The Decay of Magnetohydrodynamic Turbulence

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Summary. Using the hypothesis of zero fourth order cumulants, the equations governing the kinetic and magnetic energy spectrum functions in isotropic hydromagnetic turbulence are derived from the Navier-Stokes equation and the equation of motional induction. The role of the correlation between kinetic and magnetic fields is discussed in detail. The decay of a state in which the energy is initially divided among eddies of a particular size is studied and the asymptotic ratio of magnetic to kinetic energy is calculated for small Reynolds numbers.

1. Introduction. The general study of turbulent motion of an electrically conducting fluid in the presence of a magnetic field (that is, magnetohydrodynamic turbulence) is a necessary first step towards the understanding of a number of geophysical and astrophysical problems. For example, it is now generally accepted that the geomagnetic field is caused by some form of "turbulence" in the earth's liquid, electrically conducting core. The theory of sunspots, which are highly magnetic, must involve the discussion of highly conducting gas (photosphere) in the presence of magnetic fields. The existence of an interstellar magnetic field of strength 10^{-5} gauss has been inferred from observations of the polarization of starlight; the corresponding magnetic energy is comparable with both the gravitational and kinetic energy of the matter in the spiral arms of the galaxy, and, according to a discussion of the stability of the spiral arms, it is this magnetic field that is responsible for maintaining them against the action of gravity (CHANDRASEKHAR & FERMI, [8]). It is believed that the acceleration of cosmic rays is in some way associated with the interaction between charged particles and the turbulent magnetic fields in interstellar space (FERMI, [9, 10]). It is the purpose of this paper to investigate several aspects of magnetohydrodynamic turbulence in the simple case of a homogeneous incompressible fluid in which the turbulence is assumed to be both homogeneous and isotropic.

Perhaps the best way of visualizing hydromagnetic interaction is to consider the effect of turbulence on an initially weak magnetic field in an ideal conductor.

It is easy to see that there can be no relative motion between field lines and the fluid because, otherwise, induced e.m.f.'s. would arise and generate infinite currents. In turbulent motion, the average distance between any two particles increases with time, and therefore there will be a general stretching of the field lines and a consequent amplification of the field strength. In a non-ideal conductor (conductivity σ , permeability β , e.m.u.), it is possible for the field lines to move relative to the fluid. If U is a typical velocity and L is a characteristic length, it may be shown that the lines can slip at a velocity which increases as the parameter $4\pi\sigma\beta UL$ decreases. In a similar way, vortex lines move with an inviscid fluid but in a non-ideal fluid (kinematical viscosity ν) the vortex lines can move relative to the fluid with a velocity which increases as the Reynolds number UL/ν decreases. In a system of stationary turbulence interacting with a weak magnetic field, the diffusion of the vortex lines is exactly balanced by their stretching, and therefore the diffusion of the magnetic field lines will be dominated by their stretching if

$$4\pi\sigma\beta UL > UL/\nu,$$

that is, the initially weak field will be amplified if

$$4\pi\sigma\beta\nu > 1.$$

This argument is due to BATCHELOR [1] who also discussed the later stages of the amplification process. As the magnetic energy increases, it becomes necessary to consider its reaction on the turbulent motions themselves which will be modified. BATCHELOR conjectured that, in the final equilibrium state, there would be approximately equal amounts of kinetic energy in the smallest eddies and total magnetic energy.

CHANDRASEKHAR [3, 4] derived a magnetohydrodynamic theory analogous to that of VON KÁRMÁN & HOWARTH [14] in hydrodynamic turbulence. This followed the classical treatment involving statistical correlations between physical quantities at different points of the fluid at the same instant of time. In a later paper, CHANDRASEKHAR [5] introduced the novel feature of correlations taken both at different points and at different instants of time. This technique, in conjunction with the hypothesis of zero fourth order cumulants, enabled CHANDRASEKHAR to obtain an explicit representation of all non-linear effects of interaction between eddies. In a subsequent paper [7] he presented solutions to this deductive theory. From the detailed structure of his basic equations, CHANDRASEKHAR [6] was able to develop and solve a heuristic theory of magnetohydrodynamic turbulence that is the direct analogue of HEISENBERG's theory [11, 12] of hydrodynamic turbulence.

In the present investigation, the authors have reverted to the classical study of correlations taken at the same instant of time and have shown that, by considering correlations between three different points, the hypothesis of zero fourth order cumulants can be used to close the equations governing magnetohydrodynamic turbulence. The investigation is, therefore, the analogue of the

studies of hydrodynamic turbulence by PROUDMAN & REID [16] and TATSUMI [18, 19, 20]. Three equations are derived that relate the kinetic spectrum and magnetic spectrum with the correlation between velocity and magnetic fields, and the particular nature of this correlation is carefully studied. In the last section, results are presented for the final period of decay at low Reynolds numbers of turbulence whose initial hydrodynamic and magnetic energies are isotropically and homogeneously distributed in eddies of one particular size. The authors hope to return to the more important questions of steady magneto-hydrodynamic turbulence in a future paper, and some preliminary results will appear soon (TATSUMI, [21]).

2. Basic tensors. Owing to the algebraic complexity of the problem, it is necessary to make simplifying assumptions. The stress tensor will be supposed to depend on the mean particle velocity $\mathbf{u}$ and the pressure p only. In this way the precise distribution of particle velocity need not be considered and the Navier-Stokes equation replaces Boltzmann's equation. Second, the velocity will be considered to be very much less than the velocity of sound in the fluid, so that $\mathbf{u}$ is closely solenoidal. Since the fluid velocity is much less than the velocity of light, both the convection $\rho_e \mathbf{u}$ of electric charge density ρ_e and displacement currents may be neglected. Further, the force density $\rho_e \mathbf{E}$ due to the action of electric fields $\mathbf{E}$ may be ignored. Third, the electronic collision time will be taken as short compared to their Larmor gyration time. The fluid will then obey Ohm's law in a frame moving locally with the fluid. These assumptions are unrealistic for many astrophysical problems but are tenable for turbulence in the earth's core.

If we measure the magnetic field $\mathbf{h}$ in units of $(4\pi\rho/\beta)^{\frac{1}{2}}$ where ρ is the fluid's density, we find (see CHANDRASEKHAR, [3])

$$(2.1) \quad \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k} (u_i u_k - h_i h_k) = -\frac{\partial \varpi}{\partial x_i} + \nu \nabla^2 u_i ,$$

$$(2.2) \quad \frac{\partial h_i}{\partial t} + \frac{\partial}{\partial x_k} (h_i u_k - h_k u_i) = \lambda \nabla^2 h_i ,$$

where

$$(2.3) \quad \lambda = \frac{1}{4\pi\sigma\beta} , \quad \varpi = \frac{p}{\rho} + \frac{1}{2} |\mathbf{h}|^2 ,$$

and σ is the electrical conductivity of the fluid and all electromagnetic quantities are measured in e.m.u. In deriving equations (2.1), (2.2) use has been made of the continuity conditions

$$(2.4) \quad \frac{\partial u_i}{\partial x_i} = 0, \quad \frac{\partial h_i}{\partial x_i} = 0.$$

During the course of the analysis, it is necessary to refer to a number of correlation and spectrum tensors. For convenience these are tabulated below. Unprimed quantities are to be evaluated at the point $\mathbf{x}$, primed at $\mathbf{x}'$, doubly

primed at $\mathbf{x}''$, and all at the time t . Bars imply statistical averages (over ensembles) of the expressions below them. The vectors $\mathbf{r}$, $\mathbf{r}'$ are defined by

$$(2.5) \quad \mathbf{r} = \mathbf{x}' - \mathbf{x}, \quad \mathbf{r}' = \mathbf{x}'' - \mathbf{x},$$

and, in view of the assumed homogeneity of the turbulence, the basic tensors depend on $\mathbf{x}$, $\mathbf{x}'$, $\mathbf{x}''$ only as functions of $\mathbf{r}$, $\mathbf{r}'$. They also depend on t but this variable will be suppressed except where confusion might otherwise arise.

