A Non-linear Boundary Value Problem in Hypersonic Gas Dynamics

D. NAYLOR

Communicated by G. TEMPLE

1. Introduction. The analytical determination of the flow behind an attached shock wave has resisted theoretical solution in all but the simplest cases. In this paper a boundary value problem for the non-linear equation of hypersonic flow associated with the determination of the position of the curved part of an attached shock wave in plane flow is reduced to the solution of an integral equation of the Volterra type. The equation of hypersonic flow considered is that developed on the basis of a small disturbance theory due to VAN DYKE, GOLDSWORTHY and others. In addition, in order to obtain a reasonable boundary value problem the shock conditions have been simplified by following [7] and taking the value unity for the adiabatic exponent γ . This assumption is used noting that the known solutions of simpler problems, for example the flow behind an infinite cone with a straight attached shock, were not sensitive to relevant variations in the value of γ . Therefore it is desirable to take $\gamma = 1$ at the outset. Even this assumption does not render the resulting boundary value problem capable of direct solution (although the transformed partial differential equation becomes linearised in a modified hodograph plane) by the Riemann-Green integration technique. Instead, a suitable limiting process must be applied to obtain the desired integral equation, from which may be determined the shape of the shock. The linearised transformed boundary value problem is broadly similar to that occurring in oscillatory supersonic wing theory, and the method here employed follows that developed by TEMPLE, [4], to obtain the solution of that problem.

The flow problem considered, illustrated in Fig. 1 in which the lateral dimensions are exaggerated, represents supersonic flow at high Mach number over a wedge possessing a shoulder at O . The assumption is made of an attached shock at the vertex Q . In the region QOC bounded by the straight shock segment QC and the straight characteristic OC the flow variables are constant. A simple wave, modified for the hypersonic flow equation, is centered at O , and the characteristics emanating from O intersect the shock which subsequently becomes curved. The problem is to determine the flow in the interaction region

$$(4) \quad \frac{\rho}{\rho_0} = \lambda^2 M^2.$$

Here U is the free stream velocity, M , p_0 and ρ_0 the free stream Mach number, pressure and density, $(U + u, v)$ are the resolute of velocity, and p , ρ the pressure and density just behind the shock. λ denotes the local shock slope. From (3), (4)

$$\frac{p}{\rho} = \frac{p_0}{\rho_0}.$$

Therefore as a consequence of the basic assumptions both the entropy

$$(5) \quad S = C_v \ln \frac{p}{\rho} = C_v \ln \frac{p_0}{\rho_0}$$

and the acoustic speed

$$(6) \quad a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma p_0}{\rho_0}} = a_0$$

remain constant in the field of flow, see [7].

A different approximation to simplify the shock conditions is employed by OSWATITSCH [1] which corresponds to a hypersonic theory in the limiting case $M \rightarrow \infty$.

3. Hypersonic Flow Equations. Following GOLDSWORTHY [2] and VAN DYKE [7] the equations of flow are taken in the form

$$(7) \quad U \rho_x + (\rho v)_y = 0,$$

$$(8) \quad \rho(U v_x + v v_y) + p_y = 0,$$

$$(9) \quad \rho(U u_x + v u_y) + p_x = 0$$

where now

$$(10) \quad p = \rho a_0^2.$$

Equation (7) shows that a modified stream function $\psi(x, y)$ exists with the properties

$$(11) \quad \psi_x = -\rho v, \quad \psi_y = \rho U,$$

in terms of which the equation of motion (8) transforms into the quasi-linear partial differential equation

$$(12) \quad U^2 \psi_{xx} + 2Uv \psi_{xy} + (v^2 - a_0^2) \psi_{yy} = 0.$$

It is of interest to note that from this equation may be asserted the existence of a function $\phi(x, y)$ which is actually a potential function with the properties

$$U \phi_x = u, \quad U \phi_y = v$$

and satisfying

$$U^2\phi_{xx} + 2Uv\phi_{xy} + (v^2 - a_0^2)\phi_{yy} = 0.$$

To show this define $\phi(x, y)$ by means of the relations

$$-\phi_x = \frac{1}{2}\left(\frac{\psi_x}{\psi_y}\right)^2 + \frac{1}{M^2} \ln \psi_y ,$$

$$-\phi_y = \frac{\psi_x}{\psi_y}.$$

Elimination of ϕ leads to (12) and elimination of ψ produces the above potential equation. To show that $U\phi_x = u$ use (8), (9), (10). Then

$$Uu + \frac{1}{2}v^2 + a_0^2 \ln \frac{\rho}{\rho_0} = 0$$

which replaces one of GOLDSWORTHY's equations. Substituting

$$U\rho = \psi_y , \quad v = -\frac{\psi_x}{\psi_y}$$

yields $u = U\phi_x$.

