The Characteristic System for Three-dimensional, Steady, Isentropic, Compressible, Hydromagnetic Flows

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1. Introduction. FRIEDRICHS & KRANZER [1] have derived the partial differential equation for the characteristic manifolds in hydromagnetic flows by use of the weak shock technique. In the present paper, we shall (1) study this equation in order to determine the properties of the normals to characteristic surfaces in the steady three-dimensional case and (2) determine, by use of these results, the characteristic system for steady, three-dimensional, compressible, isentropic, hydromagnetic flows.

The normal cone with vertex at any point P is studied by determining the projection onto the local plane formed by the velocity and magnetic field vectors at P of the intersection of the cone of normals and the unit sphere with center at P . The resulting locus L furnishes a geometric picture of the normal cone and is easily analyzed. It is shown that *the normal cone exists for all steady, three-dimensional, isentropic, compressible (or incompressible) hydromagnetic flows*. The locus L depends upon two positive scalars: M , the Mach number, and K^2 , the ratio of the magnetic and kinetic energies. Further, *there exist at least two distinct normals n^λ , $'n^\lambda$ which lie in the plane of the distinct velocity and magnetic field vectors at each point P , and there exists one normal k^λ orthogonal to this plane* in any steady, three-dimensional, compressible (or incompressible) hydromagnetic flow. These normals, n^λ , $'n^\lambda$, k^λ , determine three families of characteristic surfaces which will be used in formulating the characteristic system for three-dimensional hydromagnetic flows. The intersections of these characteristic surfaces lie along k^λ and l^λ , $'l^\lambda$, which are the bicharacteristics corresponding to n^λ , $'n^\lambda$, respectively, and lie in the local plane formed by the velocity and magnetic field vectors (v^λ and H^λ , respectively).

The characteristic system consists of three types of relations. First, the relations expressing $'n^\lambda$, n^λ , l^λ , $'l^\lambda$ in terms of v^λ , H^λ are determined. If *the de-*

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derivatives of the density are continuous on the initial surface and hence throughout the flow space, the relations between t^λ , $'t^\lambda$, n^λ , $'n^\lambda$, v^λ , H^λ depend on determining the roots of a biquadratic, and hence these relations are explicitly determined. When some of the derivatives of the density ρ are discontinuous, these relations depend on determining the roots of a biquartic and these relations are *not* explicitly determined. Second, the equations of continuity and motion and the Maxwell relations are expressed in terms of the directional derivatives of ρ , v^λ , H^λ , $\omega^{\lambda\mu}$ (the vorticity tensor), and $\mathbf{H}^{\lambda\mu}$ (the current bivector) along t^λ , $'t^\lambda$, k^λ . Finally, the integrability conditions for $\omega^{\lambda\mu}$ and $\mathbf{H}^{\lambda\mu}$ are expressed in terms of directional derivatives along t^λ , $'t^\lambda$, k^λ . This leads to a system of twenty-six quasilinear, homogeneous, partial differential equations in eighteen unknowns which are necessary consequences of our basic system of equations. In the axi-symmetric case this reduces to a system of twelve quasilinear partial differential equations in eleven unknowns. It is to be noted that ten of the equations contain derivatives with respect to only one of the variables. The remaining problems are: (1) to determine sufficient equations for the cases discussed in the present paper; (2) to determine the necessary and sufficient conditions for flows with discontinuities in the derivatives of the density on the boundary surface and hence throughout the flow-space. A possible application of our present results is in the study of simple waves.

Recently, L. WOLTJER [2] has generalized the method used by S. CHANDRASEKHAR [3] to deal with the case of axial-symmetric compressible fluids. WOLTJER's equations consist of three linear second order and one non-linear first order partial differential equations for the determination of the velocity and magnetic field vectors. These equations are shown to be related to the extrema of physically significant calculus of variations integrals, and should be useful in studying properties of various velocity and magnetic field vectors. The three linear second order partial differential equations can be replaced by six linear first order differential equations. Thus, for numerical work, WOLTJER's system of equations consists of six linear plus one non-linear partial differential equations. These equations contain derivatives with respect to two independent variables.

2. Basic Equations of the Theory. By use of the magneto-hydrodynamical approximation, the equations of continuity and motion and the Maxwell relations for an inviscid, infinitely conductive, compressible (isentropic) or incompressible fluid become [4]

$$(2.1) \quad \nabla_\mu(\rho v^\mu) = 0,$$

$$(2.2) \quad \rho v^\mu \nabla_\mu v_\lambda + \nabla_\lambda(p + \tfrac{1}{2}\eta H^2) - \nabla_\mu(\eta H^\mu H_\lambda) = 0,$$

$$(2.3) \quad v^\mu \nabla_\mu H_\lambda - H^\mu \nabla_\mu v_\lambda + H_\lambda \nabla_\mu v^\mu = 0,$$

$$(2.4) \quad \nabla_\mu H^\mu = 0$$

where ∇_μ is the covariant derivative with respect to x^μ and x^μ , $\mu = 1, 2, 3$, are general curvilinear coordinates; η is the magnetic permeability; p is the pressure; $a^2 = dp/d\rho$ is the local sound speed; q is the magnitude of the velocity vector v^λ ; H is the magnitude of the magnetic field vector H^λ .

Applying the weak shock technique to (2.1)–(2.4), it has been shown [1] that the unit normal vector n^λ of a characteristic manifold in the *steady* flow of a hydromagnetic fluid satisfies one of the following two equations:

$$(2.5) \quad (v^\lambda n_\lambda)^2 \left[(v^\mu n_\mu)^2 - a^2 - \frac{\eta}{\rho} H^2 \right] + \frac{\eta}{\rho} a^2 (H^\lambda n_\lambda)^2 = 0,$$

$$(2.6) \quad (v^\lambda n_\lambda)^2 = \frac{\eta}{\rho} (H^\lambda n_\lambda)^2.$$

The first of the above equations is valid in the general, compressible case (discontinuities permitted in derivatives of ρ , v^λ , H^λ); the second equation is valid for incompressible flow and for those compressible flows [5] in which the derivatives of ρ are continuous (discontinuities permitted in derivatives of v^λ , H^λ). The bicharacteristic directions associated with (2.5), (2.6), respectively, are [6]

$$(2.7) \quad \frac{dx^\lambda}{d\sigma} = v^\lambda (v^\mu n_\mu) \left[2(v^\alpha n_\alpha)^2 - a^2 - \frac{\eta}{\rho} H^2 \right] - (v^\alpha n_\alpha)^4 n^\lambda + \frac{\eta}{\rho} a^2 (H^\alpha n_\alpha) H^\lambda,$$

$$(2.8) \quad \frac{dx^\lambda}{d\sigma} = v^\lambda (v^\alpha n_\alpha) - \frac{\eta}{\rho} H^\lambda (H^\alpha n_\alpha)$$

where σ is a parameter along any bicharacteristic.