TABLE

Definitions of basic tensors.

First order:

$$\left. \begin{aligned} \overline{\varpi u_i'} &= L_{0i}^{(1)}(\mathbf{r}) \\ \overline{\varpi h_i'} &= L_{0i}^{(2)}(\mathbf{r}) \end{aligned} \right\} = \int \Lambda_{0i}^{(n)}(\mathbf{\kappa}) e^{i\mathbf{\kappa} \cdot \mathbf{r}} d\mathbf{\kappa}.$$

Second order:

$$\left. \begin{aligned} \overline{u_i u_i'} &= B_{ii}^{(1)}(\mathbf{r}) \\ \overline{u_i h_i'} &= B_{ii}^{(2)}(\mathbf{r}) \\ \overline{h_i h_i'} &= B_{ii}^{(3)}(\mathbf{r}) \end{aligned} \right\} = \int \Phi_{ii}^{(n)}(\mathbf{\kappa}) e^{i\mathbf{\kappa} \cdot \mathbf{r}} d\mathbf{\kappa},$$

$$\left. \begin{aligned} \overline{\varpi u_i' u_i''} &= P_{0ii}^{(1)}(\mathbf{r}, \mathbf{r}') \\ \overline{\varpi u_i' h_i''} &= P_{0ii}^{(2)}(\mathbf{r}, \mathbf{r}') \\ \overline{\varpi h_i' h_i''} &= P_{0ii}^{(3)}(\mathbf{r}, \mathbf{r}') \end{aligned} \right\} = \iint \Pi_{0ii}^{(n)}(\mathbf{\kappa}, \mathbf{\kappa}') e^{i(\mathbf{\kappa} \cdot \mathbf{r} + \mathbf{\kappa}' \cdot \mathbf{r}')} d\mathbf{\kappa} d\mathbf{\kappa}'.$$

Third order:

$$\left. \begin{aligned} \overline{u_i u_i' u_k''} &= T_{iik}^{(1)}(\mathbf{r}, \mathbf{r}') \\ \overline{u_i h_i' u_k''} &= T_{iik}^{(2)}(\mathbf{r}, \mathbf{r}') \\ \overline{h_i u_i' h_k''} &= T_{iik}^{(3)}(\mathbf{r}, \mathbf{r}') \\ \overline{h_i h_i' h_k''} &= T_{iik}^{(4)}(\mathbf{r}, \mathbf{r}') \end{aligned} \right\} = \iint \Psi_{iik}^{(n)}(\mathbf{\kappa}, \mathbf{\kappa}') e^{i(\mathbf{\kappa} \cdot \mathbf{r} + \mathbf{\kappa}' \cdot \mathbf{r}')} d\mathbf{\kappa} d\mathbf{\kappa}'.$$

Fourth order:

$$\left. \begin{aligned} \overline{u_i u_i u_i' u_k''} - \overline{u_i u_i} \cdot \overline{u_i' u_k''} &= Q_{(iil)ik}^{(1)}(\mathbf{r}, \mathbf{r}') \\ \overline{u_i u_i u_i' h_k''} - \overline{u_i u_i} \cdot \overline{u_i' h_k''} &= Q_{(iil)ik}^{(2)}(\mathbf{r}, \mathbf{r}') \\ \overline{u_i h_i u_i' u_k''} - \overline{u_i h_i} \cdot \overline{u_i' u_k''} &= Q_{(iil)ik}^{(3)}(\mathbf{r}, \mathbf{r}') \\ \overline{u_i u_i h_i' h_k''} - \overline{u_i u_i} \cdot \overline{h_i' h_k''} &= Q_{(iil)ik}^{(4)}(\mathbf{r}, \mathbf{r}') \\ \overline{u_i h_i u_i' h_k''} - \overline{u_i h_i} \cdot \overline{u_i' h_k''} &= Q_{(iil)ik}^{(5)}(\mathbf{r}, \mathbf{r}') \\ \overline{h_i h_i u_i' u_k''} - \overline{h_i h_i} \cdot \overline{u_i' u_k''} &= Q_{(iil)ik}^{(6)}(\mathbf{r}, \mathbf{r}') \\ \overline{u_i h_i h_i' h_k''} - \overline{u_i h_i} \cdot \overline{h_i' h_k''} &= Q_{(iil)ik}^{(7)}(\mathbf{r}, \mathbf{r}') \\ \overline{h_i h_i u_i' h_k''} - \overline{h_i h_i} \cdot \overline{u_i' h_k''} &= Q_{(iil)ik}^{(8)}(\mathbf{r}, \mathbf{r}') \\ \overline{h_i h_i h_i' h_k''} - \overline{h_i h_i} \cdot \overline{h_i' h_k''} &= Q_{(iil)ik}^{(9)}(\mathbf{r}, \mathbf{r}') \end{aligned} \right\} = \iint \Omega_{(iil)ik}^{(n)}(\mathbf{\kappa}, \mathbf{\kappa}') e^{i(\mathbf{\kappa} \cdot \mathbf{r} + \mathbf{\kappa}' \cdot \mathbf{r}')} d\mathbf{\kappa} d\mathbf{\kappa}'.$$

3. Tensor equations. By multiplying the Navier-Stokes equations (2.1) for u_i, u'_i by u'_j, u_i respectively, adding, and averaging the result over ensembles, we obtain an equation relating $B_{ij}^{(1)}$ to the first and third order tensors:

$$(3.1) \quad \left[\frac{\partial}{\partial t} - 2\nu \nabla^2 \right] B_{ij}^{(1)}(\mathbf{r}) = \frac{\partial}{\partial r_k} \{ T_{ijk}^{(1)}(\mathbf{r}, \mathbf{0}) - T_{ijk}^{(1)}(-\mathbf{r}, \mathbf{0}) - T_{ijk}^{(3)}(\mathbf{r}, \mathbf{0}) \\ + T_{ijk}^{(3)}(-\mathbf{r}, \mathbf{0}) \} + \frac{\partial}{\partial r_i} L_{0j}^{(1)}(\mathbf{r}) - \frac{\partial}{\partial r_j} L_{0i}^{(1)}(-\mathbf{r}).$$

The Fourier transform of this result is

$$(3.2) \quad \left[\frac{\partial}{\partial t} + 2\nu \kappa^2 \right] \Phi_{ij}^{(1)}(\kappa) = i\kappa_k \int \{ \Psi_{ijk}^{(1)}(\kappa, \kappa') - \Psi_{ijk}^{(1)}(-\kappa, \kappa') - \Psi_{ijk}^{(3)}(\kappa, \kappa') \\ + \Psi_{ijk}^{(3)}(-\kappa, \kappa') \} d\kappa' + i\kappa_i \Lambda_{0j}^{(1)}(\kappa) - i\kappa_j \Lambda_{0i}^{(1)}(-\kappa).$$

In a similar manner, making use of both equations (2.1) and (2.2), we obtain

$$(3.3) \quad \left[\frac{\partial}{\partial t} + (\nu + \lambda) \kappa^2 \right] \Phi_{ij}^{(2)}(\kappa) = i\kappa_k \int \{ \Psi_{ijk}^{(2)}(\kappa, \kappa') - \Psi_{ijk}^{(4)}(\kappa, \kappa') \\ - \Psi_{kji}^{(2)}(\kappa', -\kappa) + \Psi_{kji}^{(2)}(\kappa', -\kappa) \} d\kappa' + i\kappa_i \Lambda_{0j}^{(2)}(\kappa),$$

$$(3.4) \quad \left[\frac{\partial}{\partial t} + 2\lambda \kappa^2 \right] \Phi_{ij}^{(3)}(\kappa) = i\kappa_k \int \{ \Psi_{ijk}^{(3)}(\kappa', \kappa) - \Psi_{ijk}^{(3)}(\kappa', -\kappa) \\ - \Psi_{kji}^{(3)}(\kappa', \kappa) + \Psi_{kji}^{(3)}(\kappa', -\kappa) \} d\kappa'.$$

Equations (3.2) to (3.4) are the basic equations for the second order spectrum tensors.