To linearise (12) characteristic variables must be used. Let r, s denote the characteristic variables of (12), then this equation may be replaced by the following system:

$$(13) \quad Uy_r = (v + a_0)x_r ,$$

$$(14) \quad Uy_s = (v - a_0)x_s ,$$

$$(15) \quad \rho v_r + a_0 \rho_r = 0 ,$$

$$(16) \quad \rho v_s - a_0 \rho_s = 0 .$$

From (15), (16) one may select for r, s the quantities

$$(17) \quad r = \frac{1}{4}\left(\frac{v}{a_0} - \sigma\right) , \quad s = \frac{1}{4}\left(\frac{v}{a_0} + \sigma\right)$$

where

$$(18) \quad \sigma = \ln \frac{\rho}{\rho_0} ,$$

so that

$$(19) \quad v = 2a_0(r + s), \quad \sigma = 2(s - r).$$

Since $d\psi = \rho(Udy - vdx)$ it follows from (13), (14) that

$$(20) \quad \psi_r = \rho a_0 x_r , \quad \psi_s = -\rho a_0 x_s$$

or

$$(21) \quad \psi_r = \rho_0 a_0 e^{2(s-r)} x_r , \quad \psi_s = -\rho_0 a_0 e^{2(s-r)} x_s .$$

Eliminating x produces

$$(22) \quad \psi_{rs} - \psi_r + \psi_s = 0;$$

elimination of ψ gives

$$(23) \quad x_{rs} + x_r - x_s = 0.$$

Similarly the characteristic form of the potential equation is

$$\phi_{rs} + \phi_r - \phi_s = 0.$$

In terms of the physical variables v, ρ

$$a_0^2 \psi_{vv} - \rho^2 \psi_{\rho\rho} = 0, \quad a_0^2 x_{vv} - \rho^2 x_{\rho\rho} - 2\rho x_\rho = 0.$$

4. Hypersonic Simple Wave Theory. Referring to the flow diagram of Fig. 1, it is known that the flow in the region OCD' adjacent to the region of constant state QOC must correspond to a degenerate solution of the partial differential equation (12). The classical Prandtl-Meyer expansion theory [3] could be employed in this region, but we prefer to apply the small disturbance theory to the entire region of flow to avoid any possible inconsistencies in ultimately formulating the boundary value problem in the interaction region bounded by the curved part of the shock and the simple wave characteristics extended.

In the simple wave region solutions are sought for which v and ρ are reducible to functions of s only. Then $v_r = \rho_r = 0$, and the characteristic system reduces to

$$(24) \quad Uy_s = (v - a_0)x_s,$$

$$(25) \quad Uy_r = (v + a_0)x_r,$$

$$(26) \quad \rho v_s - a_0 \rho_s = 0.$$

From (26) it follows that, in a simple wave region $v(s), \rho(s)$,

$$(27) \quad \frac{v}{a_0} - \ln \frac{\rho}{\rho_0} = 4r_0,$$

r_0 being some constant.

Since $v = v(s)$ the equation (25) can be integrated at once to yield

$$(28) \quad Uy = (v + a_0)x + f(s).$$

For varying s this represents a one-parameter family of straight characteristics in the (x, y) plane along each of which v and ρ are constant. We shall be interested only in a centered wave for which these characteristics are concurrent through the edge O , which is selected to be the origin of coordinates in the (x, y) plane, so that $f(s) = 0$ throughout the simple wave region. Therefore

$$(29) \quad y = \left(\frac{v}{U} + \frac{1}{M} \right) x$$

in the simple wave region.

To obtain the equation of the characteristic $r = r_0$ emanating from C differentiate (29) with respect to s and employ (24). Then

$$\frac{2}{M}x_s + \frac{x}{U}v_s = 0,$$

giving

$$(30) \quad \frac{2}{M} \ln x + \frac{v}{U} = g(r)$$

in terms of some function of integration $g(r)$.