We note one important result: *If n^λ lies in the plane of v^λ and H^λ , then the bicharacteristic directions in both the compressible and incompressible cases lie in this same plane.* The importance of this result will become apparent in the next section where we shall prove that in every steady three-dimensional hydromagnetic flow, there always exist two normals which lie in the plane of v^λ and H^λ . Thus, *as far as the decomposition of the velocity and magnetic field vectors with respect to the normal and bicharacteristic of a characteristic manifold is concerned, the three-dimensional and the two-dimensional cases are equivalent.*

3. The Geometry of the Normal Cones. In this section, we shall study the normals at any point P which satisfy (2.5) or (2.6). At P , the normals which satisfy (2.5) form the *normal cone* for general steady compressible hydromagnetic flows and the normals which satisfy (2.6) determine the *normal cone* for steady incompressible hydromagnetic flows as well as compressible flows possessing continuous derivatives of ρ . If we introduce the angles α , β where α is the angle between any normal vector and the velocity vector at P and β is the angle between *this* normal and the magnetic field vector, then

$$(3.1) \quad q \cos \alpha = v^\lambda n_\lambda, \quad H \cos \beta = H^\lambda n_\lambda.$$

Further, if we introduce the scalars M , K where M is the Mach number and K^2 is the ratio of the magnetic and kinetic energies, then

$$(3.2) \quad M^2 = \frac{q^2}{a^2}, \quad K^2 = \frac{\eta H^2}{\rho q^2}.$$

By use of (3.1), (3.2), the equations (2.5) and (2.6) become, after division by $q^2 a^2$,

$$(3.3) \quad M^2 \cos^4 \alpha - (1 + M^2 K^2) \cos^2 \alpha + K^2 \cos^2 \beta = 0,$$

$$(3.4) \quad \cos^2 \alpha = K^2 \cos^2 \beta.$$

Let us project the *extremities of the unit normal vector* with origin at any point P onto the local plane N through v^λ and H^λ at P . Evidently, the resulting curve is the projection on N of the curve of intersection of the unit sphere with center at P and the cone of normals determined by either (2.5) or (2.6) with vertex at P . If Q denotes a point of the curve of intersection of the above sphere and cone, and if $'Q$ denotes the projection of Q on the plane N , then the projection of $'Q$ on v^λ (or H^λ) coincides with the projection of Q on v^λ (or H^λ). This projection is $\cos \alpha$ (or $\cos \beta$). Hence, if a local coordinate system is chosen so that the origin is at P and the coordinate axes x^1, x^2, x^3 coincide with v^λ, H^λ , and the vector k^λ orthogonal to both of these vectors, respectively, then x_1, x_2 denote two of the *covariant components* of any unit normal vector n^λ which are the projections of n^λ on x^1, x^2 , respectively, or

$$(3.5) \quad \begin{aligned} x_1 &= \cos \alpha, & x_2 &= \cos \beta, \\ x_1^2 &\leq 1, & x_2^2 &\leq 1. \end{aligned}$$

Thus, (3.3), (3.4) become, respectively,

$$(3.6) \quad M^2 x_1^4 - (1 + M^2 K^2) x_1^2 + K^2 x_2^2 = 0,$$

$$(3.7) \quad x_1^2 - K^2 x_2^2 = 0.$$

The metric tensor in the above x^1, x^2, x^3 coordinate system is

$$(3.8) \quad g_{11} = g_{22} = g_{33} = 1, \quad g_{12} = \cos^* \theta, \quad g_{13} = g_{23} = 0,$$

where $^* \theta$ is the angle between H^λ and v^λ at P . Hence the unit circle in the x^1, x^2 plane is determined by

$$(3.9) \quad g^{\lambda\mu} x_\lambda x_\mu = 1, \quad x_3 = 0.$$

By use of (3.8), the above becomes

$$(3.10) \quad x_1^2 - 2x_1 x_2 \cos^* \theta + x_2^2 = \sin^2^* \theta, \quad x_3 = 0.$$

We shall show that *each of the loci* (3.6), (3.7) *must intersect the unit circle* (3.10) *in at least one point 'P in each quadrant*. First, we note that $x_1 = x_2 = 0$ satisfies both (3.6), (3.7). That is, both (3.6) and (3.7) pass through the point

P . Since (3.7) consists of two intersecting lines, the desired result is valid for (3.7). Second, we write (3.6) in the form

$$(3.11) \quad \left(x_1^2 - \frac{1 + M^2 K^2}{2M^2}\right)^2 = -\frac{K^2}{M^2} \left(x_2^2 - \frac{(1 + M^2 K^2)^2}{4M^2 K^2}\right).$$

Since for all non-zero M, K

$$\frac{(1 + M^2 K^2)^2}{4M^2 K^2} \geq 1,$$

the points determined by

$$x_1^2 = \frac{1 + M^2 K^2}{2M^2}, \quad x_2^2 = \frac{(1 + M^2 K^2)^2}{4M^2 K^2}$$

have projections on the x^2 -axis which are larger than or equal to one. Hence these points fall on or outside the unit circle. Thus the locii (3.6) and (3.10) intersect at least once in each quadrant. The locus (3.6) is a "butterfly" type curve [7].

The point P corresponds to the normal k^λ which is orthogonal to the plane of v^λ, H^λ ; the points $'P$ of intersection of (3.6), (3.10) or (3.7), (3.10) determine at least two distinct normals $n^\lambda, 'n^\lambda$ in the plane of v^λ, H^λ . Thus $n^\lambda, 'n^\lambda, k^\lambda$ determine three families of characteristic surfaces which intersect in k^λ and the bicharacteristics $t^\lambda, 't^\lambda$, orthogonal to $n^\lambda, 'n^\lambda$, respectively. These bicharacteristics lie in the plane of v^λ and H^λ . The three families of characteristic surfaces will be used to form a coordinate system. The coordinate curves fall along $k^\lambda, t^\lambda, 't^\lambda$.

We now determine the points $'P$ of intersections of (3.7) and (3.10), and we find

$$(3.12) \quad \begin{aligned} x_1 &= \frac{K \sin^* \theta}{(K^2 - 2K \cos^* \theta + 1)^{\frac{1}{2}}}, & x_2 &= \frac{\sin^* \theta}{(K^2 - 2K \cos^* \theta + 1)^{\frac{1}{2}}}, \\ 'x_1 &= \frac{K \sin^* \theta}{(K^2 + 2K \cos^* \theta + 1)^{\frac{1}{2}}}, & 'x_2 &= \frac{-\sin^* \theta}{(K^2 + 2K \cos^* \theta + 1)^{\frac{1}{2}}}. \end{aligned}$$

The first point lies on the normal n^λ ; the second lies on the normal $'n^\lambda$. The solutions of (3.6) and (3.10) lead to a biquartic equation. Due to the algebraic complexities in analyzing the various roots of this equation we shall not discuss this problem.

4. The Characteristic System for Steady, Compressible, Hydromagnetic Flows. In this section we shall consider steady, three-dimensional, hydromagnetic flows. The basic equations are (2.1)–(2.4). These constitute eight equations in seven unknowns for the isentropic case since p is a function of only ρ . However, as every permissible solution to the hydromagnetic equations must be such that $\nabla_\lambda H^\lambda = 0$, we shall treat this equation as part of the system. Furthermore, we restrict ourselves to the case in which the density has continuous derivatives.

It should be noted that the resulting theory is also valid for the incompressible case, provided that the pressure is retained in the original equations.

