*Proper Time, Apparent Time, and Formal Time in the Twin Paradox**

V. HLAVATÝ

Introduction. In this paper we shall study in more detail some phenomena connected with the so-called twin paradox,† distinguishing strictly between the *purely geometrical aspect* of this problem and its *physical application*. Using the non-relativistic language (which is not adequate) we may describe the paradox as follows (Fig. 1). While one observer P_1 stays at o , the other observer P_2 first recedes from P_1 , till h , and then returns to o . This motion is uniform (rectilinear) and the velocity in the $\begin{cases} \text{first} \\ \text{second} \end{cases}$ phase of the experiment is $\begin{cases} v = \text{Tg } \omega \\ -v = -\text{Tg } \omega. \end{cases}$ If (P_i) ($i = 1, 2$) denotes the biological time spent by P_i during the whole experiment then

$$(A) \qquad (P_1) = (P_2) \text{Cos } \omega.$$

In this paper we shall consider only the geometrical aspect of the paradox. From the point of view of geometry the only paradoxical thing about this problem is its name. In fact, in the geometrical interpretation of (A) the symbol (P_i) stands for the proper time of P_i , i.e., for the Lorentz length of the world line w_i of P_i . Then (A) is simply a consequence of the elementary statement that for the Lorentz lengths (ab) , (bc) , (ca) of the edges of a time-like triangle abc we have

$$(ab) > (bc) + (ca).$$

This is a fundamental relation in the Lorentz geometry, and there is nothing paradoxical about it.

On the other hand, there is something deeper in this problem than meets the eye on first glance. From the relativistic point of view the statement that

* This paper was prepared with the support of the Aeronautical Research Laboratory, Wright Air Development Center.

† There is a recent comprehensive article describing this paradox, A. SCHILD, The clock paradox in relativity theory, *American Mathematical Monthly*, **66**, pp. 1–18, 1959.

the event on the world line of $\left\{P_i\right\}$ which is simultaneous with $\left\{E_i\right\}$ from the point of view of $\left\{P_i\right\}$. Then

$$(C) \quad \frac{[E_i^i]_i}{[E_i]_i} = \frac{[E_i^i]_i}{[E_i]_i} = \text{Cos } \Omega(t_i)$$

where t_i is the first coordinate of $E_i = E_i^i$ in the inertial frame of reference of P_i . In the first phase of the experiment we have

$$(C') \quad \Omega = \omega,$$

so that (C) and (C') describe analytically the known statement that each observer sees the other observer's clock *slowing down* by a constant factor $1/\text{Cos } \omega$. In the second phase where P_i changed his inertial frame of reference, Ω is a decreasing function of t_i :

$$(C'') \quad \omega \geq \Omega \geq 0.$$

The formulae (C), (C'') describe analytically the fact that each observer sees the other observer's clock* accelerating so that when they finally meet their clocks are synchronized.

Let us return to the formula (A). One switches from the geometrical interpretation to the physical application by identifying the (geometrically defined) proper time with the biological time. Although some interesting conclusions could be drawn from this identification we shall not deal with them, because in this paper *we are interested only in the geometrical aspect of the problem*. In other words, *we shall investigate only the geometrical invariants* without attempting to identify them with physical (biological) notions.

* * *

The paper contains three sections. The first section deals with the necessary prerequisites. The second section is devoted to the investigations of the proper time and of the apparent time. In the third section we study the formal time.

Throughout this paper we assume the velocity of light to be 1 and we confine ourselves to the Lorentz plane described by the coordinates t and x . These agreements do not restrict the generality of the problem.

The indices i, j take the values 1, 2. If not stated otherwise, we assume $i \neq j$.

The symbols Sin, Cos, Tg denote hyperbolic functions.

1. Prerequisites. I. Let f be an inertial frame of reference, described by means of the coordinates t, x . Let P be an observer at rest with respect to f . Assume that at the time τ the observer P sends a light signal which reaches an event $E(\tau_0, \xi_0)$ and comes back at the time τ' . The world lines of these signals are†

$$t - \tau_0 = \epsilon(x - \xi_0), \quad \epsilon = \pm 1,$$

* Of course, we assume that these clocks indicate the formal times under consideration.