Employing (29) now gives

$$\frac{y}{x} + \frac{2}{M} \ln x = g(r) + \frac{1}{M}.$$

Therefore the equation of the characteristic CD , $r = r_0$, is

$$\frac{y}{x} + \frac{2}{M} \ln x = \text{constant},$$

the value of the constant being determined from the condition that this characteristic passes through the known point C . Since $v = 2a_0(r_0 + s)$ in the simple wave region, it follows from (30) that the value of x on the characteristic $r = r_0$ is given by

$$(31) \quad x = ce^{-s},$$

c being some constant whose value is determined by considering conditions at the point C . The x -coordinate of this point is known, say x_0 . Also C is on the boundary of the region of constant state and the variables r and s both reduce to constants in this region. Their values are obtained by using equations (17) with the known constant values of v and ρ as derived from shock theory over the straight segment QC . In region QOC , $r = r_0$ and $s = s_0$; therefore

$$c = x_0 e^{s_0}.$$

5. Shock Conditions in the Characteristic Plane. It is now necessary to formulate the boundary value problem for the equations (22), (23) in the (r, s) plane. First of all the shock conditions must be interpreted in the (r, s) plane. From (1), (4)

$$\frac{v}{a_0} = \sqrt{\frac{\rho}{\rho_0}} - \sqrt{\frac{\rho_0}{\rho}}.$$

Using (18) this becomes

$$(32) \quad v = 2a_0 \sinh \frac{\sigma}{2}$$

or, in terms of the variables r, s defined by (19),

$$(33) \quad r + s = \sinh (s - r).$$

From this one obtains the slope of the shock image

$$(34) \quad \frac{dr}{ds} = \tanh^2 \frac{\sigma}{4}.$$

For a displacement along the shock $dy = \lambda dx$, so that

$$y_r dr + y_s ds = \lambda(x_r dr + x_s ds).$$

Using (34) this becomes

$$y_r \tanh^2 \frac{\sigma}{4} + y_s = \lambda \left(x_r \tanh^2 \frac{\sigma}{4} + x_s \right).$$

From (13), (14)

$$(v + a_0 - \lambda U)x_r \tanh^2 \frac{\sigma}{4} + (v - a_0 - \lambda U)x_s = 0.$$

Substituting

$$v = 2a_0 \sinh \frac{\sigma}{2}, \quad \lambda U = a_0 e^{\frac{3}{2}\sigma}$$

yields finally

$$(35) \quad x_s = x_r \tanh^3 \frac{\sigma}{4},$$

which is the shock condition to be satisfied along the curve (33).

The equivalent condition on the function ψ is obtained by combining (35) with (21):

$$\psi_s + \psi_r \tanh^3 \frac{\sigma}{4} = 0.$$

On the shock the flow variables r, s are given by

$$(36) \quad 4r = 2 \sinh \frac{\sigma}{2} - \sigma,$$

$$(37) \quad 4s = 2 \sinh \frac{\sigma}{2} + \sigma$$

which are obtained from (17) and (32).

6. Boundary Value Problem. In the region QOC of Figure 1 the flow variables v, σ are constants whose values are known from shock theory in terms of the required wedge deflection δ . In fact, the shock slope λ_0 for QC is given by

$$\lambda_0 = \frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + \frac{1}{M^2}},$$

obtained from (1) by putting $v = U\delta$. Then the associated value of σ is given by $\sigma_0 = 2 \ln (M\lambda_0)$. The quantities r and s are constant in this region and take the values

$$r_0 = \frac{1}{4}[M\delta - 2 \ln (M\lambda_0)],$$

$$s_0 = \frac{1}{4}[M\delta + 2 \ln (M\lambda_0)].$$

The line OC separating the region of constant state from the simple wave region COD' is the straight characteristic $s = s_0$ of the system (24). Throughout the simple wave region the quantity r is constant and equal to r_0 . Also the simple wave region is bounded on the right by the straight characteristic OD' specified by $s = -r_0$, which is obtained from (19) after substituting the value $r = r_0$ and $v = 0$ corresponding to the zero slope of the boundary OF . The constants r_0, s_0 can be shown to be positive. Since at the shock $\rho > \rho_0$, then by (18) $\sigma > 0$. Also $M\lambda > 1$. It follows that on the shock the quantity s defined by (37) is positive. In addition, the function r defined by (36) is monotonic increasing and vanishes when $\sigma = 0$; therefore $r > 0$ for $\sigma > 0$. In particular, $r_0 > 0$ and $s_0 > 0$. It follows that in the simple wave region $r = r_0 > 0$ and $-r_0 \leq s \leq s_0$ ($s_0 > 0$).

The region bounded by $SCD'D''$ is mapped out by characteristics $s = \text{constant}$, continued out of the simple wave region so that $-r_0 \leq s \leq s_0$ in the interaction region. Also in this region $0 < r \leq r_0$. To prove this assume that the slope λ of the curved part CS of the shock is less than λ_0 , that of the straight part QC . This is required physically. That is, take $0 < \lambda \leq \lambda_0$ over CS . Let σ_0 be the value of σ associated with the straight section QC . Then $e^{\sigma_0} = M^2\lambda_0^2$. Since $e^\sigma = M^2\lambda^2$ for CS , then, along CS , $0 < \sigma \leq \sigma_0$ since σ is essentially positive. Therefore

$$0 < 2 \sinh \frac{\sigma}{2} - \sigma \leq 2 \sinh \frac{\sigma_0}{2} - \sigma_0,$$

as the function

$$2 \sinh \frac{\sigma}{2} - \sigma$$

is monotonic increasing and positive for $\sigma > 0$. Therefore along the curved part of the shock $0 < r \leq r_0$ as required. Now since the variable r is constant along each characteristic of the r -family which originates at the shock and penetrates the interaction region, it follows that the above inequality holds throughout this region.

