A Martingale Representation Theorem

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In recent years martingale theory has been applied to the study of a variety of stochastic processes by use of the following technique: Let $\{X_t\}$ be a stochastic process. Let $f(x, t)$ be a function of x and t such that $\{f(X_t, t)\}$ is a martingale, that is, such that

$$E \{f(X_t, t) \mid X_u, u \leq s\} = f(X_s, s)$$

with probability one. We say that $f(x, t)$ is a *martingale function* for $\{X_t\}$. If $f(x, t)$ is non-negative, the convergence theorem for discrete time non-negative martingales and the theorems concerning continuity properties of sample functions for continuous time martingales may then be exploited to obtain information about the sample functions of the original process. This technique has been employed by DOOB in a sequence of papers relating Markov processes and potential theory. See DOOB [1]. The first use of the technique was made by VILLE [7] to obtain a proof of one part of the law of the iterated logarithm for Bernoulli trials. His procedure was the following: Let S_n be the number of successes in n trials with probability p for success. He wished to prove that if $\{a_n\}$ is a sequence tending to infinity faster than

$$\{np + \lambda \sqrt{2npq \log \log n}\}, \quad \lambda > 1,$$

then

$$\Pr [S_n = a_n \text{ infinitely often}] = 0.$$

To do this he first verified that any function of the form

$$(1) \quad f(x, n) = \int_0^1 \left(\frac{s}{p}\right)^x \left(\frac{1-s}{1-p}\right)^{n-x} dF(s)$$

is a martingale function for the process $\{S_n\}$. He showed, in fact, that every martingale function for this process may be represented in the form (1). He then showed that if $\{a_n\}$ tends to infinity faster than

$$\{np + \lambda \sqrt{2npq \log \log n}\}, \quad \lambda > 1,$$

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he could construct a measure $F(t)$ such that $f(a_n, n) \rightarrow \infty$. But since the sample paths of a non-negative martingale are bounded with probability one,

$$\Pr [X_n = a_n \text{ infinitely often}] = 0.$$

VILLE also proved the corresponding theorem for one-dimensional Brownian motion using the fact that for this process

$$(2) \quad f(x, t) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}ts^2 + xs} dF(s)$$

is a martingale function. He did not prove that every such non-negative martingale comes from such a function.² In this paper we shall prove that this is true and consider such a representation theorem for an important class of processes. Specifically we consider sums of independent random variables with common Pólya type distribution.

Let $S_n = S_1 + X_2 + \cdots + X_n$ where $X_1, X_2, \cdots, X_n$ are independent with the common distribution having Pólya type density $p(x)$. A Pólya density [6] is characterized by the fact that its Laplace transform is of the form

$$(3) \quad F(s) = Ce^{\gamma s^2 + \delta s} \prod_{n=1}^{\infty} (1 + \delta_n s)^{-1} e^{\delta_n s},$$

where the δ 's are real, $\sum_{n=1}^{\infty} \delta_n^2 < \infty$, and $\gamma > 0$. Let $p(x, n)$ be the density for S_n . Then for $u(x, n)$ to be a martingale function we must have

$$(4) \quad u(x, k) = \int_{-\infty}^{\infty} u(x + y, n + k) p(y, n) dy.$$

Thus we may also consider the problem to be that of representing non-negative solutions of (4).

The corresponding problem for lattice valued random variables was considered in [3]. There the following result was proved (in the terminology of the present paper). Let $S_n = Y_1 + Y_2 + \cdots + Y_n$ be the sum of independent random variables with a common lattice distribution. Let $p_i^{(n)} = \Pr [S_n = j]$. Then every non-negative solution of

$$(5) \quad u(x, k) = \sum_y u(x + y, n + k) p_k^{(n)}$$

with $u(0, 0) = 1$ can be represented in the form

$$(6) \quad u(x, k) = \int_I \frac{e^{-sx}}{F(s)^k} d\mu$$

where $F(s)$ is the Laplace transform of the distribution for Y_i , I is the interval on which this transform exists (with ∞ adjoined when Y has a finite maximum value and 0 when it has a finite minimum value), and μ is a probability measure.

² However, a proof of this fact was communicated to us by Professor D. V. WIDDER.

This result was proved using Martin boundary theory developed by Doob in [2] for denumerable Markov chains. Boundary theory shows that every solution of (5) is a convex combination of minimal solutions. A solution u is minimal if it is non-negative and if any other non-negative solution less than or equal to u is a constant multiple of u . Hence when boundary theory is available it is necessary only to identify the minimal solutions, and it was shown that the minimal functions are of the form

$$u(x, k) = \frac{e^{-sx}}{F(s)^k}.$$

The methods of [3] apply also to the density case to show that the minimal functions are again of this form. However, there is not yet developed a boundary theory which gives the representation theorem. We therefore give a direct proof of this fact in this paper for Pólya densities.