In order to obtain equations for the third order moments, we start from the Navier-Stokes equations for u_i, u'_i, u''_i , multiply them by $u'_i u''_k, u_i u''_k, u_i u'_i$ respectively, add, and average over ensembles:

$$(3.5) \quad \left[\frac{\partial}{\partial t} - \nu \left\{ \left(\frac{\partial}{\partial r_i} + \frac{\partial}{\partial r'_i} \right)^2 + \frac{\partial^2}{\partial r_i^2} + \frac{\partial^2}{\partial r'_i{}^2} \right\} \right] T_{ijk}^{(1)}(\mathbf{r}, \mathbf{r}') \\ = \left(\frac{\partial}{\partial r_i} + \frac{\partial}{\partial r'_i} \right) \{ Q_{(il)jk}^{(1)}(\mathbf{r}, \mathbf{r}') - Q_{(il)jk}^{(6)}(\mathbf{r}, \mathbf{r}') \} \\ - \frac{\partial}{\partial r_i} \{ Q_{(il)ki}^{(1)}(\mathbf{r}' - \mathbf{r}, -\mathbf{r}) - Q_{(il)ki}^{(6)}(\mathbf{r}' - \mathbf{r}, -\mathbf{r}) \} \\ - \frac{\partial}{\partial r'_i} \{ Q_{(kl)ij}^{(1)}(-\mathbf{r}', \mathbf{r} - \mathbf{r}') - Q_{(kl)ij}^{(6)}(-\mathbf{r}', \mathbf{r} - \mathbf{r}') \} \\ + \left(\frac{\partial}{\partial r_i} + \frac{\partial}{\partial r'_i} \right) P_{0ik}^{(1)}(\mathbf{r}, \mathbf{r}') \\ - \frac{\partial}{\partial r_i} P_{0ik}^{(1)}(-\mathbf{r}, \mathbf{r}' - \mathbf{r}) - \frac{\partial}{\partial r'_i} P_{0ik}^{(1)}(-\mathbf{r}', \mathbf{r} - \mathbf{r}').$$

Expressed in terms of Fourier transforms this result is

$$(3.6) \quad \left[\frac{\partial}{\partial t} + \nu(\kappa^2 + \kappa'^2 + \kappa''^2) \right] \Psi_{ijk}^{(1)}(\mathbf{\kappa}, \mathbf{\kappa}') \\ = -i \sum_{\mathbf{\kappa}, \mathbf{\kappa}', \mathbf{\kappa}''}^{ijk} [\kappa_l'' \{ \Omega_{(il)jk}^{(1)}(\mathbf{\kappa}, \mathbf{\kappa}') - \Omega_{(il)jk}^{(6)}(\mathbf{\kappa}, \mathbf{\kappa}') \} + \kappa_l'' \Pi_{0jk}^{(1)}(\mathbf{\kappa}, \mathbf{\kappa}')],$$

where $\mathbf{\kappa}''$ is defined by

$$(3.7) \quad \mathbf{\kappa} + \mathbf{\kappa}' + \mathbf{\kappa}'' = 0$$

and the summation runs over simultaneous cyclic changes of $\mathbf{\kappa}$, $\mathbf{\kappa}'$, $\mathbf{\kappa}''$ and j, k, i ; *i.e.* if $S_{ijk}(\mathbf{\kappa}, \mathbf{\kappa}', \mathbf{\kappa}'')$ is the ijk component of a tensor involving $\mathbf{\kappa}$, $\mathbf{\kappa}'$, $\mathbf{\kappa}''$, then

$$\sum_{\mathbf{\kappa}, \mathbf{\kappa}', \mathbf{\kappa}''}^{ijk} S_{ijk}(\mathbf{\kappa}, \mathbf{\kappa}', \mathbf{\kappa}'') = S_{ijk}(\mathbf{\kappa}, \mathbf{\kappa}', \mathbf{\kappa}'') + S_{jki}(\mathbf{\kappa}', \mathbf{\kappa}'', \mathbf{\kappa}) + S_{kij}(\mathbf{\kappa}'', \mathbf{\kappa}, \mathbf{\kappa}').$$

The solenoidal relationship (2.4) implies that

$$(3.8) \quad \kappa_i'' \Psi_{ijk}^{(1)}(\mathbf{\kappa}, \mathbf{\kappa}') = 0,$$

so that, on multiplying equation (3.6) by κ_i'' we obtain

$$(3.9) \quad \Pi_{0jk}^{(1)}(\mathbf{\kappa}, \mathbf{\kappa}') = -\frac{\kappa_i'' \kappa_l''}{\kappa''^2} \{ \Omega_{(il)jk}^{(1)}(\mathbf{\kappa}, \mathbf{\kappa}') - \Omega_{(il)jk}^{(6)}(\mathbf{\kappa}, \mathbf{\kappa}') \}.$$

On using this result to eliminate $\Pi_{0jk}^{(1)}$ from equation (3.6), we obtain

$$(3.10) \quad \left[\frac{\partial}{\partial t} + \nu(\kappa^2 + \kappa'^2 + \kappa''^2) \right] \Psi_{ijk}^{(1)}(\mathbf{\kappa}, \mathbf{\kappa}') \\ = -i \sum_{\mathbf{\kappa}, \mathbf{\kappa}', \mathbf{\kappa}''}^{i,j,k,i} \kappa_l'' \Delta_{im}(\mathbf{\kappa}'') \{ \Omega_{(ml)jk}^{(1)}(\mathbf{\kappa}, \mathbf{\kappa}') - \Omega_{(ml)jk}^{(6)}(\mathbf{\kappa}, \mathbf{\kappa}') \},$$

where $\Delta_{im}(\mathbf{\kappa}'')$ is the abbreviation introduced by PROUDMAN & REID (1955):

$$(3.11) \quad \Delta_{ij}(\mathbf{\kappa}) = \delta_{ij} - \frac{\kappa_i \kappa_j}{\kappa^2}, \quad \kappa = |\mathbf{\kappa}|.$$

In a similar manner we obtain the equations

$$(3.12) \quad \left[\frac{\partial}{\partial t} + \lambda \kappa^2 + \nu(\kappa'^2 + \kappa''^2) \right] \Psi_{ijk}^{(2)}(\mathbf{\kappa}, \mathbf{\kappa}') \\ = -i \sum_{\mathbf{\kappa}, \mathbf{\kappa}'}^{k,i} \kappa_l'' \Delta_{im}(\mathbf{\kappa}'') \{ \Omega_{(ml)kj}^{(2)}(\mathbf{\kappa}', \mathbf{\kappa}) - \Omega_{(ml)kj}^{(8)}(\mathbf{\kappa}', \mathbf{\kappa}) \} \\ - i \kappa_l \{ \Omega_{(il)jk}^{(3)}(\mathbf{\kappa}'', \mathbf{\kappa}') - \Omega_{(il)jk}^{(3)}(\mathbf{\kappa}'', \mathbf{\kappa}') \}, \\ \left[\frac{\partial}{\partial t} + \nu \kappa^2 + \lambda(\kappa'^2 + \kappa''^2) \right] \Psi_{ijk}^{(3)}(\mathbf{\kappa}, \mathbf{\kappa}')$$

$$\begin{aligned}
(3.13) \quad &= -i \sum_{\mathbf{k}, \mathbf{k}'}^{k, i} \kappa_l'' \{ \Omega_{(li)jk}^{(5)}(\mathbf{k}, \mathbf{k}') - \Omega_{(li)jk}^{(5)}(\mathbf{k}, \mathbf{k}) \} \\
&\quad - i \kappa_l \Delta_{jm}(\mathbf{k}) \{ \Omega_{(ml)ik}^{(4)}(\mathbf{k}'', \mathbf{k}') - \Omega_{(ml)ik}^{(9)}(\mathbf{k}'', \mathbf{k}') \}, \\
(3.14) \quad &\left[\frac{\partial}{\partial t} + \lambda(\kappa^2 + \kappa'^2 + \kappa''^2) \right] \Psi_{ijk}^{(4)}(\mathbf{k}, \mathbf{k}') \\
&= -i \sum_{\mathbf{k}, \mathbf{k}', \mathbf{k}''}^{j, k, i} \kappa_l'' \{ \Omega_{(li)jk}^{(7)}(\mathbf{k}, \mathbf{k}') - \Omega_{(li)jk}^{(7)}(\mathbf{k}, \mathbf{k}') \}.
\end{aligned}$$

Equations (3.10), (3.12) to (3.14) are the basic equations for the third order spectrum tensors.