When these results are transferred onto the (r, s) plane the configuration of Fig. 2 results. The image of the entire simple wave region is degenerate and appears as the segment $D'C$ of the characteristic $r = r_0$ for which $-r_0 \leq s \leq s_0$. The shock image comprises the appropriate segment of the curve $r + s = \sinh (s - r)$. The interaction region is bounded by the shock wave and the simple wave characteristic $D'C$ and its location is determined by the conditions $0 < r \leq r_0$ and $-r_0 \leq s \leq s_0$.

FIGURE 2.

To obtain the solution at a point such as $A(s_1, r_1)$, Riemann's method must be employed. In the physical plane, Fig. 1, the characteristics AB , AD are drawn through A . Then the solution of the equation (23) is determined by the values of x_r and x_s along BC and CD . Since CD is characteristic it is sufficient to know only x along CD . The values of x_r and x_s along CD follow from this information together with the differential equation (23) itself. The value of x along CD is supplied by the boundary condition (31), that is,

$$(38) \quad x = ce^{-s}$$

along the characteristic $r = r_0$, $s_1 \leq s \leq s_0$.

Let (s_2, r_1) denote the intersection of the characteristic $r = r_1$ with the shock image. This corresponds to the point B in Fig. 1. Then the shock boundary condition is

$$(39) \quad x_s = x_r \tanh^3 \left(\frac{s - r}{2} \right)$$

to be applied along that part of BC ,

$$r + s = \sinh(s - r),$$

for which $r_1 \leq r \leq r_0$, $s_2 \leq s \leq s_0$. As the shock is not a characteristic of the equation (23) the above condition cannot yield the values of x_r and x_s that are required for Riemann's integration technique to be directly successful. It is the problem of the "oblique derivative" for hyperbolic equations, with subsidiary information provided along a characteristic to render the solution unique.

A similar problem arises in the theory of two dimensional oscillatory supersonic wing theory. Complete information is provided along a characteristic but only the normal derivative is prescribed along a curve which is not a character-

istic. In [4] this problem is solved completely by applying the Riemann-Green method over an adjacent region in such a way that the tangential derivative over the non-characteristic is removed. This method does not seem applicable directly to the present problem. Instead we shall follow a device employed in [5] to set up an integral equation for the value of x along the non-characteristic BC , that is, along the shock itself. This involves a limiting process in which the field point A is moved along the characteristic $r = r_1$ to coincide ultimately with the point B on the shock image.

Once this integral equation for the value of x along CS has been solved, the Riemann-Green technique may be used again since complete information is then available along both the supports BC and CD . Here we shall concentrate on obtaining the value of x along BC which is sufficient to provide the actual shape of the curved part of the shock wave.

7. Riemann-Green Integral. The function $x(r, s)$ satisfies the equation

$$L(x) = x_{rs} + x_r - x_s = 0,$$

and the characteristic function satisfies the adjoint equation

$$M(G) = G_{rs} - G_r + G_s = 0$$

together with the conditions

- (a) $G = 1$ at $A(s_1, r_1)$,
- (b) $G_s - G = 0$ along the characteristic AB , $r = r_1$, $s_1 \leq s \leq s_2$,
- (c) $G_r + G = 0$ along the characteristic AD , $s = s_1$, $r_1 \leq r \leq r_0$.

To construct G write

$$G(r, s) = e^{s-r} H(r, s).$$

Then $H_{rs} + H = 0$, together with the conditions $H = e^{r_1-s_1}$ at $A(s_1, r_1)$, $H = \text{constant}$ along $r = r_1$ and $H = \text{constant}$ along $s = s_1$. Therefore the function $H(r, s)$ must take the constant value $\exp\{r_1 - s_1\}$ along the characteristics through A , and, as we are interested in the region for which $r \geq r_1$, $s \geq s_1$, the suitable choice for H is known to be