† The velocity of light is assumed to be 1.

so that

$$\tau = \tau_0 - \xi_0, \quad \tau' = \tau_0 + \xi_0.$$

The observer P who reads the times τ and τ' is able to find out the coordinates τ_0, ξ_0 of E :

$$(1.1) \quad \tau_0 = \frac{1}{2}(\tau' + \tau), \quad \xi_0 = \frac{1}{2}(\tau' - \tau),$$

and, in particular, he is able to define the event $E'(\tau_0, 0)$ which is f -simultaneous* with E and incident with his world line.

II. Let an observer P_i be at rest with an inertial frame f_i described by means of cartesian coordinates t_i, x_i ($i = 1, 2$). Assume that f_1 and f_2 have the common origin $o(0, 0)$. Then either one of the observers P_1 and P_2 recedes from the other by a rectilinear motion of the constant velocity $v = \text{Tg } \omega$. The relationship of the two frames is given by

$$(1.2) \quad \begin{aligned} t_i &= t_j \cos \omega - x_j \sin \omega, \\ x_i &= t_j \sin \omega - x_j \cos \omega, \end{aligned} \quad i, j = 1, 2,$$

so that the world line w_i of P_i has the equations

$$(1.3) \quad \begin{aligned} (a) \quad x_i &= 0 && \text{in } f_i, \\ (b) \quad x_i &= t_i \text{Tg } \omega && \text{in } f_i. \end{aligned}$$

Let $h_i \neq O$ be an event on w_i and on the isochrone hyperbola

$$t_i^2 - x_i^2 = k^2 \quad (= t_i^2 - x_i^2), \quad k = \text{real const.} \neq 0.$$

Then the coordinates of h_i are

$$(1.4) \quad \begin{aligned} (a) \quad h_i &(k, 0) && \text{in } f_i. \\ (b) \quad h_i &(k \cos \omega, k \sin \omega) && \text{in } f_i. \end{aligned}$$

Denote by $\begin{Bmatrix} h_i^i \\ h_i^i \end{Bmatrix}$ the $\begin{Bmatrix} f_i^- \\ f_i^- \end{Bmatrix}$ -simultaneous event with h_i on w_i . Then the coordinates of these events are

$$(1.5) \quad \begin{aligned} (a) \quad h_i^i &(k, k \text{Tg } \omega) && \text{in } f_i, \\ (b) \quad h_i^i &(k/\cos \omega, 0) && \text{in } f_i. \end{aligned}$$

and

$$(1.6) \quad \begin{aligned} (a) \quad h_i^i &(k \cos^2 \omega, k \sin \omega \cos \omega) && \text{in } f_i, \\ (b) \quad h_i^i &(k \cos \omega, 0) && \text{in } f_i. \end{aligned}$$

* By f -simultaneity we understand the simultaneity with respect to the frame f . All f -simultaneous events with $E(\tau_0, \xi_0)$ are incident with the line parallel with the t -axis through E .

III. Assume that in h_i the observer P_i changes his frame of reference f_i into f_i^* , reversing the velocity $v = \text{Tg } \omega$ into $v = -\text{Tg } \omega$. His new world line w_i^* has the equation

$$(1.7a) \quad x_i \cos \omega + t_i \sin \omega = 2k \sin \omega \cos \omega \quad \text{in } f_i$$

with the t_i -intercept

$$(1.8a) \quad e_i(2k \cos \omega, 0) \quad \text{in } f_i.$$

The world line w_i^* is incident with the t^* -axis of f_i^* . The x^* -axis of f_i^* is incident with