To close the equations, we make use of the hypothesis of zero fourth order cumulants; namely, if X_1, X_2, X_3, X_4 are four statistical variables, then

$$(3.15) \quad \overline{X_1^* X_2^* X_3^* X_4^*} = \overline{X_1^* X_2^*} \cdot \overline{X_3^* X_4^*} + \overline{X_1^* X_3^*} \cdot \overline{X_2^* X_4^*} + \overline{X_1^* X_4^*} \cdot \overline{X_2^* X_3^*},$$

where

$$(3.16) \quad X_n^* = X_n - \overline{X}_n.$$

The mathematical statement of this hypothesis and its physical basis given by one of us (TATSUMI, [19], pp. 23–24) is not precise (see ROBERTS, [17].) The correct argument appears to be the following. The product $\overline{X_1 \cdot X_2}$ of the mean values of two physical variables will, in general, differ from the mean value $\overline{X_1 X_2}$ of their product. The deviation, or cumulant,

$$(3.17) \quad \varepsilon_2(X_1, X_2) = \overline{X_1 X_2} - \overline{X_1} \cdot \overline{X_2}$$

arises from the proximity of the two points at which the variables are measured and falls to zero as their separation increases. In the same way, we can estimate the net effect of the closeness of three points upon a triple moment $\overline{X_1 X_2 X_3}$ by considering the cumulant

$$\begin{aligned}
(3.18) \quad \varepsilon_3(X_1, X_2, X_3) = &\overline{X_1 X_2 X_3} - \overline{X_1} \cdot \overline{X_2 X_3} - \overline{X_2} \cdot \overline{X_3 X_1} - \overline{X_3} \cdot \overline{X_1 X_2} \\
&+ 2\overline{X_1} \cdot \overline{X_2} \cdot \overline{X_3},
\end{aligned}$$

and, for four points, similarly

$$\begin{aligned}
(3.19) \quad \varepsilon_4(X_1, X_2, X_3, X_4) = &\overline{X_1 X_2 X_3 X_4} - \overline{X_1} \cdot \overline{X_2 X_3 X_4} - \overline{X_2} \cdot \overline{X_3 X_4 X_1} \\
&- \overline{X_3} \cdot \overline{X_4 X_1 X_2} - \overline{X_4} \cdot \overline{X_1 X_2 X_3} - \overline{X_1 X_2} \cdot \overline{X_3 X_4} - \overline{X_1 X_3} \cdot \overline{X_4 X_2} \\
&- \overline{X_1 X_4} \cdot \overline{X_2 X_3} + 2\overline{X_1} \cdot \overline{X_2} \cdot \overline{X_3 X_4} + 2\overline{X_1} \cdot \overline{X_3} \cdot \overline{X_4 X_2} \\
&+ 2\overline{X_1} \cdot \overline{X_4} \cdot \overline{X_2 X_3} + 2\overline{X_1 X_2} \cdot \overline{X_3} \cdot \overline{X_4} + 2\overline{X_1 X_3} \cdot \overline{X_4} \cdot \overline{X_2} \\
&+ 2\overline{X_1 X_4} \cdot \overline{X_2} \cdot \overline{X_3} - 6\overline{X_1} \cdot \overline{X_2} \cdot \overline{X_3} \cdot \overline{X_4}.
\end{aligned}$$

Now we make the hypothesis that the effect of the proximity of four points upon their moment $\overline{X_1 X_2 X_3 X_4}$ is negligible, that is,

$$(3.20) \quad \varepsilon_4 = 0,$$

and this is identical to equation (3.15). It is worth noting that equation (3.20) is *identically* true if any *one* or *two* of the four points is far from the others. Thus inaccuracies can only arise from the simultaneous closeness of all four points. In κ -space, this corresponds to the high wave number regions. Therefore our approximation will give good results if and only if the main body of energy is distributed among the small wave number components. Recently KRAICHNAN [15] has severely criticised the hypothesis. The majority of his arguments, however, are relevant only to the case of correlations taken at different instants of time. For the case of simultaneous correlations, such as are considered in the present paper, despite his claim he offers only an analogy that is far from convincing.

Applied to the tensor $Q_{(il)jk}^{(1)}(\mathbf{r}, \mathbf{r}')$, the hypothesis takes the form

$$(3.21) \quad Q_{(il)jk}^{(1)}(\mathbf{r}, \mathbf{r}') = B_{ij}^{(1)}(\mathbf{r})B_{lk}^{(1)}(\mathbf{r}') + B_{ik}^{(1)}(\mathbf{r}')B_{lj}^{(1)}(\mathbf{r}).$$

This and the similar results for the other quartic tensors lead to the following equations for the spectrum tensors:

$$(3.22) \quad \left. \begin{aligned} \Omega_{(il)jk}^{(1)}(\kappa, \kappa') &= \Phi_{ij}^{(1)}(\kappa)\Phi_{lk}^{(1)}(\kappa') + \Phi_{ik}^{(1)}(\kappa')\Phi_{lj}^{(1)}(\kappa), \\ \Omega_{(il)jk}^{(2)}(\kappa, \kappa') &= \Phi_{ij}^{(1)}(\kappa)\Phi_{lk}^{(2)}(\kappa') + \Phi_{ik}^{(2)}(\kappa')\Phi_{lj}^{(1)}(\kappa), \\ \Omega_{(il)jk}^{(3)}(\kappa, \kappa') &= \Phi_{ij}^{(1)}(\kappa)\Phi_{kl}^{(2)}(-\kappa') + \Phi_{ik}^{(1)}(\kappa')\Phi_{lj}^{(2)}(-\kappa), \\ \Omega_{(il)jk}^{(4)}(\kappa, \kappa') &= \Phi_{ij}^{(2)}(\kappa)\Phi_{lk}^{(2)}(\kappa') + \Phi_{ik}^{(2)}(\kappa')\Phi_{lj}^{(2)}(\kappa), \\ \Omega_{(il)jk}^{(5)}(\kappa, \kappa') &= \Phi_{ij}^{(1)}(\kappa)\Phi_{lk}^{(3)}(\kappa') + \Phi_{ik}^{(2)}(\kappa')\Phi_{lj}^{(2)}(-\kappa), \\ \Omega_{(il)jk}^{(6)}(\kappa, \kappa') &= \Phi_{ij}^{(2)}(-\kappa)\Phi_{kl}^{(2)}(-\kappa') + \Phi_{ik}^{(2)}(-\kappa')\Phi_{lj}^{(2)}(-\kappa), \\ \Omega_{(il)jk}^{(7)}(\kappa, \kappa') &= \Phi_{ij}^{(2)}(\kappa)\Phi_{lk}^{(3)}(\kappa') + \Phi_{ik}^{(2)}(\kappa')\Phi_{lj}^{(3)}(\kappa), \\ \Omega_{(il)jk}^{(8)}(\kappa, \kappa') &= \Phi_{ij}^{(2)}(-\kappa)\Phi_{lk}^{(3)}(\kappa') + \Phi_{ik}^{(3)}(\kappa')\Phi_{lj}^{(2)}(-\kappa), \\ \Omega_{(il)jk}^{(9)}(\kappa, \kappa') &= \Phi_{ij}^{(3)}(\kappa)\Phi_{lk}^{(3)}(\kappa') + \Phi_{ik}^{(3)}(\kappa')\Phi_{lj}^{(3)}(\kappa). \end{aligned} \right\}$$

4. Isotropic forms. The vector $\mathbf{u}$ must be polar since it measures the rate of displacement of material particles and displacement is essentially a polar vector. Similarly, the electric current $\mathbf{j}$, measuring as it does the rate of displacement of electric charge, must also be a polar vector.