Now we consider the problem of decomposing H^λ , v^λ with respect to the ordered right-handed triads $(t^\lambda, n^\lambda, k^\lambda)$ and $(t'^\lambda, n'^\lambda, k'^\lambda)$. The resulting relations form the *first set of characteristic equations*. We shall show that it is permissible to write

$$(4.1) \quad \begin{aligned} v^\lambda &= bn^\lambda + dt^\lambda; & v^\lambda &= 'b'n^\lambda - 'd't^\lambda, \\ H^\lambda &= en^\lambda - ft^\lambda; & H^\lambda &= -'e'n^\lambda - 'f't^\lambda \end{aligned}$$

where b, d, e, f are scalar functions of q, H , and ${}^*\theta$ as expressed by (see 3.12)

$$(4.2) \quad \begin{aligned} b &= q |x_1|, & e &= H |x_2|, \\ d &= q(1 - x_1^2)^{\frac{1}{2}} = q \frac{|K \cos {}^*\theta - 1|}{(K^2 + 1 - 2K \cos {}^*\theta)^{\frac{1}{2}}}, \\ f &= \pm H(1 - x_2^2)^{\frac{1}{2}} = H \frac{K - \cos {}^*\theta}{(K^2 + 1 - 2K \cos {}^*\theta)^{\frac{1}{2}}} \end{aligned}$$

and $'b, 'd, 'e, 'f$ are similar relations to those given above except that the denominators contain $(K^2 + 1 + 2K \cos {}^*\theta)^{\frac{1}{2}}$ instead of $(K^2 + 1 - 2K \cos {}^*\theta)^{\frac{1}{2}}$ and the numerators of $'d$ and $'f$ are $|K \cos {}^*\theta + 1|, |K + \cos {}^*\theta|$, respectively. Two cases arise. In the first case $0 < {}^*\theta < \frac{1}{2}\pi$, $|\alpha|, |\beta|$ both less than $\frac{1}{2}\pi$ (see (3.5) for the definitions of α and β). This situation is shown in Figure 1, where the dotted lines make angles of $\frac{1}{2}\pi$ with H^λ, v^λ . For normals lying in regions I

FIGURE 1.

and III we have $^*\theta = |\alpha| - |\beta|$. For n^λ , $x_1 = Kx_2$ (see (3.12)) or $\cos \alpha = K \cos \beta$. By writing

$$\cos |\alpha| = \cos |\beta| \cos ^*\theta - \sin |\beta| \sin ^*\theta$$

and dividing both sides by $\cos |\beta|$ we get

$$K = \cos ^*\theta - \tan |\beta| \sin ^*\theta.$$

This implies that if $K < \cos ^*\theta$, the projection of H^λ on t^λ is positive. This condition will be satisfied for the normals of region I but will *not* be satisfied by any normal lying in region III. Furthermore, normals in region IV are not permitted since the projections on n^λ of H^λ and v^λ should both be positive. It is now easily seen that (4.1) are satisfied for v^λ , H^λ in terms of n^λ , t^λ (for normals of region I) by the coefficients given in (4.2). For normals lying in region II we have $^*\theta = |\alpha| + |\beta|$. By similar arguments we see that in this case $K > \cos ^*\theta$. Hence permissible normals exist in this region which satisfy (4.1). In the second case $\frac{1}{2}\pi \leq ^*\theta < \pi$ but $|\alpha|$, $|\beta|$ both are less than π . This situation is shown in

FIGURE 2.

Figure 2 where again the dotted lines are perpendicular to both H^λ and v^λ . Since the projection on n^λ of H^λ and v^λ must both be positive, only normals in region II are permissible. This is consistent with (4.1) and (4.2). We note that the projection of v^λ on $'n^\lambda$ is positive and the projection of H^λ on $'n^\lambda$ is negative. If $'n^\lambda$ lies in region IV of Figures 1, 2, then (4.1), (4.2) will be consistent with these figures.

Our aim is to express the basic equations (2.1)–(2.4) in terms of the directional derivatives of p , v^λ , H^λ along t^λ , $'t^\lambda$, k^λ . In this process we find it necessary to

introduce the fluid vorticity tensor $\omega_{\lambda\mu}$ and the current bivector $\mathbf{H}_{\lambda\mu}$. First, the metric tensor $g^{\lambda\mu}$ may be expanded as follows [8]:

$$(4.3) \quad g^{\lambda\mu} = n^\lambda n^\mu + t^\lambda t^\mu + k^\lambda k^\mu$$

and a similar decomposition in terms of $'n^\lambda$, $'t^\lambda$, k^λ . Solving for $\nabla_\lambda \ln \rho$ in (2.2) and substituting this into (2.1) we get

$$(4.4) \quad g^{\lambda\mu} \nabla_\mu v_\lambda + \frac{v^\mu}{a^2} \left[\frac{\eta}{\rho} H^\lambda \nabla_\lambda H_\mu - \frac{\eta}{2\rho} \nabla_\mu H^2 - v^\lambda \nabla_\lambda v_\mu \right] = 0.$$

Using (4.1), (4.3), the equation (4.4), which is a form of (2.1), becomes

$$(4.5) \quad (n^\lambda n^\mu + t^\lambda t^\mu + k^\lambda k^\mu) \nabla_\mu v_\lambda + \frac{\eta}{\rho a^2} (bn^\mu + dt^\mu)(en^\lambda - ft^\lambda) \nabla_\lambda H_\mu \\ - \frac{\eta}{2\rho a^2} (bn^\mu + dt^\mu) \nabla_\mu H^2 - \frac{1}{a^2} (bn^\mu + dt^\mu)(bn^\lambda + dt^\lambda) \nabla_\lambda v_\mu = 0.$$

Furthermore, by using (4.1), (4.3), the equations (2.2)–(2.4) become

$$(4.6) \quad (bn^\mu + dt^\mu) \nabla_\mu v_\lambda + a^2 \nabla_\lambda \ln \rho + \frac{\eta}{2\rho} \nabla_\lambda H^2 - \frac{\eta}{\rho} (en^\mu - ft^\mu) \nabla_\mu H_\lambda = 0,$$

$$(4.7) \quad (bn^\mu + dt^\mu) \nabla_\mu H_\lambda - (en^\mu - ft^\mu) \nabla_\mu v_\lambda + (en_\lambda - ft_\lambda) \nabla_\mu v^\mu = 0,$$

$$(4.8) \quad (n^\lambda n^\mu + t^\lambda t^\mu + k^\lambda k^\mu) \nabla_\mu H_\lambda = 0.$$

From (4.8) we get

$$(4.9) \quad n^\lambda n^\mu \nabla_\mu H_\lambda = -(t^\lambda t^\mu + k^\lambda k^\mu) \nabla_\mu H_\lambda.$$

We now introduce the vorticity tensor and current bivector by the following relations:

$$(4.10) \quad 2\omega_{\lambda\mu} = \nabla_\mu v_\lambda - \nabla_\lambda v_\mu,$$

$$(4.11) \quad 2\mathbf{H}_{\lambda\mu} = \nabla_\mu H_\lambda - \nabla_\lambda H_\mu.$$

From these relations, we obtain

$$(4.12) \quad n^\mu t^\lambda \nabla_\mu v_\lambda = n^\mu t^\lambda \nabla_\lambda v_\mu + 2n^\mu t^\lambda \omega_{\lambda\mu},$$

$$(4.13) \quad n^\mu t^\lambda \nabla_\mu H_\lambda = n^\mu t^\lambda \nabla_\lambda H_\mu + 2n^\mu t^\lambda \mathbf{H}_{\lambda\mu},$$

$$(4.14) \quad k^\mu t^\lambda \nabla_\mu v_\lambda = k^\mu t^\lambda \nabla_\lambda v_\mu + 2k^\mu t^\lambda \omega_{\lambda\mu},$$

$$(4.15) \quad k^\mu t^\lambda \nabla_\mu H_\lambda = k^\mu t^\lambda \nabla_\lambda H_\mu + 2k^\mu t^\lambda \mathbf{H}_{\lambda\mu}$$

and similar relations involving $'n^\lambda$, $'t^\lambda$, k^λ . Further, we have the following integrability conditions [9]:

$$(4.16) \quad \nabla_\nu \omega_{\lambda\mu} + \nabla_\mu \omega_{\nu\lambda} + \nabla_\lambda \omega_{\mu\nu} = 0, \\ \nabla_\nu \mathbf{H}_{\lambda\mu} + \nabla_\mu \mathbf{H}_{\nu\lambda} + \nabla_\lambda \mathbf{H}_{\mu\nu} = 0.$$