$$(1.8b) \quad o^*(0, 2k \sin \omega) \quad \text{in } f_i$$

and parallel to the tangent of the hyperbola

$$t_i^2 - (x_i - 2k \sin \omega)^2 = k^2.$$

at h_i . Hence the equation of the x^* -axis is

$$(1.7b) \quad t_i \cos \omega + x_i \sin \omega = 2k \sin^2 \omega \quad \text{in } f_i,$$

and the relationship between f_i^* and f_i is described by

$$(1.9a) \quad t_i^* = t_i \cos \omega + x_i \sin \omega - 2k \sin^2 \omega,$$

$$x_i^* = -t_i \sin \omega - x_i \cos \omega + 2k \sin \omega \cos \omega,$$

$$(1.9b) \quad t_i = t_i^* \cos \omega + x_i^* \sin \omega,$$

$$x_i = -t_i^* \sin \omega - x_i^* \cos \omega + 2k \sin \omega,$$

so that the equation of w_i is

$$(1.10) \quad t_i^* \sin \omega + x_i^* \cos \omega = 2k \sin \omega \quad \text{in } f_i^*.$$

Moreover, we have from (1.4b) and (1.9a)

$$(1.11) \quad h_i(k, 0) \quad \text{in } f_i^*,$$

as was to be expected. If h_i^{i*} is the f_i^* -simultaneous event with h_i on w_i , then we have from (1.9) and (1.10)

$$(1.12) \quad h_i^{i*}(k, k \text{Tg } \omega) \quad \text{in } f_i^*.$$

Note that h_i^i and h_i^{i*} have the same coordinates in f_i and f_i^* respectively. Besides this result we shall need later the following one. If $\begin{Bmatrix} t_i^* \\ t_i \end{Bmatrix}$ is the first coordinate in $\begin{Bmatrix} f_i^* \\ f_i \end{Bmatrix}$ of an event on w_i^* , then we obtain from (1.9b) and (1.7a)

$$(1.13) \quad t_i^* = \frac{t_i}{\cos \omega}.$$

Note that the event e_i , incident with w_i^* and w_i , has also the coordinates

$$(1.14) \quad e_i(2k, 0) \quad \text{in} \quad f_i^*.$$

IV. If instead of (1.9) we use the transformation

$$(1.15) \quad t_i^* = t_i, \quad x_i^* = x_i,$$

then the observer P_i does not change his frame and therefore the equations (1.15) and (1.10) define a new world line w_i' :

$$(1.16) \quad t_i \sin \omega + x_i \cos \omega = 2k \sin \omega \quad \text{in} \quad f_i.$$

This world line intersects w_i in the element e_i

$$(1.17) \quad \begin{aligned} \text{(a)} \quad & e_i(2k, 0) \quad \text{in} \quad f_i, \\ \text{(b)} \quad & e_i(2k \cos \omega, 2k \sin \omega) \quad \text{in} \quad f_i. \end{aligned}$$

Hence e_i and e_i are f_i -simultaneous. Moreover, e_i and e_i have the same coordinates in f_i^* and f_i respectively. The intersection of w_i' and w_i is the event h_i^i (see (1.5a)).

Remark. It follows from (1.17) that e_i is also the intersection of w_i and the world line $x_i = 2k \sin \omega$ incident with o^* .

2. Proper time and apparent time. I. Let

$$(2.1a) \quad \varphi(t, x) = 0$$

be the equation of a world line w in an inertial reference frame f which is described by the coordinates t, x . Let o, e be two events on w . Then

$$(oe) \stackrel{\text{def}}{=} \int_o^e \sqrt{dt^2 - dx^2}$$

is called *the proper time* of the segment oe of w . The proper time is clearly a Lorentz invariant.

Denote by f^* another inertial frame. Assume that there is an event h incident with w such that the event h^* , which has in f^* the same coordinates as h has in f , is also incident with w . Denote by

$$(2.1b) \quad \varphi^*(t^*, x^*) = 0$$

the equation of w in f^* and by P an observer whose world line is composed of the time axes of f and f^* . Then

$$\{oe\}_P \stackrel{\text{def}}{=} (oh) + (h^*e)^{**}$$

* Hence P is first at rest with f and then with f^* .