Since the relationship between $\mathbf{h}$ and $\mathbf{j}$ is skew:

$$(4.1) \quad \mathbf{j} = \text{curl } \mathbf{h}$$

($\mathbf{j}$ measured in units of $\sqrt{4\pi\beta/\rho}$), it follows that $\mathbf{h}$ must be skew. Therefore those correlation tensors between $\mathbf{u}$ and $\mathbf{h}$ and the corresponding Fourier transforms of these tensors that involve $\mathbf{h}$ an odd number of times are skew and those involving $\mathbf{h}$ an even number of times are polar. Since both $\mathbf{u}$ and $\mathbf{h}$ are solenoidal, the isotropic forms of the basic tensors are

$$\begin{aligned}
\Delta_{0i}^{(1),(2)}(\mathbf{\kappa}) &= 0, \\
\Phi_{ij}^{(1),(3)}(\mathbf{\kappa}) &= \Phi^{(1),(3)}(\kappa) \Delta_{ij}(\mathbf{\kappa}), \\
\Phi_{ij}^{(2)}(\mathbf{\kappa}) &= \Phi^{(2)}(\kappa) E_{ij}(\mathbf{\kappa}), \\
\Psi_{(ik)i}^{(1),(31)}(\mathbf{\kappa}) &= \int \Psi_{ijk}^{(1),(3)}(\mathbf{\kappa}, \mathbf{\kappa}') d\mathbf{\kappa}' \\
&= -\frac{i}{2} \Psi^{(1),(31)}(\kappa) \left\{ \frac{\kappa_i}{\kappa} \Delta_{kj}(\mathbf{\kappa}) + \frac{\kappa_k}{\kappa} \Delta_{ij}(\mathbf{\kappa}) \right\}, \\
(4.2) \quad \Psi_{(ik)i}^{(32)}(\mathbf{\kappa}) &= \int \Psi_{ijk}^{(3)}(\mathbf{\kappa}', \mathbf{\kappa}) d\mathbf{\kappa}' \\
&= -\frac{i}{2} \left\{ \Psi_1^{(32)}(\kappa) \frac{\kappa_i}{\kappa} \Delta_{kj}(\mathbf{\kappa}) + \Psi_2^{(32)}(\kappa) \frac{\kappa_k}{\kappa} \Delta_{ij}(\mathbf{\kappa}) \right\}, \\
\Psi_{(ik)i}^{(21),(4)}(\mathbf{\kappa}) &= \int \Psi_{ijk}^{(2),(4)}(\mathbf{\kappa}, \mathbf{\kappa}') d\mathbf{\kappa}' \\
&= -\frac{i}{2} \Psi^{(21),(4)}(\kappa) \left\{ \frac{\kappa_i}{\kappa} E_{kj}(\mathbf{\kappa}) + \frac{\kappa_k}{\kappa} E_{ij}(\mathbf{\kappa}) \right\}, \\
\Psi_{(ik)i}^{(22)}(\mathbf{\kappa}) &= \int \Psi_{ijk}^{(2)}(\mathbf{\kappa}', \mathbf{\kappa}) d\mathbf{\kappa}' \\
&= -\frac{i}{2} \left\{ \Psi_1^{(22)}(\kappa) \frac{\kappa_i}{\kappa} E_{kj}(\mathbf{\kappa}) + \Psi_2^{(22)}(\kappa) \frac{\kappa_k}{\kappa} E_{ij}(\mathbf{\kappa}) \right\}.
\end{aligned}$$

where $E_{ij}(\mathbf{\kappa})$ is defined by

$$(4.3) \quad E_{ij}(\mathbf{\kappa}) = \frac{i\kappa_k}{\kappa} \epsilon_{ijk},$$

and ϵ_{ijk} is the permutation ϵ which is $+1$ if i, j, k is an even permutation of $1, 2, 3$, is -1 if i, j, k is an odd permutation of $1, 2, 3$, and is zero otherwise. The isotropic forms (4.2) which involve $\Delta_{ij}(\mathbf{\kappa})$ have already been derived and discussed by TATSUMI [19]. Those which involve the less familiar $E_{ij}(\mathbf{\kappa})$ can be derived in an exactly analogous fashion.

It is worth noticing that an inequality exists between the scalars $\Phi^{(n)}$. If we denote by u_i, j'_k any two particular components of $\mathbf{u}, \mathbf{j}$ at two different points, it follows from Schwartz' inequality that

$$(4.4) \quad |\overline{u_i j'_k}| \leq \frac{1}{3} u j$$

where u, j are the absolute values of $\mathbf{u}, \mathbf{j}$. Now, from equations (4.1) and (4.2), we have

$$(4.5) \quad \overline{u_i j'_k} = \epsilon_{klm} \frac{\partial}{\partial x_l} B_{im}^{(2)}(\mathbf{r}) = \int \kappa \Phi^{(2)}(\kappa) \Delta_{ik}(\mathbf{\kappa}) e^{i\mathbf{\kappa} \cdot \mathbf{r}} d\mathbf{\kappa},$$

so that

$$(4.6) \quad \overline{u_i j'_i} = \frac{8\pi}{r} \int_0^\infty \Phi^{(2)}(\kappa) \kappa^2 \sin \kappa r d\kappa.$$

Hence, from inequality (4.4),

$$(4.7) \quad \left[\frac{8\pi}{r} \int_0^\infty \Phi^{(2)}(\kappa) \kappa^2 \sin \kappa r d\kappa \right]^2 \leq \left[8\pi \int_0^\infty \Phi^{(1)}(\kappa) \kappa^2 d\kappa \right] \left[8\pi \int_0^\infty \Phi^{(3)}(\kappa) \kappa^4 d\kappa \right]$$

for all values of r , and this implies

$$(4.8) \quad \left[\int_0^\infty \Phi^{(2)}(\kappa) \kappa^3 d\kappa \right]^2 \leq \left[\int_0^\infty \Phi^{(1)}(\kappa) \kappa^2 d\kappa \right] \left[\int_0^\infty \Phi^{(3)}(\kappa) \kappa^4 d\kappa \right].$$

5. Equations for defining scalars. In terms of the defining scalars introduced in the last section, equations (3.2) to (3.4) become

$$(5.1) \quad \left[\frac{\partial}{\partial t} + 2\nu\kappa^2 \right] \Phi^{(1)}(\kappa) = \kappa \{ \Psi^{(1)}(\kappa) - \Psi^{(31)}(\kappa) \},$$

$$(5.2) \quad \left[\frac{\partial}{\partial t} + (\nu + \lambda)\kappa^2 \right] \Phi^{(2)}(\kappa) = \frac{\kappa}{2} \{ \Psi^{(21)}(\kappa) + \Psi^{(22)}(\kappa) - \Psi^{(4)}(\kappa) \},$$

$$(5.3) \quad \left[\frac{\partial}{\partial t} + 2\lambda\kappa^2 \right] \Phi^{(3)}(\kappa) = \kappa \Psi^{(32)}(\kappa),$$

where

$$(5.4) \quad \Psi^{(22), (32)}(\kappa) = \Psi_2^{(22), (32)}(\kappa) - \Psi_1^{(22), (32)}(\kappa).$$

