has the other before him.

1. A correction. It was kindly pointed out by REID [1] that the last sentence in Theorem XII of [2] is incorrect as it stands but becomes correct if one deletes the words "with $\rho = 0$ ". REID's proof of the latter fact (communicated privately to the author) depends on ideas different from those of [2]. Here it is shown that the method of [2] also yields the correct result, provided algebraic errors are avoided.

As on p. 367, the condition " $VV^* = 1$ whenever $KK^* = 1$ " gives

$$(1) \quad (AV^* + D + VBV^* + VC) + (\text{same})^* = 0$$

for all unitary matrices V . The choice $V = \pm 1$ leads to

$$(A + C^*) + (A^* + C) = 0, \quad (B + B^*) + (D + D^*) = 0,$$

whereas² the choice $V = \pm i$ gives

$$(A + C^*) - (A^* + C) = 0, \quad (B + B^*) + (D + D^*) = 0.$$

Hence (1) becomes

$$(D + D^*) - V(D + D^*)V^* = 0,$$

so that $D + D^*$ commutes with the arbitrary unitary matrix V . Therefore $D + D^*$ must be a scalar matrix ρ . This gives the desired result

$$(2) \quad \begin{bmatrix} D & A \\ C & B \end{bmatrix} + \begin{bmatrix} D & A \\ C & B \end{bmatrix}^* = \begin{bmatrix} \rho & 0 \\ 0 & -\rho \end{bmatrix}.$$

¹ The preparation of this paper was sponsored by the Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the United States Government.

² The error arose from a wrong sign in the equation for $V = -i$.

Conversely, if (2) holds, then the matrix $Y = 1 - VV^*$ satisfies

$$\frac{\partial Y}{\partial y} = Y(D^* + C^*V^*) + (D + VC)Y - \rho Y$$

and $Y \equiv 0$ by uniqueness, as in [2].

2. A simplified proof. Theorem 3.1. of REID's paper [3] gives four functional equations, which not only are interesting in themselves but have technological applications. It seems desirable to give as simple a proof as possible. Theorems I and II of [2] take a step in this direction, but there the emphasis is on a comparison of REID's equations with the closure condition for U alone. If only REID's equations are desired, the following procedure is still simpler.

Recall that REID's matrix $H(x, y)$ is nonsingular and satisfies

$$H_y(x, y) = G(y)H(x, y), \quad H(x, x) = 1.$$

It follows that the inverse matrix $L = H^{-1}$ satisfies

$$\frac{\partial}{\partial z} L(y, z)L^{-1}(x, z) = 0$$

and hence the familiar relation

$$(3) \quad L(x, y)L(y, z) = L(x, z).$$

By Lemma 2.1 of [3],

$$L = \begin{bmatrix} R - WS^{-1}U & WS^{-1} \\ -S^{-1}U & S^{-1} \end{bmatrix}.$$

Now it was shown in [2], Lemma 1, that the functional equations are equivalent to

$$(4) \quad M(x, z) = M(x, y)*M(y, z)$$

where

$$M = \begin{bmatrix} S & U \\ W & R \end{bmatrix}$$

and where the star product is as there defined. Since $M \leftrightarrow L$ is an isomorphism of matrix algebra under star multiplication with that under ordinary multiplication, (3) gives (4) and, thereby, the functional equations.³

³ For $n = 1$ the fact that the functional equations are equivalent to (3) was already pointed out in [4].

3. The isomorphism. The isomorphism was established by direct computation in [2], but the following method gives more insight. The equation

$$\begin{bmatrix} V_1 \\ V_4 \end{bmatrix} = \begin{bmatrix} S & U \\ W & R \end{bmatrix} \begin{bmatrix} V_3 \\ V_2 \end{bmatrix}$$

can be written equally well as

$$\begin{bmatrix} V_4 \\ V_3 \end{bmatrix} = \begin{bmatrix} R - WS^{-1}U & WS^{-1} \\ -S^{-1}U & S^{-1} \end{bmatrix} \begin{bmatrix} V_2 \\ V_1 \end{bmatrix},$$

the matrix S being nonsingular. The relations defining the star product, (21) of [2], now correspond to similar relations defining a matrix product:

$$\begin{aligned} \begin{bmatrix} V_1 \\ V_4 \end{bmatrix} &= M_2 \begin{bmatrix} V_3 \\ V_2 \end{bmatrix} \leftrightarrow \begin{bmatrix} V_4 \\ V_3 \end{bmatrix} = L_2 \begin{bmatrix} V_2 \\ V_1 \end{bmatrix}, \\ \begin{bmatrix} V_3 \\ V_6 \end{bmatrix} &= M_1 \begin{bmatrix} V_5 \\ V_4 \end{bmatrix} \leftrightarrow \begin{bmatrix} V_6 \\ V_5 \end{bmatrix} = L_1 \begin{bmatrix} V_4 \\ V_3 \end{bmatrix}, \\ \begin{bmatrix} V_1 \\ V_6 \end{bmatrix} &= M_1 * M_2 \begin{bmatrix} V_5 \\ V_2 \end{bmatrix} \leftrightarrow \begin{bmatrix} V_6 \\ V_5 \end{bmatrix} = L_1 L_2 \begin{bmatrix} V_2 \\ V_1 \end{bmatrix}. \end{aligned}$$

Because the vectors V_i are unrestricted, these transformations wholly determine the corresponding matrices. Hence $M \leftrightarrow L$ is an isomorphism.

4. The star product generalized. If a one-to-one correspondence $M \leftrightarrow L$ is denoted by $L = f(M)$, we can define an associative star product by

$$M_1 * M_2 = f^{-1}[f(M_1)f(M_2)].$$

It is easy to see that the particular star product of [2] is generated by

$$f(M) = [(1 - M) + (1 + M)J]^{-1}[(M - 1) + (M + 1)J]$$

where

$$J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The mapping f is an involution; that is, $H = f(M)$ implies $M = f(H)$, so that $f = f^{-1}$. The relations

$$f(M^{-1}) = [f(M)]^{-1}, \quad f(1) = 1$$

agree with the fact that the star-product identity and inverse are the same as the matrix-product identity and inverse. In terms of Cayley transforms, f admits the decomposition $f = f_3 f_2 f_1$ where

$$\begin{aligned} f_1(M) &= (M + 1)^{-1}(M - 1), \\ f_2(M) &= Jf_1(M), \\ f_3(M) &= [1 - f_2(M)]^{-1}[1 + f_2(M)]. \end{aligned}$$

By specifying not the precise form but only suitable properties of f , one can map the linear differential system onto a variety of nonlinear ones admitting a theory like that of [2] and [3].

REFERENCES

- [1] W. T. REID, Properties of solutions of a Riccati matrix differential equation, Abstract 559-99, *Am. Math. Soc. Notices* **6** (Aug. 1959), 438.
- [2] R. M. REDHEFFER, Inequalities for a matrix Riccati equation, *J. Math. Mech.* **8** (1959), 349-367.
- [3] W. T. REID, Solutions of a Riccati matrix differential equation as functions of initial values, *J. Math. Mech.* **8** (1959), 221-230.
- [4] R. M. REDHEFFER, On the solutions of Riccati's equation as functions of the initial values, *J. Rat. Mech. Anal.* **5** (1956), 835-848.

University of California
Los Angeles, California