$$H(r, s) = e^{r_1-s_1} J_0[2 \sqrt{(r-r_1)(s-s_1)}],$$

so that

$$(40) \quad G(r, s) = e^{s-s_1+r_1-r} J_0[2 \sqrt{(r-r_1)(s-s_1)}].$$

Writing

$$\Lambda = Gx_s - xG_s + 2xG,$$

$$\Gamma = Gx_r - xG_r - 2xG,$$

then

$$2[GL(x) - xM(G)] = \Lambda_r + \Gamma_s,$$

so that

$$\Lambda_r + \Gamma_s = 0.$$

Applying Green's theorem to the region bounded by the characteristics CD , DA , AB and the shock image BC , there results

$$\oint (\Lambda ds - \Gamma dr) = 0.$$

Along the characteristic CD , $r = r_0$, $x_s = -x$ (from (38)) and the line integral reduces to

$$\int_C^D \Lambda ds = - \int_C^D (xG_s + Gx_s) ds = x(C)G(C) - x(D)G(D)$$

with an obvious notation.

Along the characteristic DA , $s = s_1$, $G_r + G = 0$ and

$$- \int_D^A \Gamma dr = - \int_D^A (xG_r + Gx_r) dr = x(D)G(D) - x(A).$$

Along AB , $r = r_1$, $G_s = G$, therefore

$$\int_A^B \Lambda ds = \int_A^B (xG_s + Gx_s) ds = x(B)G(B) - x(A).$$

Collecting these results it follows that

$$2x(A) = x(B)G(B) + x(C)G(C)$$

$$+ \int_B^C [(Gx_s - xG_s + 2xG) ds - (Gx_r - xG_r - 2xG) dr].$$

As indicated in §6, this expression cannot be evaluated explicitly since the quantities x , x_r , x_s are unknown along BC . The only information provided along BC is embodied in the relation (39) which is equivalent to specifying only an oblique derivative of x over the curve BC . Keeping r_0 , r_1 and s_2 fixed we allow $s_1 \rightarrow s_2$, so that, from (40), with $r = r_1$, $s = s_2$, $G(B) \rightarrow 1$. Therefore

$$(41) \quad x(B) = x(C)G(C) + \int_B^C [(Gx_s - xG_s + 2xG) ds - (Gx_r - xG_r - 2xG) dr]$$

and

$$(42) \quad G(C) = e^{s_0 - s_2 + r_1 - r_0} J_0 [2 \sqrt{(r_0 - r_1)(s_0 - s_2)}]$$

where, along BC ,

$$(43) \quad G(r, s) = e^{s - s_2 + r_1 - r} J_0 [2 \sqrt{(r - r_1)(s - s_2)}].$$

8. Integral Equation for $x(\sigma)$. From the equations

$$\sigma = 2(s - r), \quad s + r = \sinh \frac{\sigma}{2}$$

by differentiation there follow

$$(44) \quad dr = \frac{d\sigma}{2} \sinh^2 \frac{\sigma}{4}, \quad ds = \frac{d\sigma}{2} \cosh^2 \frac{\sigma}{4}.$$

Along the shock x will be expressed in terms of σ only, so that, along BC ,

$$x_s d\sigma = x_r dr + x_s ds.$$

Substituting for dr, ds from (44) and for x_s from (39), one obtains

$$(45) \quad \begin{cases} x_r = 2x_s e^{-\frac{1}{2}\sigma} \cosh \frac{\sigma}{4} \operatorname{cosech}^2 \frac{\sigma}{4}, \\ x_s = 2x_s e^{-\frac{1}{2}\sigma} \sinh \frac{\sigma}{4} \operatorname{sech}^2 \frac{\sigma}{4}. \end{cases}$$

From these it follows that

$$(46) \quad x_r dr - x_s ds = x_s e^{-\frac{1}{2}\sigma} d\sigma.$$

Let σ_0, σ_1 denote the values of σ associated with the extremities C and B , so that

$$\sigma_0 = 2(s_0 - r_0), \quad \sigma_1 = 2(s_2 - r_1).$$

Then using (46)

$$\begin{aligned} \int_B^C G(x_s ds - x_r dr) &= - \int_{\sigma_1}^{\sigma_0} e^{-\frac{1}{2}\sigma} G(\sigma) x'(\sigma) d\sigma \\ &= e^{-\frac{1}{2}\sigma_1} x(\sigma_1) - e^{-\frac{1}{2}\sigma_0} x(C) G(C) + \int_{\sigma_1}^{\sigma_0} x e^{-\frac{1}{2}\sigma} (G_\sigma - \frac{1}{2}G) d\sigma \end{aligned}$$

after an integration by parts and using $G(B) = 1$. Substituting the value of $x(C)$ from (38) and of $G(C)$ from (42) the equation (41) becomes

$$(47) \quad \begin{aligned} (1 - e^{-\frac{1}{2}\sigma_1}) x(\sigma_1) &= c(e^{-r_0} - e^{-s_0}) e^{-\frac{1}{2}\sigma_1} J_0[2 \sqrt{(r_0 - r_1)(s_0 - s_2)}] \\ &+ \int_{\sigma_1}^{\sigma_0} x(\sigma) [e^{-\frac{1}{2}\sigma} (G_\sigma - \frac{1}{2}G) d\sigma + (2G - G_s) ds + (2G + G_r) dr]. \end{aligned}$$