Forming the scalar product of these integrability conditions with $t''t^\mu k^\lambda$, we obtain

$$(4.17) \quad \begin{aligned} k^\lambda t^\mu t^\nu (\nabla_\nu \omega_{\lambda\mu} + \nabla_\mu \omega_{\nu\lambda} + \nabla_\lambda \omega_{\mu\nu}) &= 0, \\ k^\lambda t^\mu t^\nu (\nabla_\nu \mathbf{H}_{\lambda\mu} + \nabla_\mu \mathbf{H}_{\nu\lambda} + \nabla_\lambda \mathbf{H}_{\mu\nu}) &= 0. \end{aligned}$$

In order to express the continuity, motion, and Maxwell relations in terms of characteristic variables, we shall find it necessary to eliminate the directional derivatives in the normal direction. This means that we must eliminate terms of the type $n^\lambda n^\mu \nabla_\lambda v_\mu$ and $n^\lambda n^\mu \nabla_\lambda H_\mu$. This will be done by use of (4.5) and (4.9). Further, terms of the types $n^\mu t^\lambda \nabla_\mu v_\lambda$, $n^\mu t^\lambda \nabla_\mu H_\lambda$, $n^\mu k^\lambda \nabla_\mu v_\lambda$, $n^\mu k^\lambda \nabla_\mu H_\lambda$ can be expressed in terms of directional derivatives along t^λ and k^λ by use of (4.12), (4.13) and two similar equations involving k^λ and n^μ .

Solving for $n^\lambda n^\mu \nabla_\mu v_\lambda$ in (4.5), we find, when $b \neq a$, by use of (4.12), (4.13)

$$(4.18) \quad \begin{aligned} n^\lambda n^\mu \nabla_\mu v_\lambda &= \frac{1}{a^2 - b^2} \left\{ (d^2 - a^2) t^\lambda t^\mu \nabla_\mu v_\lambda - a^2 k^\lambda k^\mu \nabla_\mu v_\lambda \right. \\ &\quad \left. + 2b dn^\mu t^\lambda \omega_{\lambda\mu} - 2 \frac{\eta}{\rho} (de + bf) t^\lambda n^\mu \mathbf{H}_{\lambda\mu} + 2b dn^\mu t^\lambda \nabla_\lambda v_\mu \right\}. \end{aligned}$$

The case $b = a$ corresponds to the "slow speed" wave and will not be considered in our work. Forming the scalar product of (4.6) with t^λ and simplifying the resulting equation by using (4.12), (4.13), we get

$$(4.19) \quad b(n^\mu t^\lambda \nabla_\lambda v_\mu + 2n^\mu t^\lambda \omega_{\lambda\mu}) + dt^\mu t^\lambda \nabla_\lambda v_\mu + a^2 t^\lambda \nabla_\lambda \ln \rho + 2 \frac{\eta}{\rho} en^\mu t^\lambda \mathbf{H}_{\lambda\mu} = 0.$$

Taking the scalar product of (4.6) with n^λ , we obtain a relation for $n^\lambda \nabla_\lambda \ln \rho$ which is of no interest for the characteristic system desired. However, forming the scalar product of (4.6) with k^λ and using (4.13)–(4.15) and equations similar to (4.14), (4.15) in $n^\mu k^\lambda$ leads to

$$(4.20) \quad \begin{aligned} b(n^\mu k^\lambda \nabla_\lambda v_\mu + 2n^\mu k^\lambda \omega_{\lambda\mu}) + d(t^\mu k^\lambda \nabla_\lambda v_\mu + 2t^\mu k^\lambda \omega_{\lambda\mu}) \\ + a^2 k^\lambda \nabla_\lambda \ln \rho - 2 \frac{\eta}{\rho} (en^\mu k^\lambda \mathbf{H}_{\lambda\mu} - fk^\lambda t^\mu \mathbf{H}_{\lambda\mu}) = 0. \end{aligned}$$

In addition to eliminating the directional derivative of v^λ , H^λ in the normal direction, we shall find it desirable, in order to form a simple quasilinear system, at times to replace $t^\lambda k^\mu \nabla_\lambda v_\mu$, $t^\lambda k^\mu \nabla_\lambda H_\mu$ by $t^\lambda k^\mu \nabla_\mu v_\lambda$, $t^\lambda k^\mu \nabla_\mu H_\lambda$ through use of (4.14), (4.15). This will be done in discussing (4.7). Taking the scalar product of (4.7) with k^λ and using (4.12)–(4.15) we obtain

$$(4.21) \quad \begin{aligned} b(n^\mu k^\lambda \nabla_\lambda H_\mu + 2n^\mu k^\lambda \mathbf{H}_{\lambda\mu}) + d(t^\mu k^\lambda \nabla_\lambda H_\mu + 2k^\lambda t^\mu \mathbf{H}_{\lambda\mu}) \\ - e(n^\mu k^\lambda \nabla_\lambda v_\mu + 2n^\mu k^\lambda \omega_{\lambda\mu}) + f(t^\mu k^\lambda \nabla_\lambda v_\mu + 2k^\lambda t^\mu \omega_{\lambda\mu}) = 0. \end{aligned}$$

Forming the scalar product of (4.7) with n^λ and making use of (4.9) to eliminate $n^\lambda n^\mu \nabla_\lambda H_\mu$, we get by use of (4.3)

$$(4.22) \quad \begin{aligned} e(t^\lambda t^\mu \nabla_\mu v_\lambda + k^\lambda k^\mu \nabla_\mu v_\lambda) + fn^\lambda t^\mu \nabla_\mu v_\lambda \\ - b(t^\lambda t^\mu \nabla_\mu H_\lambda + k^\lambda k^\mu \nabla_\mu H_\lambda) + dn^\lambda t^\mu \nabla_\mu H_\lambda = 0. \end{aligned}$$

Taking the scalar product of (4.7) with t^λ and using (4.18) to eliminate $n^\lambda n^\mu \nabla_\lambda v_\mu$, we get

$$(4.23) \quad \begin{aligned} & dt^\lambda t^\mu \nabla_\mu H_\lambda + bt^\mu n^\lambda \nabla_\mu H_\lambda + 2 \left[b + \frac{f}{a^2 - b^2} \frac{\eta}{\rho} (de + bf) \right] t^\lambda n^\mu \mathbf{H}_{\lambda\mu} \\ & + f \frac{b^2}{a^2 - b^2} k^\lambda k^\mu \nabla_\mu v_\lambda - f \frac{1}{a^2 - b^2} (d^2 - a^2) t^\lambda t^\mu \nabla_\mu v_\lambda \\ & - \left(\frac{2b}{a^2 - b^2} \frac{df}{b^2} + e \right) t^\mu n^\lambda \nabla_\mu v_\lambda - 2 \left(\frac{b}{a^2 - b^2} \frac{df}{b^2} + e \right) t^\lambda n^\mu \omega_{\lambda\mu} = 0. \end{aligned}$$