In examining the third order equations, we will assume that, at the instant $t = 0$, all third order tensors and, therefore, their defining scalars are zero. This, of course, is not generally true, but it will be shown later that such is the case for our particular choice of initial conditions for $\Phi^{(n)}(\kappa)$. Moreover, the modifications that are necessary if this assumption is rejected are perfectly straightforward. The solution to equations (3.10), (3.12) to (3.14), using equations (3.22) and the isotropic forms introduced in the last section, are then found to be

$$\begin{aligned} \Psi^{(1)}(\kappa) = & -\frac{4\pi}{\kappa} \iiint \exp [-\nu(\kappa^2 + \kappa'^2 + \kappa''^2)(t - t')] \\ & \cdot \left[\{ \Phi^{(1)}(\kappa) - \Phi^{(1)}(\kappa') \} \Phi^{(1)}(\kappa'') \left(\frac{\kappa^2 \kappa'^2}{\kappa''^2} + \mu \kappa \kappa' \right) + F_1 \right]_t, \\ & \cdot \kappa'^2 (1 - \mu^2) dt' d\kappa' d\mu, \end{aligned}$$

$$\begin{aligned} \Psi^{(21)}(\kappa) = & -4\pi \iiint \exp [-\{\lambda\kappa^2 + \nu(\kappa'^2 + \kappa''^2)\}(t - t')] \\ & \cdot \left[\left\{ \frac{1}{\kappa} \Phi^{(2)}(\kappa) - \frac{1}{\kappa'} \Phi^{(2)}(\kappa') \right\} \Phi^{(1)}(\kappa'') \frac{\kappa^2 \kappa'^2}{\kappa''^2} \right. \\ & + \left\{ \frac{1}{\kappa} \Phi^{(1)}(\kappa'') \Phi^{(2)}(\kappa) - \frac{1}{\kappa''} \Phi^{(1)}(\kappa') \Phi^{(2)}(\kappa'') - \frac{1}{\kappa} \Phi^{(3)}(\kappa) \Phi^{(2)}(\kappa') \right\} \mu \kappa \kappa' \Big]_t, \\ & \cdot \kappa'^2 (1 - \mu)^2 dt' d\kappa' d\mu, \end{aligned}$$

$$\begin{aligned}\Psi^{(22)}(\kappa) = & -4\pi \iiint \exp [-\{\nu(\kappa^2 + \kappa'^2) + \lambda\kappa'^2\}(t - t')] \\ & \cdot \left[\{\kappa\Phi^{(1)}(\kappa')\Phi^{(2)}(\kappa) - \kappa'\Phi^{(1)}(\kappa)\Phi^{(2)}(\kappa')\} \right. \\ & + \frac{1}{\kappa'} \Phi^{(2)}(\kappa') \{\kappa'(\kappa' + \mu\kappa)\Phi^{(1)}(\kappa) - \kappa(\kappa + \mu\kappa')\Phi^{(1)}(\kappa')\} \\ & \left. - \frac{\mu\kappa\kappa'}{\kappa'^{3/2}} \Phi^{(3)}(\kappa') \{\kappa\Phi^{(2)}(\kappa) - \kappa'\Phi^{(2)}(\kappa')\} \right]_{t'}, \kappa'^2(1 - \mu^2) dt' d\kappa' d\mu,\end{aligned}$$

$$\begin{aligned}\Psi^{(31)}(\kappa) = & -\frac{4\pi}{\kappa} \iiint \exp [-\{\nu\kappa^2 + \lambda(\kappa'^2 + \kappa'^2)\}(t - t')] \\ & \cdot \left[\{\Phi^{(1)}(\kappa) - \Phi^{(3)}(\kappa')\}\Phi^{(3)}(\kappa') \frac{\mu\kappa\kappa'^3}{\kappa'^{3/2}} - F_2 \right]_{t'}, \kappa'^2(1 - \mu^2) dt' d\kappa' d\mu,\end{aligned}$$

$$\begin{aligned}\Psi^{(32)}(\kappa) = & -\frac{4\pi}{\kappa} \iiint \exp [-\{\lambda(\kappa^2 + \kappa'^2) + \nu\kappa'^2\}(t - t')] \\ & \cdot \left[\{\Phi^{(3)}(\kappa) - \Phi^{(3)}(\kappa')\}\Phi^{(1)}(\kappa') \frac{\kappa^2\kappa'^2}{\kappa'^{3/2}} - F_1 \right]_{t'}, \kappa'^2(1 - \mu^2) dt' d\kappa' d\mu \\ & + \frac{4\pi}{\kappa} \iiint \exp [-\{\lambda(\kappa^2 + \kappa'^2) + \nu\kappa'^2\}(t - t')] \\ & \cdot \left[\{\Phi^{(3)}(\kappa) - \Phi^{(1)}(\kappa')\}\Phi^{(3)}(\kappa') \frac{\mu\kappa^3\kappa'}{\kappa'^{3/2}} + F_2 \right]_{t'}, \kappa'^2(1 - \mu^2) dt' d\kappa' d\mu,\end{aligned}$$

$$\begin{aligned}\Psi^{(4)}(\kappa) = & -4\pi \iiint \exp [-\lambda(\kappa^2 + \kappa'^2 + \kappa'^2)(t - t')] \\ & \cdot \left[\left\{ \frac{1}{\kappa} \Phi^{(2)}(\kappa) - \frac{1}{\kappa'} \Phi^{(2)}(\kappa') \right\} \Phi^{(3)}(\kappa') \frac{\mu\kappa\kappa'^3}{\kappa'^{3/2}} \right]_{t'}, \kappa'^2(1 - \mu^2) dt' d\kappa' d\mu,\end{aligned}$$

with

$$\begin{aligned}(5.5) \quad 2F_1 = & \Phi^{(2)}(\kappa') \left\{ \frac{1}{\kappa} \Phi^{(2)}(\kappa) - \frac{1}{\kappa'} \Phi^{(2)}(\kappa') \right\} \frac{\kappa^2\kappa'^2}{\kappa'^{3/2}} \\ & - \Phi^{(2)}(\kappa)\Phi^{(2)}(\kappa') \frac{\kappa\kappa'}{\kappa'^{3/2}} (\kappa'^2 - \kappa^2), \\ 2F_2 = & \Phi^{(2)}(\kappa) \left\{ \frac{1}{\kappa'} \Phi^{(2)}(\kappa') - \frac{1}{\kappa} \Phi^{(2)}(\kappa') \right\} \kappa\kappa'^2 \\ & - \Phi^{(2)}(\kappa)\Phi^{(2)}(\kappa') \frac{\kappa^2\kappa'}{\kappa'^{3/2}},\end{aligned}$$

where $\mu = (\kappa \cdot \kappa')/\kappa\kappa'$ and the ranges of integration are the following: t' from 0 to t ; κ' from 0 to ∞ ; μ from -1 to $+1$.

The $\Phi^{(1)}$ products in $\Psi^{(1)}$ redistribute hydrodynamic energy and the first two $\Phi^{(1)}\Phi^{(3)}$ terms of $\Psi^{(32)}$ similarly redistribute magnetic energy. The third $\Phi^{(1)}\Phi^{(3)}$ term of $\Psi^{(32)}$ and the $\Phi^{(1)}\Phi^{(3)}$ product in $\Psi^{(31)}$ change kinetic energy into magnetic energy, whereas the $\Phi^{(3)}$ products reverse this trend. The $\Phi^{(2)}$ products in F_1 , F_2 transfer energy from the magnetic to the hydrodynamic spectrum.

In the inviscid, infinite conductivity case, the equations (5.1) to (5.5) possess a steady solution:

$$(5.6) \quad \Phi^{(1)}(\kappa) = \Phi^{(3)}(\kappa) = \text{constant}, \quad \Phi^{(2)}(\kappa) = 0.$$

This equipartition case corresponds to the solution

$$(5.7) \quad \Phi^{(1)}(\kappa) = \text{constant} \quad [\Phi^{(2)}(\kappa) = \Phi^{(3)}(\kappa) = 0],$$

discovered for the purely hydrodynamic case by HOFF & TITT [13] by entirely different methods.