Writing

$$\mu = \sqrt{(r - r_1)(s - s_2)}, \quad \tau = \sqrt{\frac{(s - s_2)}{(r - r_1)}}, \quad \mu_0 = \sqrt{(r_0 - r_1)(s_0 - s_2)}$$

the following expressions may be calculated from (43):

$$(48) \quad \begin{aligned} G_r + G &= \tau e^{\frac{1}{2}(\sigma - \sigma_1)} J'_0(2\mu), \\ G_s - G &= \frac{1}{\tau} e^{\frac{1}{2}(\sigma - \sigma_1)} J'_0(2\mu), \\ 2G_\sigma - G &= \left(\tau \sinh^2 \frac{\sigma}{4} + \frac{1}{\tau} \cosh^2 \frac{\sigma}{4} \right) e^{\frac{1}{2}(\sigma - \sigma_1)} J'_0(2\mu). \end{aligned}$$

When these results are substituted in (47) the following equation results:

$$(49) \quad \begin{aligned} (e^{\frac{1}{2}\sigma_1} - 1)x(\sigma_1) &= c(e^{-r_0} - e^{-s_0})J_0(2\mu_0) \\ &+ \frac{1}{2} \int_{\sigma_1}^{\sigma_0} x(\sigma) \left[e^{\frac{1}{2}\sigma} \left(\tau \sinh \frac{\sigma}{4} - \frac{1}{\tau} \cosh \frac{\sigma}{4} \right) \sinh \frac{\sigma}{2} J'_0(2\mu) \right. \\ &\left. + e^{\frac{1}{2}\sigma} \cosh \frac{\sigma}{2} J_0(2\mu) \right] d\sigma \end{aligned}$$

where, in the integral, μ and τ are to be expressed in terms of σ , σ_1 by means of the formulas

$$(50) \quad \mu = \frac{1}{4} \sqrt{\left(2 \sinh \frac{\sigma}{2} - 2 \sinh \frac{\sigma_1}{2} + \sigma_1 - \sigma \right) \left(2 \sinh \frac{\sigma}{2} - 2 \sinh \frac{\sigma_1}{2} + \sigma - \sigma_1 \right)},$$

$$(51) \quad \tau = \sqrt{\frac{2 \sinh \frac{\sigma}{2} - 2 \sinh \frac{\sigma_1}{2} + \sigma - \sigma_1}{2 \sinh \frac{\sigma}{2} - 2 \sinh \frac{\sigma_1}{2} + \sigma_1 - \sigma}}$$

obtained by combining (36), (37) with (48).

In the equation (49) the quantities r_0 , s_0 , σ_0 are fixed, μ_0 is a function of σ_0 and σ_1 obtained by putting $\sigma = \sigma_0$ in equation (50). Equation (49) is a Volterra type integral equation to determine $x(\sigma_1)$, which is the x -coordinate of the point of the curved part of the shock wave at which the local slope is given by

$$(52) \quad \frac{dy}{dx} = \lambda = \frac{1}{M} e^{\frac{1}{2}\sigma_1},$$

which follows from (4) and (18). Solving (49) numerically will provide the variation of x with dy/dx . This information is sufficient to obtain the shape of the shock wave.

9. Approximate Solution of the Integral Equation. To illustrate the approximate solution of (49) we consider the typical case of a wedge in hypersonic conditions taking a wedge angle $\delta = 11\frac{1}{2}^\circ$ and a free stream Mach number $M = 5$. The associated shock angle for the initial straight section is, in hypersonic theory, given by

$$\lambda_0 = \frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + \frac{1}{M^2}},$$

giving $\lambda_0 = 0.324$ or $18\frac{1}{2}^\circ$ nearly. From §6 the variables r_0 and s_0 for the region of constant state are found to take the values $r_0 = 0.01$, $s_0 = 0.49$. Also $\sigma_0 = 2 \ln (M\lambda_0) = 0.96$.