Our system of equations (4.19)–(4.23) and similar equations in $'t^\lambda$, $'n^\lambda$, k^λ are consequences of the equations of continuity and motion and the Maxwell relations (2.1)–(2.4). It should be noted that we have used the continuity equation and the vanishing of the divergence of the magnetic field in deriving (4.19)–(4.23). Further, we have decomposed the equations of motion and the Maxwell relation (2.3) in terms of three independent directions. Through these considerations we see that the five equations (4.19)–(4.23) and the corresponding five equations in $'t^\lambda$, $'n^\lambda$, k^λ should imply that (2.1)–(2.4) are satisfied. With the aid of (4.2), (4.12)–(4.15), and the corresponding equations in $'n^\lambda$, $'t^\lambda$, k^λ , this fact may be established by the following argument: Let us consider (2.1) through (2.4) with the zeros in the right hand sides of these equations replaced by E , F_λ , G_λ , L respectively. For instance, instead of (2.1), we consider

$$\nabla_\lambda (\rho v^\lambda) = E.$$

By use of (4.1), (4.13)–(4.15), we find by reversing the order of the steps in computing (4.19), (4.20) that these last equations lead to

$$t^\lambda F_\lambda = k^\lambda F_\lambda = 0.$$

The equation corresponding to (4.19) in $'t^\lambda$, $'n^\lambda$ leads to

$$'t^\lambda F_\lambda = 0.$$

Hence we see that (4.19), (4.20), and the equation corresponding to (4.19) in $'t^\lambda$ imply $F_\lambda = 0$. From (4.21), (4.12)–(4.15), (4.1) we see that

$$k^\lambda G_\lambda = 0.$$

Again, from (4.22) and the corresponding equation in $'t^\lambda$, n^λ , k^λ we find

$$n^\lambda G_\lambda - L = 0, \quad 'n^\lambda G_\lambda - L = 0.$$

Finally, from (4.23) and the corresponding equation in $'t^\lambda$, n^λ , k^λ we obtain

$$t^\lambda G_\lambda - \frac{fa^2}{a^2 - b^2} E = 0, \quad 't^\lambda G_\lambda - \frac{fa^2}{a^2 - b^2} E = 0.$$

Thus

$$k^\lambda G_\lambda = (n^\lambda - 'n^\lambda) G_\lambda = (t^\lambda - 't^\lambda) G_\lambda = 0.$$

Since $n^\lambda - 'n^\lambda$ and $t^\lambda - 't^\lambda$ are independent (see Figures 1, 2), we find

$$G_\lambda = 0, \quad L = 0, \quad E = 0.$$

Thus the ten equations ((4.19)–(4.23) plus similar relations in $'t^\lambda, 'n^\lambda, k^\lambda$) together with the six independent equations of the type (4.12)–(4.15) are equivalent to our original system of equations (2.1)–(2.4). The equations (4.14), (4.15) and similar relations in $k^\lambda t^\mu \nabla_\lambda v_\mu$, $k^\lambda t^\mu \nabla_\lambda H_\mu$, $'t^\lambda t^\mu \nabla_\lambda v_\mu$, $'t^\lambda t^\mu \nabla_\lambda H_\mu$ will be taken as the six independent equations which define $\omega_{\lambda\mu}$ and $\mathbf{H}_{\lambda\mu}$. Finally, by forming the scalar product of the integrability conditions (4.16) with $'t^\lambda t^\mu k^\lambda$ we obtained two additional equations (4.17) which must be added to the system. Hence our new system replacing the equations of continuity and motion and the Maxwell relations will consist of eighteen equations in thirteen unknowns H^λ , v^λ , ρ , $\omega_{\lambda\mu}$, $\mathbf{H}_{\lambda\mu}$. In addition, we must add to this system the equations defining the directions $t^\lambda, 't^\lambda, k^\lambda$.

We will now express (1) the first characteristic relations, (2) the equations (4.19)–(4.23), (3) the relations defining the vorticity $\omega_{\lambda\mu}$ and the current bivector $\mathbf{H}_{\lambda\mu}$, and (4) their integrability conditions in terms of partial derivatives along a curvilinear coordinate system determined by $t^\lambda, 't^\lambda, k^\lambda$ (see Section 3).

Let $\alpha =$ variable denote the curves along t^λ , $\beta =$ variable denote the curves along $'t^\lambda$, and $\gamma =$ variable denote the curves along k^λ . Then, if A, B, C , and D are the metric coefficients, we may express the element of arc in space by the following relation:

$$(4.24) \quad ds^2 = A^2 d\alpha^2 + B^2 d\beta^2 + C^2 d\gamma^2 + 2D d\alpha d\beta,$$

where

$$(4.25) \quad \begin{aligned} A^2 &= g_{\lambda\mu} \frac{\partial x^\lambda}{\partial \alpha} \frac{\partial x^\mu}{\partial \alpha}, & B^2 &= g_{\lambda\mu} \frac{\partial x^\lambda}{\partial \beta} \frac{\partial x^\mu}{\partial \beta}, \\ C^2 &= g_{\lambda\mu} \frac{\partial x^\lambda}{\partial \gamma} \frac{\partial x^\mu}{\partial \gamma}, & D &= g_{\lambda\mu} \frac{\partial x^\lambda}{\partial \alpha} \frac{\partial x^\mu}{\partial \beta}, \end{aligned}$$

and x^λ , $\lambda = 1, 2, 3$, are general curvilinear coordinates of the space of the flow. Then the components of $t^\lambda, 't^\lambda$, and k^λ may be written as

$$(4.26) \quad t^\lambda = \frac{1}{A} \frac{\partial x^\lambda}{\partial \alpha}, \quad 't^\lambda = \frac{1}{B} \frac{\partial x^\lambda}{\partial \beta}, \quad k^\lambda = \frac{1}{C} \frac{\partial x^\lambda}{\partial \gamma}.$$

We now define the following quantities: $T^\lambda, 'T^\lambda, K^\lambda, N^\lambda$, and $'N^\lambda$, whose components will be shown to depend only on H^λ, v^λ :

$$\begin{aligned} T^\lambda &= A t^\lambda \left/ \frac{\partial x^1}{\partial \alpha} \right/ \frac{\partial x^1}{\partial \alpha} = t^\lambda / t^1, \\ 'T^\lambda &= B 't^\lambda \left/ \frac{\partial x^2}{\partial \beta} \right/ \frac{\partial x^2}{\partial \beta} = 't^\lambda / 't^2, \end{aligned}$$

$$\begin{aligned}
(4.27) \quad K^\lambda &= Ck^\lambda \left/ \frac{\partial x^3}{\partial \gamma} \right/ = \frac{\partial x^\lambda}{\partial \gamma} \left/ \frac{\partial x^3}{\partial \gamma} \right/ = k^\lambda / k^3, \\
N^\lambda &= ACn^\lambda \left/ \frac{\partial x^1}{\partial \alpha} \frac{\partial x^3}{\partial \gamma} \right/ = \frac{-e^{\lambda\alpha\beta}}{\sqrt{g}} T_\alpha K_\beta = -\sqrt{g} e_{\lambda\alpha\beta} T^\alpha K^\beta, \\
{}'N^\lambda &= BC'n^\lambda \left/ \frac{\partial x^2}{\partial \beta} \frac{\partial x^3}{\partial \gamma} \right/ = \frac{-e^{\lambda\alpha\beta}}{\sqrt{g}} {}'T_\alpha K_\beta = -\sqrt{g} e_{\lambda\alpha\beta} {}'T^\alpha K^\beta.
\end{aligned}$$

Furthermore, we define scalars $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ which depend only on H^λ , v^λ :

$$\begin{aligned}
(4.28) \quad \mathfrak{A} &= A \left/ \frac{\partial x^1}{\partial \alpha} \right/, \quad \mathfrak{A}^2 = g_{\lambda\mu} T^\lambda T^\mu, \\
\mathfrak{B} &= B \left/ \frac{\partial x^2}{\partial \beta} \right/, \quad \mathfrak{B}^2 = g_{\lambda\mu} {}'T^\lambda {}'T^\mu, \\
\mathfrak{C} &= C \left/ \frac{\partial x^3}{\partial \gamma} \right/, \quad \mathfrak{C}^2 = g_{\lambda\mu} K^\lambda K^\mu.
\end{aligned}$$

We will now find the relations determining T^λ , $'T^\lambda$, K^λ and the first set of characteristic relations. From (4.1) we obtain

$$v_\lambda t^\lambda = d, \quad H_\lambda t^\lambda = -f$$

or, using (4.26),

$$(4.29) \quad v_\lambda \frac{\partial x^\lambda}{\partial \alpha} = A d, \quad H_\lambda \frac{\partial x^\lambda}{\partial \alpha} = -A f.$$