** Of course, we have

$$(h^*e) = \int_{h^*}^e \sqrt{(dt^*)^2 - (dx^*)^2}.$$

is termed *the apparent time* of the segment oe of w . This name is justified by the following consideration. If P does not realize the change $f \rightarrow f^*$, he identifies h with h^* , and therefore $\{oe\}_P$ appears to him to be the proper time of the segment oe of w . If and only if $f = f^*$ (which results in $h = h^*$) the apparent time $\{oe\}_P$ is equal to the proper time (oe) :

$$\{oe\}_P = (oe).$$

II. Let us now apply these notions to the observers P_1 and P_2 described in the first section (sub II). Then we have, according to (1.3b), (1.4b), and (1.6b),

$$(2.3a) \quad (oh_i) = \int_o^{h_i} \frac{dt_i}{\text{Cos } \omega} = k,$$

$$(2.4a) \quad (oh_i^i) = \int_o^{h_i^i} dt_i = k \text{ Cos } \omega.$$

Consider now the second phase of our experiment, described in the first section (sub III). Then we obtain, by virtue of (1.7a), (1.8a), (1.4b), and (1.6b),

$$(2.3b) \quad (h_i e_i) = \int_{h_i}^{e_i} \frac{dt_i}{\text{Cos } \omega} = k,$$

$$(2.4b) \quad (h_i^i e_i) = \int_{h_i^i}^{e_i} dt_i = k \text{ Cos } \omega.$$

Hence if we denote by $\left\{ \begin{smallmatrix} (P_i) \\ (P_i) \end{smallmatrix} \right\}$ the proper time spent by the observer $\left\{ \begin{smallmatrix} P_i \\ P_i \end{smallmatrix} \right\}$ during the whole experiment:

$$(2.5) \quad \begin{aligned} (P_i) &\stackrel{\text{def}}{=} (oh_i) + (h_i e_i), \\ (P_i) &\stackrel{\text{def}}{=} (oh_i^i) + (h_i^i e_i), \end{aligned}$$

we obtain

$$(2.6) \quad \begin{aligned} (a) \quad (P_i) &= 2k, \\ (b) \quad (P_i) &= 2k \text{ Cos } \omega \end{aligned}$$

and

$$(2.7) \quad \frac{(P_i)}{(P_i)} = \text{Cos } \omega.$$

The formula (2.7) expresses analytically the twin paradox. We shall see that *this result describes only a part of the problem*.

III. Let us consider now the same situation from the point of view of the observer P_i who is in the $\left\{ \begin{smallmatrix} \text{first} \\ \text{second} \end{smallmatrix} \right\}$ phase of the experiment at rest with the

inertial frame $\left\{f_i^*\right\}$. Then we have, by virtue of (1.5a), (1.3b) (with the interchanged role of i and j), (1.12), (1.10), and (1.14),

$$(2.8) \quad \begin{aligned} (a) \quad (oh_i^i) &= \int_o^{h_i^i} \frac{dt_i}{\text{Cos } \omega} = \frac{k}{\text{Cos } \omega}, \\ (b) \quad (h_i^{i*} e_i) &= \int_{h_i^{i*}}^{e_i} \frac{dt_i^*}{\text{Cos } \omega} = \frac{k}{\text{Cos } \omega}. \end{aligned}$$

If we define by

$$(2.9) \quad \{P_i\}_{P_i} \stackrel{\text{def}}{=} (oh_i^i) + (h_i^{i*} e_i)$$

the apparent time spent by the observer P_i during the whole experiment and by

$$\{P_i\}_{P_i} = (P_i)$$

the apparent time spent by the observer P_i during the whole experiment, we have by virtue of (2.8) and (2.6a)

$$(2.10) \quad \begin{aligned} (a) \quad \{P_i\}_{P_i} &= 2k, \\ (b) \quad \{P_i\}_{P_i} &= \frac{2k}{\text{Cos } \omega}, \\ (c) \quad \frac{\{P_i\}_{P_i}}{\{P_i\}_{P_i}} &= \text{Cos } \omega. \end{aligned}$$