Consider now the argument of the Bessel function occurring in the integral equation. Since $0 < r \leq r_0$, $-r_0 \leq s \leq s_0$ with the same inequalities valid when r and s are replaced by r_1 and s_2 , then

$$\mu^2 = (r - r_1)(s - s_2) \leq 2r_0s_0,$$

For the case considered $0 \leq \mu/\sqrt{2} \leq \sqrt{r_0 s_0} = 0.07$; then $\frac{1}{2}\mu^2 \leq 0.0049$. Without incurring further loss of accuracy than that already involved in the hypersonic small disturbance approximation, the functions $J_0(2\mu)$ and $J'_0(2\mu)$ may be approximated by using $J_0(2\mu) = 1$ and $J'_0(2\mu) = -\mu$. Then

$$\left(\tau \sinh \frac{\sigma}{4} - \frac{1}{\tau} \cosh \frac{\sigma}{4}\right) J'_0(2\mu) = -\frac{1}{4} \left[(\sigma - \sigma_1) e^{\frac{1}{2}\sigma} - 2e^{-\frac{1}{2}\sigma} \left(\sinh \frac{\sigma}{2} - \sinh \frac{\sigma_1}{2} \right) \right]$$

after substituting for μ and τ from (50), (51).

The integral equation now reduces to

$$(53) \quad (e^{\frac{1}{2}\sigma_1} - 1)x(\sigma_1) = c(e^{-r_0} - e^{-s_0}) + \frac{1}{2} \int_{\sigma_1}^{\sigma_0} x(\sigma) \left[e^{\frac{1}{2}\sigma} \cosh \frac{\sigma}{2} - \frac{1}{4} (\sigma - \sigma_1) e^{\frac{1}{2}\sigma} \sinh \frac{\sigma}{2} + \frac{1}{2} \left(\sinh \frac{\sigma}{2} - \sinh \frac{\sigma_1}{2} \right) \sinh \frac{\sigma}{2} \right] d\sigma.$$

Since $0 \leq \sigma_1 \leq \sigma \leq \sigma_0 < 1$ the above equation will be further simplified by taking $\sinh \sigma/2 = \sigma/2$, the error involved here is always less than 1 % of the dominant term $e^{\frac{1}{2}\sigma} \cosh \frac{1}{2}\sigma$ in the integrand of (53). Then (53) will be replaced by

$$(54) \quad (e^{\frac{1}{2}\sigma_1} - 1)\bar{x}(\sigma_1) = 1 + \int_{\sigma_1}^{\sigma_0} \bar{x}(\sigma) \left[\frac{1}{4} (1 + e^\sigma) + \frac{1}{10} \sigma (\sigma - \sigma_1) (1 - e^{\frac{1}{2}\sigma}) \right] d\sigma$$

where

$$(55) \quad x(\sigma) = c(e^{-r_0} - e^{-s_0})\bar{x}(\sigma).$$

It is seen from the equation (54) that $\bar{x}(\sigma_1) \rightarrow \infty$ as $\sigma_1 \rightarrow 0$. Therefore to approximate the solution by means of a power series it is first convenient to write

$$(56) \quad \xi(\sigma) = (e^{\frac{1}{2}\sigma} - 1)\bar{x}(\sigma)$$

and obtain the integral equation for $\xi(\sigma)$. This integral equation is

$$(57) \quad \xi(\sigma_1) = 1 + \int_{\sigma_1}^{\sigma_0} \xi(\sigma) \left[\frac{1}{4} \left(\frac{e^\sigma + 1}{e^{\frac{1}{2}\sigma} - 1} \right) - \frac{1}{16} \sigma (\sigma - \sigma_1) \right] d\sigma.$$

Now write $\xi(\sigma) = u(\sigma)v(\sigma)$ where $u(\sigma)$ satisfies

$$(58) \quad u(\sigma_1) = 1 + \frac{1}{4} \int_{\sigma_1}^{\sigma_0} u(\sigma) \left(\frac{e^\sigma + 1}{e^{\frac{1}{2}\sigma} - 1} \right) d\sigma,$$

so that

$$(59) \quad 4(e^{\frac{1}{2}\sigma} - 1)u'(\sigma) + (e^\sigma + 1)u(\sigma) = 0$$

for $0 \leq \sigma_1 \leq \sigma \leq \sigma_0 < 1$. Integrating (59), subject to the condition $u(\sigma_0) = 1$ obtained from (58), yields

$$(60) \quad u(\sigma) = \left(\frac{z_0 - 1}{z - 1} \right) \sqrt{\frac{z}{z_0}} \cdot e^{\frac{1}{2}(z_0 - z)}$$

where $z = e^{\frac{1}{2}\sigma}$, $z_0 = e^{\frac{1}{2}\sigma_0}$.