It may be verified that

$$(5.8) \quad \int_0^\infty \{\Psi^{(1)}(\kappa) - \Psi^{(31)}(\kappa) + \Psi^{(32)}(\kappa)\} \kappa^3 d\kappa = 0.$$

Thus, introducing the expressions $E(\kappa)$ and $H(\kappa)$ for the hydrodynamic and magnetic energy spectrum functions,

$$(5.9) \quad E(\kappa) = 4\pi\kappa^2\Phi^{(1)}(\kappa) \quad \text{and} \quad H(\kappa) = 4\pi\kappa^2\Phi^{(3)}(\kappa),$$

we can establish that there is no dissipation of energy by the non-linear terms:

$$(5.10) \quad \frac{\partial}{\partial t} \left[\int_0^\infty E(\kappa) d\kappa + \int_0^\infty H(\kappa) d\kappa \right] = -2\nu \int_0^\infty E(\kappa) \kappa^2 d\kappa \\ - 2\lambda \int_0^\infty H(\kappa) \kappa^2 d\kappa.$$

This result is not unexpected in view of the analogous equation that can be deduced, assuming reasonable boundary conditions, from equations (2.1) and (2.2), namely,

$$(5.11) \quad \frac{\partial}{\partial t} \left[\int \mathbf{u}^2 dV + \int \mathbf{h}^2 dV \right] = -2\nu \int \boldsymbol{\omega}^2 dV - 2\lambda \int \mathbf{j}^2 dV,$$

where $\boldsymbol{\omega}$ is the vorticity and dV denotes an element of the volume over which the integrations are performed.

6. The decay process. In this section we will investigate the decay of magneto-hydrodynamic turbulence and, as the initial state of the system we will take

$$(6.1) \quad \Phi^{(n)}(\kappa, 0) = a_n \delta(\kappa - \kappa_0),$$

where the quantities a_n , κ_0 are constants and $\delta(x)$ is the Dirac δ -function:

$$\delta(x) = 0 \quad \text{if } x \neq 0; \quad \int_{-\infty}^{\infty} \delta(x) dx = 1.$$

These conditions correspond to a state composed of an infinite number of kinetic and magnetic plane waves having entirely random directions, amplitudes, and phases but possessing a single wave-number κ_0 in common. This supposition of a common wave-number for both the kinetic and magnetic states is not unrealistic if we suppose that the same physical mechanism generates both. Because of the randomness in phase relationship between different wave trains, the longitudinal velocity correlation $\overline{u_p u'_p}$ must be the sum of the contributions from individual wave trains. The contribution to $\overline{u_p u'_p}$ from all waves with amplitude A may be calculated in the following way. Erect moving coordinate axes in such a way that, in calculating $u_p u'_p$ for any particular wave, that wave is represented by

$$(6.2) \quad u_1 = 0, \quad u_2 = A \sin(\kappa_0 x_1 + \xi), \quad u_3 = 0,$$

where ξ is an arbitrary constant. Then the average $\langle u_p u'_p \rangle$ taken over all orientations of the axes is given by

$$(6.3) \quad \langle u_p u'_p \rangle = \frac{\kappa_0}{2\pi} \frac{A^2}{4\pi} \int_0^{2\pi/\kappa_0} \int_0^\pi \int_0^{2\pi} \sin(\kappa_0 x_1 + \xi) \sin \xi (\kappa_0 x'_1 + \xi) \\ \cdot \sin^3 \chi \cos^2 \phi \, dx_1 \, d\chi \, d\phi,$$

where

$$(6.4) \quad x'_1 - x_1 = r \cos \chi,$$

and χ is the angle between the x_1 -axis and the line joining the points P , P' concerned in the correlation. On carrying out the integrations, the result (6.3) reduces to

$$(6.5) \quad \langle u_p u'_p \rangle = \frac{A^2}{2(\kappa_0 r)^2} \left(\frac{\sin \kappa_0 r}{\kappa_0 r} - \cos \kappa_0 r \right).$$

Averaging over all amplitudes therefore leads to the result

$$(6.6) \quad \overline{u_p u'_p} = \frac{\overline{A^2}}{2(\kappa_0 r)^2} \left(\frac{\sin \kappa_0 r}{\kappa_0 r} - \cos \kappa_0 r \right).$$

Similarly, from the magnetic wave

$$(6.7) \quad h_1 = h_2 = 0, \quad h_3 = B \sin(\kappa_0 x_1 + \eta),$$

η being an arbitrary constant, we find

$$(6.8) \quad \overline{h_p h'_p} = \frac{\overline{B^2}}{2(\kappa_0 r)^2} \left(\frac{\sin \kappa_0 r}{\kappa_0 r} - \cos \kappa_0 r \right).$$

The condition of skewness for the tensor $\overline{u_i h'_j}$ implies that the velocity wave

(6.2) is correlated with the magnetic wave (6.7) only. Thus the mixed product becomes

$$(6.9) \quad \langle \epsilon_{pjk} u_j h'_k \rangle = \frac{\kappa_0}{2\pi} \frac{AB}{4\pi} \int_0^{2\pi/\kappa_0} \int_0^\pi \int_0^{2\pi} \sin(\kappa_0 x_1 + \xi) \cdot \sin(\kappa_0 x'_1 + \eta) \sin \chi \cos \chi \, dx_1 \, d\chi \, d\phi,$$

where the suffix p denotes, as before, the component along PP' . On performing the integrations, this result becomes

$$(6.10) \quad \langle \epsilon_{pjk} u_j h'_k \rangle = \frac{AB \cos(\xi - \eta)}{2\kappa_0 r} \left(\frac{\sin \kappa_0 r}{\kappa_0 r} - \cos \kappa_0 r \right),$$

so that, averaging over all amplitudes, we obtain

$$(6.11) \quad \overline{\epsilon_{pjk} u_j h'_k} = \overline{AB \cos(\xi - \eta)} \frac{1}{2\kappa_0 r} \left(\frac{\sin \kappa_0 r}{\kappa_0 r} - \cos \kappa_0 r \right).$$

Equations (6.6), (6.8), (6.11) are precisely the expressions of the conditions (6.1) in physical space and confirm our representation in terms of plane waves. The quantities a_1 , a_2 , a_3 are related to A , B by

$$(6.12) \quad a_1 = \frac{\overline{A^2}}{16\pi\kappa_0^2}, \quad a_2 = \frac{\overline{AB \cos(\xi - \eta)}}{16\pi\kappa_0^2}, \quad a_3 = \frac{\overline{B^2}}{16\pi\kappa_0^2},$$

and therefore it follows that

$$(6.13) \quad a_2^2 \leq a_1 a_3.$$

Thus the argument confirms the inequality (4.8) already established for the general case.

We can also show that the initial condition (6.1) implies that all third order moments vanish for $t = 0$. For example, using the same axes and representation (6.2) for $\mathbf{u}$, we find

$$(6.14) \quad \langle u_x^2 u'_x \rangle = \frac{\kappa_0}{2\pi} \frac{A^3}{4\pi} \iiint \sin^2(\kappa_0 x_1 + \xi) \sin(\kappa_0 x'_1 + \xi) \cdot \sin^4 \chi \cos^3 \phi \, dx_1 \, d\chi \, d\phi \equiv 0.$$

It follows that $T_{ijk}^{(1)}(\mathbf{r}, \mathbf{r}'; 0)$ is identically zero. The same argument applies to all other triple correlations involving velocity or magnetic components, giving the result that their Fourier transforms $\Psi_{ijk}^{(n)}(\boldsymbol{\kappa}, \boldsymbol{\kappa}')$ all vanish at $t = 0$.