On the other hand, we have obviously

$$\{P_i\}_{P_i} = (P_i), \quad \{P_i\}_{P_i} = (P_i),$$

so that (2.7) is equivalent to

$$(2.10d) \quad \frac{\{P_i\}_{P_i}}{\{P_i\}_{P_i}} = \text{Cos } \omega.$$

The interpretation of (2.10c, d) will be postponed till the end of the subsequent considerations.

IV. We are investigating here only the geometrical aspect of our problem, so that we have to disregard the effect of the instantaneous deceleration and acceleration of P_i at h_i . Hence the geometrical data of our problem are only two observers which first recede and then approach each other with the constant velocity. In other words, the only "landmark" left to the observer P_i is the observer P_i , and what P_i observes during the second phase of the experiment is that the distance of both observers is diminishing. Because he is at rest with respect to the inertial frame f_i^* , he does not know whether P_i approaches him or *vice versa*. This seems to indicate that P_i is not aware of the change $f_i \rightarrow f_i^*$. This is so if and only if the numerical results (2.10a, b) remain unchanged for $f_i = f_i^*$, i.e., for (1.15). If (1.15) holds then P_i sees in the second phase

of the experiment the world line of P_i as given by (1.16), *i.e.* by the segment $h_i^i e_i$. We have, by virtue of (1.16), (1.17a), and (1.5a, b),

$$(2.11a) \quad \begin{aligned} (oh_i^i) &= \int_o^{h_i^i} dt_i = \frac{k}{\text{Cos } \omega}, \\ (h_i^i e_i) &= \int_{h_i^i}^{e_i} \frac{dt_i}{\text{Cos } \omega} = \frac{k}{\text{Cos } \omega}, \end{aligned}$$

so that according to (2.10b)

$$(2.12a) \quad (oh_i^i) + (h_i^i e_i) = \frac{2k}{\text{Cos } \omega} = \{P_i\}_{P_i}.$$

On the other hand, by virtue of (1.4a) and (1.17a)

$$(2.11b) \quad (h_i e_i) = \int_{h_i}^{e_i} dt_i = k,$$

so that according to (2.3a) and (2.10a)

$$(2.12b) \quad (oh_i) + (h_i e_i) = 2k = \{P_i\}_{P_i}.$$

The equations (2.12) show that the apparent times of P_i and P_j , as seen by P_i , are the same whether P_i changes his frame of reference or not. Moreover, his proper time is also the same in both cases, namely $2k$. Hence P_i does not realize *on the basis of geometrical data* whether he changes his frame of reference or not.

Therefore the formulae (2.10c, d) describe analytically another part of the twin paradox. *Each observer sees the apparent time of the other observer smaller than his own by the factor $1/\text{Cos } \omega$.*

Note I. In the geometrical considerations of the special theory of relativity we can describe the relationship of the two observers only by stating that their distance increases (decreases), without being able to specify which one of the observers moves with respect to the other. Hence the above statement, symmetric with respect to both observers, corresponds to the spirit of the special relativity.

Note II. Let us stress that we do not identify here the biological time with the proper (or apparent) time. The proper (apparent) time is here simply a geometrical quantity (an invariant). Hence P_i cannot decide whether $\{P_i\}_{P_i}$ or (P_i) is the proper time if he does not know what his world line is, *i.e.* whether or not he changed his frame of reference during the experiment.