Differentiating (57) with respect to σ_1 produces

$$(61) \quad \xi'(\sigma_1) = \frac{1}{16} \int_{\sigma_1}^{\sigma_0} \sigma \xi(\sigma) d\sigma - \frac{1}{4} \left(\frac{e^{\sigma_1} + 1}{e^{\frac{1}{3}\sigma_1} - 1} \right) \xi(\sigma_1).$$

Inserting $\xi = uv$ in (61) gives, after using (59),

$$(62) \quad u(\sigma_1)v'(\sigma_1) = \frac{1}{16} \int_{\sigma_1}^{\sigma_0} \sigma u(\sigma)v(\sigma) d\sigma,$$

so that finally by differentiation with respect to σ_1

$$(63) \quad v''(\sigma) + \frac{u'(\sigma)}{u(\sigma)} v'(\sigma) + \frac{1}{16} \sigma v(\sigma) = 0$$

for $0 \leq \sigma \leq \sigma_0 < 1$. The boundary conditions for (63) can be found, since, from (57), $\xi(\sigma_0) = 1$, and, from (58), $u(\sigma_0) = 1$, so that $v(\sigma_0) = 1$. Also, from (62), $v'(\sigma_0) = 0$. Therefore (63) has to be integrated with the subsidiary conditions $v(\sigma_0) = 1$, $v'(\sigma_0) = 0$. The result is

$$v(\sigma) = 1 - 0.03(\sigma - \sigma_0)^2 - 0.0252(\sigma - \sigma_0)^3 - 0.0047(\sigma - \sigma_0)^4 - \dots$$

Combining (56), (60) with $\xi = uv$ the solution is now given in the form

$$(64) \quad \bar{x}(\sigma) = \frac{(z_0 - 1)}{(z - 1)^2} \sqrt{\frac{z}{z_0}} \cdot e^{\frac{1}{3}(z_0 - z)} v(\sigma)$$

where $z = e^{\frac{1}{3}\sigma}$ and $1 \leq z \leq z_0$.

Equation (64) provides the reduced x -wise coordinate of the point on the shock wave where the local slope is, from (52),

$$(65) \quad \frac{dy}{dx} = \frac{1}{M} e^{\frac{1}{3}\sigma} = \frac{z}{M}.$$

The information contained in (64), (65) is sufficient to obtain the shape of the shock wave. The variation of $\bar{x}$ with z is shown graphically in Fig. 3. For the relevant range of z , $1 \leq z \leq z_0$, $\bar{x}$ decreases monotonically. For the value $z = 1$ $\bar{x}$ becomes infinite, and by (65) $dy/dx = 1/M$ which corresponds to the shock asymptote, so that the slope of the latter is $1/M$. The maximum value of z is z_0 which corresponds to the point C (Fig. 1). As z progressively decreases, approaching the value unity, x increases. Ultimately the shock becomes straight with slope $1/M$ in accordance with linearised supersonic theory. In the physical plane the form of the shock would be as anticipated in Fig. 1.

The solution of the more exact problem of interaction of a simple wave and a shock wave in hypersonic flow could in theory be obtained by the above method provided isentropic flow is assumed. Despite the shock curvature the subsequent flow is isentropic to within a good approximation, as has been reasoned by GOLDSWORTHY [2]. With isentropic flow the equation (12) would still apply, except that the acoustic speed that occurs in this equation must be taken as a variable. This would not effect the application of the method of characteristics.

FIGURE 3.

However, the exact form of the shock relations would have to be retained. Then the shock image in the characteristic plane would take a form more complex than that of (33); therefore the details of the method would be more lengthy.

REFERENCES

- [1] OSWATITSCH, K., *Gas Dynamics*, Academic Press, New York (1956), p. 405.
- [2] GOLDSWORTHY, F. A., Two Dimensional Rotational Flow at High Mach Numbers past Thin Aerofoils, *Q. J. Mech. & App. Math.* **5** (1952), pp. 54-63.
- [3] LINNELL, R. D., Two Dimensional Airfoils in Hypersonic Flow, *J. Aero. Sci.* **16** (1949), pp. 22-30.
- [4] TEMPLE, G., *Unsteady Motion*, Aero. Res. Council (London) Fluid Motion Sub-Committee Report 1284 (1949), pp. 74-81.
- [5] TEMPLE, G. & JAHN, H. A., *Flutter at Supersonic Speeds*, Aero. Res. Council (London) Reports & Mem. 2140 (1945).
- [6] HAYES, W. D. & PROBSTEN, R. F., *Hypersonic Flow Theory*, Academic Press, New York (1959), p. 30.
- [7] VAN DYKE, M. D., *A Study of Hypersonic Small Disturbance Theory*, NACA. Report 1194 (1954), p. 17.
- [8] FORSYTH, A. R., *Theory of Differential Equations*, Part 4, Vol. 6, Camb. Univ. Press (1906), p. 10.

University of Toronto