Eliminating A and using (4.27) we get

$$(4.30) \quad (f v_\lambda + d H_\lambda) T^\lambda = 0.$$

Squaring the first relation in (4.29) we obtain

$$v_\lambda v_\mu \frac{\partial x^\lambda}{\partial \alpha} \frac{\partial x^\mu}{\partial \alpha} = A^2 d^2.$$

Using (4.25) we may write this last equation as follows:

$$(4.31) \quad (v_\lambda v_\mu - d^2 g_{\lambda\mu}) T^\lambda T^\mu = 0.$$

Hence T^λ may be determined from (4.30) and (4.31). The proper root is determined by the initial data plus (4.2). That is, we find n^λ , $'n^\lambda$, t^λ , $'t^\lambda$ and hence N^λ , $'N^\lambda$, T^λ , $'T^\lambda$ initially and then by continuity throughout the whole flow space. It is to be noted that equations (4.30), (4.31) are two equations in the two unknowns T^2 , T^3 since $T^1 \equiv 1$. In a similar way $'T^\lambda$ may be determined by the following relations:

$$\begin{aligned}
(4.32) \quad (&'f v_\lambda - 'd H_\lambda) {}'T^\lambda = 0, \\
&(v_\lambda v_\mu - 'd^2 g_{\lambda\mu}) {}'T^\lambda {}'T^\mu = 0.
\end{aligned}$$

Again, these are two equations in the two unknowns $'T^1$, $'T^3$ since $'T^2 \equiv 1$. Further, since K^λ is perpendicular to H^λ , v^λ and since $K^3 \equiv 1$, the two equations determining K^λ in terms of v^λ , H^λ are

$$(4.33) \quad v_\lambda K^\lambda = 0, \quad H_\lambda K^\lambda = 0.$$

Moreover, we have the following partial differential equations for x^λ as functions of α , β , and γ (see (4.27)):

$$(4.34) \quad \begin{aligned} \frac{\partial x^\lambda}{\partial \alpha} - T^\lambda \frac{\partial x^1}{\partial \alpha} &= 0, \\ \frac{\partial x^\lambda}{\partial \beta} - 'T^\lambda \frac{\partial x^2}{\partial \beta} &= 0, \\ \frac{\partial x^\lambda}{\partial \gamma} - K^\lambda \frac{\partial x^3}{\partial \gamma} &= 0, \quad \lambda = 1, 2, 3. \end{aligned}$$

These are six quasilinear equations for the unknown partial derivatives of x^λ with respect to α , β , and γ . *They constitute the first set of characteristic relations.*

We now express (4.19)–(4.23) in terms of the characteristic variables by making use of (4.25)–(4.28) together with the definition of the covariant derivatives. For example, we note that

$$\begin{aligned} n^\mu t^\lambda \nabla_\lambda v_\mu &= \frac{n^\mu}{A} \left(\frac{\partial v_\mu}{\partial \alpha} - \frac{\partial x^\sigma}{\partial \alpha} \Gamma_{\mu\sigma}^\nu v_\nu \right) = \frac{1}{A} \frac{1}{\mathfrak{C}} N^\mu \left(\frac{\partial v_\mu}{\partial \alpha} - \frac{\partial x^\sigma}{\partial \alpha} \Gamma_{\mu\sigma}^\nu v_\nu \right), \\ n^\mu t^\lambda \omega_{\lambda\mu} &= \frac{1}{A} \frac{N^\mu}{\mathfrak{C}} \frac{\partial x^\lambda}{\partial \alpha} \omega_{\lambda\mu}, \quad t^\lambda \nabla_\lambda \ln \rho = \frac{1}{A} \frac{\partial}{\partial \alpha} (\ln \rho), \\ n^\lambda &= \frac{N^\lambda}{\mathfrak{C}}, \quad t^\lambda = \frac{T^\lambda}{\mathfrak{C}}, \quad k^\lambda = \frac{K^\lambda}{\mathfrak{C}}. \end{aligned}$$

We use these relations to determine the characteristic form of (4.19) which is

$$(4.35) \quad \begin{aligned} bN^\mu \left(\frac{\partial v_\mu}{\partial \alpha} - \frac{\partial x^\lambda}{\partial \alpha} \Gamma_{\lambda\mu}^\nu v_\nu + 2 \frac{\partial x^\lambda}{\partial \alpha} \omega_{\lambda\mu} \right) + d\mathfrak{C}T^\mu \left(\frac{\partial v_\mu}{\partial \alpha} - \frac{\partial x^\lambda}{\partial \alpha} \Gamma_{\lambda\mu}^\nu v_\nu \right) \\ + a^2 \mathfrak{C} \frac{\partial \ln \rho}{\partial \alpha} + 2 \frac{\eta}{\rho} eN^\mu \frac{\partial x^\lambda}{\partial \alpha} \mathbf{H}_{\lambda\mu} = 0. \end{aligned}$$

In a similar way we find the characteristic form of (4.20), (4.21):

$$(4.36) \quad \begin{aligned} (bN^\mu + d\mathfrak{C}T^\mu) \left(\frac{\partial v_\mu}{\partial \gamma} - \frac{\partial x^\lambda}{\partial \gamma} \Gamma_{\lambda\mu}^\nu v_\nu + 2 \frac{\partial x^\lambda}{\partial \gamma} \omega_{\lambda\mu} \right) + a^2 \mathfrak{C} \frac{\partial \ln \rho}{\partial \gamma} \\ - 2 \frac{\eta}{\rho} \left(eN^\mu \frac{\partial x^\lambda}{\partial \gamma} \mathbf{H}_{\lambda\mu} - \mathfrak{C}fT^\mu \frac{\partial x^\lambda}{\partial \gamma} \mathbf{H}_{\lambda\mu} \right) = 0, \end{aligned}$$

$$(4.37) \quad \begin{aligned} (bN^\mu + d\mathfrak{C}T^\mu) \left(\frac{\partial H_\mu}{\partial \gamma} - \frac{\partial x^\lambda}{\partial \gamma} \Gamma_{\lambda\mu}^\nu H_\nu + 2 \frac{\partial x^\lambda}{\partial \gamma} \mathbf{H}_{\lambda\mu} \right) \\ - (eN^\mu - f\mathfrak{C}T^\mu) \left(\frac{\partial v_\mu}{\partial \gamma} - \frac{\partial x^\lambda}{\partial \gamma} \Gamma_{\lambda\mu}^\nu v_\nu + 2 \frac{\partial x^\lambda}{\partial \gamma} \omega_{\lambda\mu} \right) = 0. \end{aligned}$$

However, in order to make the characteristic form of (4.22), (4.23) quasilinear, we introduce a new unknown variable $\xi \equiv A/C$. From (4.28) we see that ξ satisfies a quasilinear partial differential equation expressed by

$$(4.38) \quad \mathfrak{C} \frac{\partial x^3}{\partial \gamma} \xi - \mathfrak{A} \frac{\partial x^1}{\partial \alpha} = 0.$$