Note III. The equations (2.10c, d) yield two invariants, namely

$$(2.13) \quad \begin{aligned} \frac{\{P_i\}_{P_i}}{\{P_i\}_{P_i}} &= \frac{\{P_i\}_{P_i}}{\{P_i\}_{P_i}} = \text{Cos } \omega, \\ \{P_i\}_{P_i} \{P_i\}_{P_i} &= \{P_i\}_{P_i} \{P_i\}_{P_i} = \{P_i\}_{P_i} \{P_i\}_{P_i} \text{Cos } \omega. \end{aligned}$$

3. Formal time. I. Let us consider the situation described in the first section (sub I). The method displayed there permits us to find the coordinates τ_0 , ξ_0 of an event E . The proper time of the segment oE will be termed *the formal time* $[E]_f$ * of E :

$$(3.1a) \quad [E]_f \stackrel{\text{def}}{=} (oE) = \sqrt{(\tau_0)^2 - (\xi_0)^2} = \sqrt{\tau \tau'}.$$

We shall assume that there is a clock connected with E which shows its formal time, and we shall compare $[E]_f$ with the formal time $[E']_f$ of E' **

$$(3.1b) \quad [E']_f = \tau_0.$$

Hence

$$(3.2) \quad \frac{[E']_f}{[E]_f} = \frac{1}{\sqrt{1 - (\xi_0/\tau_0)^2}}.$$

II. Consider the situation described in the first section (sub II). Let $K < k$ be an arbitrary positive constant and H_i an event on w_i :

$$(3.3) \quad H_i(K \cos \omega, K \sin \omega) \quad \text{in} \quad f_i$$

and H_i^i an f_i -simultaneous event to H_i on w_i :

$$(3.4) \quad H_i^i(K \cos \omega, 0) \quad \text{in} \quad f_i.$$

The formal time of $\begin{Bmatrix} H_i \\ H_i^i \end{Bmatrix}$ will be denoted by $\begin{Bmatrix} [H_i]_i \\ [H_i^i]_i \end{Bmatrix}$.† Then we have, according to (3.2),

$$(3.5a) \quad \frac{[H_i^i]_i}{[H_i]_i} = \cos \omega,$$

so that this ratio is independent of the choice of H_i on w_i . Interchanging i and j in (3.5a) we obtain

$$(3.5b) \quad \frac{[H_j^i]_i}{[H_i]_i} = \cos \omega.$$

The equations (3.5) express analytically the known fact that in the first part of the experiment *either one of the observers sees the other observer's clock, compared to his own, slowing down by a factor $1/\cos \omega$.*

III. Consider now the second phase of the experiment and define

$$(3.6) \quad \begin{aligned} (a) \quad V(t_i) &\stackrel{\text{def}}{=} \text{Tg } \omega \left(\frac{2k \cos \omega}{t_i} - 1 \right), \quad 2k \geq \frac{t_i}{\cos \omega} \geq k, \\ (b) \quad X(t_i) &\stackrel{\text{def}}{=} \frac{1}{\sqrt{1 - V^2}}. \end{aligned}$$

* The subscript f refers to the origin o of f .

** E' is f -simultaneous with E and incident with the time axis of f .

† The subscript j refers to the origin of f_j . Note that we use the f_j -parallelism.

We see from (3.6) that we could put

$$(3.7) \quad \begin{aligned} (a) \quad & X(t_i) = \text{Cos } \Omega(t_i), \\ (b) \quad & X(2k \text{ Cos } \omega) = \text{Cos } \Omega(2k \text{ Cos } \omega) = 1, \\ (c) \quad & X(k \text{ Cos } \omega) = \text{Cos } \Omega(k \text{ Cos } \omega) = \text{Cos } \omega. \end{aligned}$$

Denote by $E_i(t_i, x_i)$ an event on w_i^* between h_i and e_i (these events included) and by $E_i^i(t_i, 0)$ its f_i -simultaneous event on w_i . Then we obtain, according to (1.7a),

$$(3.8a) \quad \frac{[E_i^i]_i}{[E_i]_i} = \text{Cos } \Omega(t_i), \quad 2k \geq \frac{t_i}{\text{Cos } \omega} \geq k.$$