To follow the decay of turbulence initiated from the state (6.1), we employ a method of successive approximation which leads to a solution which converges for a particular time t if and only if the Reynolds numbers are sufficiently small. The initial hydrodynamic and magnetic energies are

$$(6.15) \quad \mathcal{E}_0 = 4\pi\kappa_0^2 a_1, \quad \mathcal{H}_0 = 4\pi\kappa_0^2 a_3,$$

and as a Reynolds number we take

$$(6.16) \quad R = \frac{\mathcal{E}_0^{\frac{1}{2}}}{\nu \kappa_0}.$$

It is also convenient to introduce the following notation:

$$(6.17) \quad \gamma = \frac{\lambda}{\nu}, \quad \alpha^2 = \frac{a_3}{a_1} = \frac{\mathcal{H}_0}{\mathcal{E}_0}, \quad \alpha\theta = \frac{a_2}{a_1} \quad (\theta^2 \leq 1),$$

and to replace κ, t by the dimensionless variables s, τ defined by

$$(6.18) \quad s = \frac{\kappa}{\kappa_0}, \quad \tau = \nu \kappa_0^2 t.$$

The expressions for $E(\kappa)$, $H(\kappa)$, and $G(\kappa)$, where

$$(6.19) \quad G(\kappa) = 4\pi \kappa^2 \Phi^{(2)}(\kappa),$$

correct to first order in R^2 are found to be

$$(6.20) \quad \begin{aligned} \frac{E(\kappa)}{\mathcal{E}_0} = & \delta(s-1)e^{-2\tau}[1 + R^2\{M_1(\tau) + \alpha^2\theta^2 M_2(\tau) - 2\alpha^2 M_3(\tau)\}] \\ & + R^2\left\{\left(\frac{e^{-2\tau} - e^{-s^2\tau}}{2 - s^2}\right)^2 + 2\alpha^2\theta^2\left(\frac{e^{-2\tau} - e^{-s^2\tau}}{2 - s^2}\right)\left(\frac{e^{-2\gamma\tau} - e^{-s^2\tau}}{2\gamma - s^2}\right)\right. \\ & \left.+ \alpha^4\left(\frac{e^{-2\gamma\tau} - e^{-s^2\tau}}{2\gamma - s^2}\right)^2\right\}N(s), \\ \frac{H(\kappa)}{\alpha^2\mathcal{E}_0} = & \delta(s-1)e^{-2\gamma\tau}[1 + R^2\{\theta^2 M_1(\tau) + \alpha^2 M_2(\tau) + 2(1 - \theta^2)M_4(\tau)\}] \\ & + R^2\left(1 - \theta^2 + \frac{s^2}{2}\right)\left(\frac{e^{-(1+\gamma)\tau} - e^{-\gamma s^2\tau}}{1 + \gamma - \gamma s^2}\right)^2 N(s), \end{aligned}$$

$$(6.21) \quad \begin{aligned} \frac{G(\kappa)}{\alpha\theta\mathcal{E}_0} = & \delta(s-1)e^{-(1+\gamma)\tau}[1 + R^2\{M_1(\tau) + \alpha^2 M_2(\tau) + \alpha^2 M_3(\tau)\}] \\ & + R^2\left(\frac{e^{-(1+\gamma)\tau} - e^{-\gamma s^2\tau}}{1 + \gamma - \gamma s^2}\right)\left\{\left(\frac{e^{-2\tau} - e^{-s^2\tau}}{2 - s^2}\right) + \alpha^2\left(\frac{e^{-2\gamma\tau} - e^{-s^2\tau}}{2\gamma - s^2}\right)\right\}N(s) \end{aligned}$$

where $N(s)$, $M_1(\tau)$ to $M_4(\tau)$ are defined by

$$N(s) = \begin{cases} \frac{1}{4}s^3\left(1 - \frac{s^2}{4}\right) & \text{for } 0 \leq s \leq 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\begin{aligned}
 M_1(\tau) &= M_2(\tau, \gamma = 1) = M_4(\tau, \gamma = 1), \\
 M_2(\tau) &= -\frac{1}{4\gamma} + \frac{1-\gamma}{4\gamma} e^{-2\gamma\tau} + \frac{1}{8\tau} e^{-2\tau} \sinh 2\tau \\
 &\quad + \frac{2-\gamma}{2} e^{-2\gamma\tau} \int_0^\tau e^{2(\gamma-1)\tau} \frac{\sinh 2\tau}{2\tau} d\tau, \\
 M_3(\tau) &= \frac{1}{48\gamma^4} (3 - 12\gamma + \gamma^2) - \frac{1}{48\gamma^4} (3 - 11\gamma^2) e^{-2\gamma\tau} \\
 &\quad - \frac{1}{8\gamma^3\tau} e^{(1-3\gamma)\tau} \cosh 2\gamma\tau + \frac{1}{8\gamma^4\tau} e^{(1-3\gamma)\tau} \left(1 - \gamma + \frac{1}{2\tau}\right) \sinh 2\gamma\tau \\
 (6.22) \quad &\quad - \frac{1}{16\gamma^4} (1 - 3\gamma)(1 + \gamma)^2 e^{-2\gamma\tau} \int_0^\tau e^{(1-\gamma)\tau} \frac{\sinh 2\gamma\tau}{2\gamma\tau} d\tau \\
 &\quad + \frac{1}{16\gamma^4} (1 - 5\gamma)(1 - \gamma)^2 \int_0^\tau e^{(1-3\gamma)\tau} \frac{\sinh 2\gamma\tau}{2\gamma\tau} d\tau, \\
 M_4(\tau) &= -\frac{1+\gamma}{8\gamma^2} - \frac{1-\gamma}{8\gamma^2} e^{-2\tau} + \frac{1}{8\gamma^3\tau} e^{-(1+\gamma)\tau} \sinh 2\gamma\tau \\
 &\quad - \frac{(1+\gamma)(1-3\gamma)}{8\gamma^2} e^{-2\tau} \int_0^\tau e^{(1-\gamma)\tau} \frac{\sinh 2\gamma\tau}{2\gamma\tau} d\tau \\
 &\quad + \frac{(1-\gamma)(1+3\gamma)}{8\gamma^2} \int_0^\tau e^{-(1+\gamma)\tau} \frac{\sinh 2\gamma\tau}{2\gamma\tau} d\tau.
 \end{aligned}$$

The fact that these spectra are of order s^3 for small s does not agree with the result of BATCHELOR & PROUDMAN [2] that the spectra are of order s^4 for small s . But this is not surprising because they assume that the spectra are regular at some hypothetical initial instant of time, while our spectra always involve non-regular δ -function parts. Indeed, the inertial transfer in the continuous part of our spectra gives rise to terms of order s^4 , in agreement with BATCHELOR & PROUDMAN. A direct integration of expressions (6.20) shows that the asymptotic distribution of energy at large times and small Reynolds numbers is

$$(6.23) \quad \frac{\mathcal{E}}{\mathcal{E}_0} \sim \frac{R^2}{128\tau^2} \left(1 + \frac{2\alpha^2\theta^2}{\gamma} + \frac{\alpha^4}{\gamma^2}\right),$$

$$(6.24) \quad \frac{\mathcal{H}}{\mathcal{H}_0} \sim \begin{cases} \frac{R^2}{32\gamma^2\tau^2} \cdot \frac{(1-\theta^2)}{(1+\gamma)^2}, & \theta^2 < 1, \\ \frac{R^2}{64\gamma^3(1+\gamma)^2\tau^3}, & \theta^2 = 1. \end{cases}$$

This asymptotic behavior of $\mathcal{E}$, $\mathcal{H}$ is in accordance with the discussion of the last section on the effects of the various terms in equation (5.5); the α^4 term in (6.23) represents magnetic energy converted into kinetic energy by the $\Phi^{(3)}$ products: The term not involving θ in (6.24) corresponds to the reverse process

due to the action of the terms $\Phi^{(1)}\Phi^{(3)}$. In both equations, the role of the $\Phi^{(2)}$ products in transferring energy from the magnetic to the velocity field is evident, and there is no *unique* ratio of the asymptotic values of the kinetic and the magnetic energies. The ratio decreases with the increasing value of θ^2 . If there is no correlation initially between the velocity fluctuation $\mathbf{u}$ and the magnetic fluctuation $\mathbf{h}$ ($\theta = 0$), the asymptotic energy ratio is maximum, and if the correlation between $\mathbf{u}$ and $\mathbf{h}$ is complete initially ($\theta^2 = 1$), the energy ratio eventually vanishes as t^{-1} leading to a purely hydrodynamic mode.

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