Using the same procedure as was employed in deriving (4.35), (4.36), (4.37) and using the new variable ξ , we find that the characteristic forms of (4.22), (4.23) are

$$(4.39) \quad \begin{aligned} & (e\mathfrak{C}T^\mu + fN^\mu) \left(\frac{\partial v_\mu}{\partial \alpha} - \frac{\partial x^\lambda}{\partial \alpha} \Gamma_{\lambda\mu}^\nu v_\nu \right) + e\mathfrak{A}\xi K^\mu \left(\frac{\partial v_\mu}{\partial \gamma} - \frac{\partial x^\lambda}{\partial \gamma} \Gamma_{\lambda\mu}^\nu v_\nu \right) \\ & - (b\mathfrak{C}T^\mu - dN^\mu) \left(\frac{\partial H_\mu}{\partial \alpha} - \frac{\partial x^\lambda}{\partial \alpha} \Gamma_{\lambda\mu}^\nu H_\nu \right) \\ & - b\mathfrak{A}\xi K^\mu \left(\frac{\partial H_\mu}{\partial \gamma} - \frac{\partial x^\lambda}{\partial \gamma} \Gamma_{\lambda\mu}^\nu H_\nu \right) = 0, \\ & (d\mathfrak{C}T^\mu + bN^\mu) \left(\frac{\partial H_\mu}{\partial \alpha} - \frac{\partial x^\lambda}{\partial \alpha} \Gamma_{\lambda\mu}^\nu H_\nu \right) + 2 \left[b + \frac{f}{a^2 - b^2} \frac{\eta}{\rho} (de + bf) \right] \\ & N^\mu \frac{\partial x^\lambda}{\partial \alpha} \mathbf{H}_{\lambda\mu} + f \frac{b^2}{a^2 - b^2} \mathfrak{A}\xi K^\mu \left(\frac{\partial v_\mu}{\partial \gamma} - \frac{\partial x^\lambda}{\partial \gamma} \Gamma_{\lambda\mu}^\nu v_\nu \right) \\ (4.40) \quad & - \left[f \frac{\mathfrak{C}}{a^2 - b^2} (d^2 - a^2) T^\mu + \left(\frac{2b}{a^2 - b^2} \frac{df}{dx} + e \right) N^\mu \right] \left(\frac{\partial v_\mu}{\partial \alpha} - \frac{\partial x^\lambda}{\partial \alpha} \Gamma_{\lambda\mu}^\nu v_\nu \right) \\ & - 2 \left(\frac{b}{a^2 - b^2} \frac{df}{dx} + e \right) N^\mu \frac{\partial x^\lambda}{\partial \alpha} \omega_{\lambda\mu} = 0. \end{aligned}$$

Similarly, we can find the second set of characteristic relations corresponding to (4.19)–(4.23) involving $'t^\lambda$, $'n^\lambda$, k^λ . In so doing we introduce another new unknown variable $\chi \equiv B/C$. The partial differential equation corresponding to (4.38) is again obtained from (4.28) and is

$$(4.41) \quad \mathfrak{C} \frac{\partial x^3}{\partial \gamma} \chi - \mathfrak{B} \frac{\partial x^2}{\partial \beta} = 0.$$

Now we consider the characteristic form of the relations defining $\omega_{\lambda\mu}$, $\mathbf{H}_{\lambda\mu}$ and their integrability conditions. From (4.14), (4.15) and the corresponding equations involving k^μ , $'t^\mu$ and t^μ , $'t^\lambda$ we find the following six quasilinear differential equations:

$$(4.42) \quad \mathfrak{C}T^\lambda \xi \left(\frac{\partial v_\lambda}{\partial \gamma} - \frac{\partial x^\mu}{\partial \gamma} \Gamma_{\lambda\mu}^\nu v_\nu \right) - \mathfrak{A}K^\mu \left(\frac{\partial v_\mu}{\partial \alpha} - \frac{\partial x^\lambda}{\partial \alpha} \Gamma_{\lambda\mu}^\nu v_\nu \right) - 2\mathfrak{A}K^\mu \frac{\partial x^\lambda}{\partial \alpha} \omega_{\lambda\mu} = 0,$$

$$(4.43) \quad \mathfrak{C}'T^\lambda \chi \left(\frac{\partial v_\lambda}{\partial \gamma} - \frac{\partial x^\mu}{\partial \gamma} \Gamma_{\lambda\mu}^\nu v_\nu \right) - \mathfrak{B}K^\mu \left(\frac{\partial v_\mu}{\partial \beta} - \frac{\partial x^\lambda}{\partial \beta} \Gamma_{\lambda\mu}^\nu v_\nu \right) - 2\mathfrak{B}K^\mu \frac{\partial x^\lambda}{\partial \beta} \omega_{\lambda\mu} = 0,$$

$$\begin{aligned}
 (4.44) \quad & \alpha {}'T^\lambda \chi \left(\frac{\partial v_\lambda}{\partial \alpha} - \frac{\partial x^\mu}{\partial \alpha} \Gamma_{\lambda\mu}^\nu v_\nu \right) - \beta T^\mu \xi \left(\frac{\partial v_\mu}{\partial \beta} - \frac{\partial x^\lambda}{\partial \beta} \Gamma_{\lambda\mu}^\nu v_\nu \right) \\
 & - 2\xi \chi \mathfrak{C} \frac{\partial x^3}{\partial \gamma} T^\mu {}'T^\lambda \omega_{\lambda\mu} = 0,
 \end{aligned}$$

and three other exactly similar equations with H_λ instead of v_λ . From the integrability conditions (4.17) we get the following additional two quasilinear differential equations:

$$(4.45) \quad \alpha K^\lambda {}'T^\mu \chi \frac{\partial \omega_{\lambda\mu}}{\partial \alpha} + \beta K^\lambda T^\nu \xi \frac{\partial \omega_{\lambda\nu}}{\partial \beta} + \mathfrak{C} {}'T^\mu T^\nu \xi \chi \frac{\partial \omega_{\mu\nu}}{\partial \gamma} = 0,$$

$$(4.46) \quad \alpha K^\lambda {}'T^\mu \chi \frac{\partial \mathbf{H}_{\lambda\mu}}{\partial \alpha} + \beta K^\lambda T^\nu \xi \frac{\partial \mathbf{H}_{\nu\lambda}}{\partial \beta} + \mathfrak{C} {}'T^\mu T^\nu \xi \chi \frac{\partial \mathbf{H}_{\mu\nu}}{\partial \gamma} = 0.$$

The relations (4.35)–(4.40) and the analogous relations for β plus the relation for χ as expressed by (4.41), the equations (4.42)–(4.44) and the corresponding ones for H_λ , the integrability conditions (4.45), (4.46), and the relations expressed by (4.34) constitute the complete characteristic system for the flow. They are in total twenty-six quasilinear partial differential equations in the eighteen unknown variables H^λ , v^λ , $\omega_{\lambda\mu}$, $\mathbf{H}_{\lambda\mu}$, x^λ , ρ , ξ , χ plus six algebraic equations (4.30)–(4.33) for T^λ , $'T^\lambda$, K^λ .

5. Axially Symmetric Flow. As a simple application we will obtain the characteristic system for the axi-symmetric flow of a compressible, nonviscous, infinitely conducting fluid. In this case we assume H^λ , v^λ , which are independent of the angle θ , to lie in meridian planes and x^1 , x^2 , x^3 are to be identified as the z , r , θ , respectively, of cylindrical coordinates. From the geometry of the flow we may choose γ to be equal to θ and the bicharacteristic curves $\alpha(x^1, x^2) = \text{constant}$, $\beta(x^1, x^2) = \text{constant}$ to lie in the meridian planes. The element of arc may be expressed by

$$(5.1) \quad ds^2 = dz^2 + dr^2 + r^2 d\theta^2 = A^2 d\alpha^2 + B^2 d\beta^2 + C^2 d\gamma^2.$$

Thus

$$(5.2) \quad A^2 = \left(\frac{\partial z}{\partial \alpha} \right)^2 + \left(\frac{\partial r}{\partial \alpha} \right)^2, \quad B^2 = \left(\frac{\partial z}{\partial \beta} \right)^2 + \left(\frac{\partial r}{\partial \beta} \right)^2, \quad C^2 = r^2.$$

From (5.1) we see that $g_{\lambda\mu} = 0$ for λ not equal to μ and

$$(5.3) \quad g_{11} = 1 = g^{11}, \quad g_{22} = 1 = g^{22}, \quad g_{33} = r^2 = \frac{1}{g^{33}}.$$