Hence, according to (1.6a), (1.4b), (1.8a), and (3.7b, c),

$$(3.8b) \quad \frac{[h_i^i]_i}{[h_i]_i} = \text{Cos } \omega,$$

$$(3.8c) \quad \frac{[e_i^i]_i}{[e_i]_i} = 1.*$$

Let us consider the same situation from the point of view of P_i . During the second phase of the experiment P_i is at rest with the inertial frame of reference f_i^* the origin of which is o^* :

$$\begin{aligned} o^*(0, 2k \text{ Sin } \omega) & \text{ in } f_i, \\ o^*(0, 0) & \text{ in } f_i^*. \end{aligned}$$

Denote by $E_i(t_i^*, x_i^*)$ an event on w_i between h_i^{i*} and e_i (these events included) and by $E_i^{i*}(t_i^*, 0)$ its f_i^* -simultaneous event on w_i^* . When we have, by virtue of (1.10),

$$\begin{aligned} \frac{[E_i^{i*}]_{i*}}{[E_i]_{i*}} &= \frac{1}{\sqrt{1 - W^2(t_i^*)}} \dagger \\ W(t_i^*) &\stackrel{\text{def}}{=} \text{Tg } \omega \left(\frac{2k}{t_i^*} - 1 \right), \quad 2k \geq t_i^* \geq k, \end{aligned}$$

where t_i^* is the first coordinate of E_i^{i*} on w_i^* , so that according to (1.13)

$$W(t_i^*) = V(t_i)$$

where t_i is the first coordinate of E_i^{i*} in f_i . Hence

$$(a) \quad \frac{[E_i^{i*}]_{i*}}{[E_i]_{i*}} = \text{Cos } \Omega(t_i),$$

* We have obviously $e_i = e_i^i$.

† The subscript i^* in $[]_{i*}$ refers to the origin o^* of f_i^* .

$$(3.9) \quad (b) \quad \left(\frac{[E_i^{i*}]_{i*}}{[E_i]_{i*}} \right)_{t_i^*=k} = \frac{[h_i]_{i*}}{[h_i^{i*}]_{i*}} = \text{Cos } \omega,$$

$$(c) \quad \frac{[e_i^{i*}]_{i*}}{[e_i]_{i*}} = 1.*$$

The results (3.8) and (3.9) may be interpreted in the following way. *In the second phase of the experiment either one of the observers sees the other observer's clock (as compared to his own clock) accelerating with the increasing time so that when the observers meet again their clocks are synchronized.*

To avoid any misunderstanding let us stress that neither one of the formal times $[E_i]_i$ and $[E_i]_{i*}$ is a proper time, while each one of the formal times $[E_i^i]_i$, $[E_i^{i*}]_{i*}$ is also a proper time.

Remark I. In the first phase of the experiment we obtain from (3.5) two invariants, namely

$$(3.10a) \quad \frac{[H_i^i]_i}{[H_i]_i} = \frac{[H_i^i]_{i*}}{[H_i]_{i*}} = \text{Cos } \omega$$

and

$$(3.10b) \quad [H_i^i]_i[H_i]_{i*} = [H_i^i]_{i*}[H_i]_i = [H_i]_i[H_i]_{i*} \text{Cos } \omega.$$

The invariant (3.10a) is independent of the choice of $H_i \neq o$ and $H_i \neq o$. In the second phase of the experiment, if $E_i^{i*} = E_i$, the parameter t_i involved in (3.8a) and (3.9a) is the same, and therefore we obtain again two invariants

$$(3.10b) \quad \frac{[E_i^i]_i}{[E_i]_i} = \frac{[E_i^{i*}]_{i*}}{[E_i]_{i*}} = \text{Cos } \Omega(t_i)$$

and

$$(3.11b) \quad [E_i^i]_i[E_i]_{i*} = [E_i^{i*}]_{i*}[E_i]_i = [E_i]_i[E_i]_{i*} \text{Cos } \Omega(t_i)$$

Either one of these invariants depends on the choice of $E_i = E_i^{i*}$ on w_i^* .

Indiana University
Bloomington, Indiana

* We have $e_i = e_i^{i*}$.