Moreover, the Christoffel symbols $\Gamma_{\alpha\beta}^\lambda$ are all equal to zero with the exception of

$$(5.4) \quad \Gamma_{33}^2 = -r, \quad \Gamma_{23}^3 = \Gamma_{32}^3 = \frac{1}{r}.$$

We can now proceed to compute the relations given by (4.27) and obtain

$$\begin{aligned}
 (5.5) \quad T^1 &= 1, & T^2 &= \frac{\partial r}{\partial \alpha} \bigg/ \frac{\partial z}{\partial \alpha}, & T^3 &= 0, \\
 {}'T^1 &= \frac{\partial z}{\partial \beta} \bigg/ \frac{\partial r}{\partial \beta}, & {}'T^2 &= 1, & {}'T^3 &= 0, \\
 K^1 &= 0, & K^2 &= 0, & K^3 &= 1, \\
 N^1 &= -T^2 r, & N^2 &= r, & N^3 &= 0, \\
 {}'N^1 &= -r, & {}'N^2 &= {}'T^1 r, & {}'N^3 &= 0.
 \end{aligned}$$

In the forthcoming work we will let $T = T^2$ and $'T = 'T^1$. In this notation we obtain from (4.28)

$$(5.6) \quad \mathfrak{A} = (1 + T^2)^{\frac{1}{2}}, \quad \mathfrak{B} = (1 + 'T^2)^{\frac{1}{2}}, \quad \mathfrak{C} = r.$$

Calculating the components of $\omega_{\lambda\mu}$, $\mathbf{H}_{\lambda\mu}$ we find that (see (4.10), (4.11))

$$(5.7) \quad \omega_{13} = \omega_{23} = \mathbf{H}_{13} = \mathbf{H}_{23} = 0.$$

From (4.30), (4.32), we find by use of (5.5)

$$\begin{aligned}
 (5.8) \quad (fv_z + dH_z) + (fv_r + dH_r)T &= 0, \\
 ({}'fv_z - {}'dH_z) {}'T + ({}'fv_r - {}'dH_r) &= 0
 \end{aligned}$$

where v_z , v_r and H_z , H_r denote the components of v_λ and H_λ , respectively. Note that the first set of characteristic relations is (see (5.5))

$$(5.9) \quad \frac{\partial r}{\partial \alpha} = T \frac{\partial z}{\partial \alpha}, \quad \frac{\partial z}{\partial \beta} = {}'T \frac{\partial r}{\partial \beta}.$$

Since T^3 , $'T^3$ are both zero, the relation (4.31) and the second of (4.32) are not needed.

Using these results we are now ready to find the characteristic system for the flow. Equation (4.35) reduces to

$$\begin{aligned}
 (5.10) \quad b \left[T \left(\frac{\partial v_z}{\partial \alpha} + 2\omega_{21} \frac{\partial r}{\partial \alpha} \right) - \left(\frac{\partial v_r}{\partial \alpha} + 2\omega_{12} \frac{\partial z}{\partial \alpha} \right) \right] - d \left(\frac{\partial v_z}{\partial \alpha} + T \frac{\partial v_r}{\partial \alpha} \right) \\
 - a^2 (1 + T^2)^{\frac{1}{2}} \frac{\partial \ln \rho}{\partial \alpha} + 2 \frac{\eta}{\rho} e \left(\frac{\partial z}{\partial \alpha} + T \frac{\partial r}{\partial \alpha} \right) \mathbf{H}_{21} = 0.
 \end{aligned}$$

Since all partial derivatives with respect to γ vanish, equations (4.36), (4.37) are identically equal to zero. Equations (4.39), (4.40) become

$$\begin{aligned}
 (5.11) \quad (e - fT) \frac{\partial v_z}{\partial \alpha} + (eT + f) \frac{\partial v_r}{\partial \alpha} - e(1 + T^2)^{\frac{1}{2}} \xi v_r - (b + dT) \frac{\partial H_z}{\partial \alpha} \\
 - (bT + d) \frac{\partial H_r}{\partial \alpha} - b(1 + T^2)^{\frac{1}{2}} \xi H_r = 0,
 \end{aligned}$$

$$\begin{aligned}
& (d + bT) \frac{\partial H_z}{\partial \alpha} + (dT - b) \frac{\partial H_r}{\partial \alpha} \\
& + 2 \left[b + \frac{f}{a^2 - b^2} \frac{\eta}{\rho} (de + bf) \right] \left(\frac{\partial z}{\partial \alpha} + T \frac{\partial r}{\partial \alpha} \right) \mathbf{H}_{21} \\
& - f \frac{b^2}{a^2 - b^2} (1 + T^2)^{\frac{1}{2}} \xi v_r \\
& - \left[f \frac{1}{a^2 - b^2} (d^2 - a^2) - \left(\frac{2b}{a^2 - b^2} \frac{df}{dz} + e \right) T \right] \frac{\partial v_z}{\partial \alpha} \\
& - \left[f \frac{1}{a^2 - b^2} (d^2 - a^2) T - \left(\frac{2b}{a^2 - b^2} \frac{df}{dz} + e \right) \right] \frac{\partial v_r}{\partial \alpha} \\
& - 2 \left(\frac{b}{a^2 - b^2} \frac{df}{dz} + e \right) \left(\frac{\partial z}{\partial \alpha} + T \frac{\partial r}{\partial \alpha} \right) \omega_{21} = 0.
\end{aligned}
\tag{5.12}$$

The equations (4.38), (4.41) for ξ and χ become

$$r\xi - (1 + T^2)^{\frac{1}{2}} \frac{\partial z}{\partial \alpha} = 0, \tag{5.13}$$

$$r\chi - (1 + T^2)^{\frac{1}{2}} \frac{\partial r}{\partial \beta} = 0. \tag{5.14}$$

The relations defining ω_{12} and $\mathbf{H}_{12}$ as given by (4.42) and the corresponding equations for the current bivector reduce to

$$\begin{aligned}
(1 + T^2)^{\frac{1}{2}} \chi \left(\frac{\partial v_r}{\partial \alpha} + T \frac{\partial v_z}{\partial \alpha} \right) - (1 + T'^2)^{\frac{1}{2}} \xi \left(\frac{\partial v_z}{\partial \beta} + T \frac{\partial v_r}{\partial \beta} \right) \\
- 2\xi\chi r(1 - T'T)\omega_{12} = 0,
\end{aligned}
\tag{5.15}$$

$$\begin{aligned}
(1 + T^2)^{\frac{1}{2}} \chi \left(\frac{\partial H_r}{\partial \alpha} + T \frac{\partial H_z}{\partial \alpha} \right) - (1 + T'^2)^{\frac{1}{2}} \xi \left(\frac{\partial H_z}{\partial \beta} + T \frac{\partial H_r}{\partial \beta} \right) \\
- 2\xi\chi r(1 - T'T)\mathbf{H}_{12} = 0.
\end{aligned}
\tag{5.16}$$

Furthermore, it is easily verified that the integrability conditions for ω_{12} and $\mathbf{H}_{12}$ are identically satisfied (see (4.45), (4.46)). Hence the complete characteristic system for the axi-symmetric flow is given by (5.10)–(5.12) plus the corresponding equations for β , together with (5.9), (5.13)–(5.16). They constitute twelve quasilinear partial differential equations in the eleven unknown variables H_z , H_r , v_z , v_r , ω_{12} , $\mathbf{H}_{12}$, r , z , ρ , ξ , and χ . Since ten of these equations contain derivatives with respect to only one of the variables α or β , the equations should be very suitable for numerical computation. Note that T , T' are determined by equation (5.8).

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