Theorem 1. *Let p be a Pólya density function having a Laplace transform $F(s)$ in a strip determined by $s_1 < 0 < s_2$. Let $[c, d]$, perhaps infinite, be the smallest interval containing the support of p . If x/k and t are in the open interval (c, d) , then*

$$\lim_{n \rightarrow \infty} \frac{p(nt - x, n - k)}{p(nt, n)}$$

exists. The convergence is uniform for all t in (c, d) , and the limit is $e^{-sx}/[F(s)]^k$ where $s = s(t)$ is the unique solution of $-F'(s)/F(s) = t$ on (s_1, s_2) and $e^{-sx}[F(s)]^{-k}$ vanishes at s_1 and s_2 .

Theorem 2. *If p is a Pólya density as in Theorem 1, then for x/k in (c, d) all non-negative solutions finite at $(0, 0)$ of*

$$u(x, k) = \int u(x + y, n + k)p(y, n) dy$$

are given by

$$\int_{s_1}^{s_2} e^{-sx}[F(s)]^{-k} d\mu$$

where μ ranges over non-negative measures of finite total mass on (s_1, s_2) .

Proof of Theorem 1. We begin by proving the last part of the statement: that $-F'(s)/F(s) = t$ is one-one from $[s_1, s_2]$ to $[c, d]$. That $-F'/F$ is decreasing on $[s_1, s_2]$ can be verified directly [see KHINCHIN, p. 77]. Also, from

$$-\frac{F'(s)}{F(s)} = \frac{\int_c^d xp(x)e^{-sx} dx}{\int_c^d p(x)e^{-sx} dx}$$

it is clear that $c < -F'(s)/F(s) < d$ for all s in (s_1, s_2) . We need to show that

all of (c, d) is covered. Consider the case in which s tends to s_2 , the proof being similar for s tending to s_1 . Let $\Delta, \Delta', \Delta''$ be arbitrary numbers for which $c < \Delta' < \Delta < \Delta''$. It is enough to show $-F'/F < \Delta''$ for s sufficiently near s_2 . We have

$$\begin{aligned} -\frac{F'(s)}{F(s)} &\leq \frac{\int_c^d xp(x)e^{-sx} dx}{\int_c^{\Delta} p(x)e^{-sx} dx} \leq \frac{\Delta \int_c^{\Delta} p(x)e^{-sx} dx + \int_{\Delta}^d xp(x)e^{-sx} dx}{\int_c^{\Delta} p(x)e^{-sx} dx} \\ &= \Delta + \frac{\int_{\Delta}^d xp(x)e^{-sx} dx}{\int_c^{\Delta} p(x)e^{-sx} dx}. \end{aligned}$$

If $s_2 = +\infty$

$$\frac{\int_{\Delta}^d xp(x)e^{-sx} dx}{\int_c^{\Delta} p(x)e^{-sx} dx} \leq \frac{\int_{\Delta}^d |x| p(x)e^{-sx} dx}{\int_c^{\Delta'} p(x)e^{-sx} dx} \leq e^{-s(\Delta-\Delta')} \frac{\int_{\Delta}^d |x| p(x) dx}{\int_c^{\Delta'} p(x) dx} = o(1)$$

as s tends to $s_2 = +\infty$. If s_2 is finite then $c = -\infty$, and

$$\int_{\Delta}^d |x| e^{-sx} p(x) dx \rightarrow \int_{\Delta}^d |x| p(x) e^{-s_2 x} dx$$

as s tends to s_2 , while

$$\int_c^{\Delta'} p(x) e^{-sx} dx \rightarrow \infty$$

since

$$\int_{\Delta'}^d e^{-sx} p(x) dx \rightarrow \int_{\Delta'}^d p(x) e^{-s_2 x} dx,$$

and by assumption

$$\int_c^d e^{-sx} p(x) dx \rightarrow \infty,$$

because of the singularity at s_2 .

To prove the existence of the limit, consider

$$p_t(x) = \frac{e^{-s(t)x} p(x+t)}{\int_{-\infty}^{\infty} e^{-s(t)y} p(y+t) dy}.$$

It follows from known properties of Pólya densities [6, p. 335] that if p is of Pólya type so is p_t .

We prove three lemmas.

Lemma 1. Let $p(x)$ be a Pólya density, and consider the family of densities, all with mean zero, given by $p_t(x)$. If $f_t(v)$ is the characteristic function of p_t , then the limit

$$f_t^n\left(\frac{v}{\sqrt{n}\sigma}\right) \rightarrow e^{-\frac{1}{2}v^2}$$

as $n \rightarrow \infty$, which holds uniformly for v in a bounded interval when t is held fixed, is also uniform for all t in the domain of $s(t)$. Here $\sigma^2 = \sigma^2(t)$ is the variance of p_t .

Proof. By Taylor expansion

$$f_t(v) = 1 - \frac{\sigma^2}{2} v^2 + \int_0^v (v-u)[f_t''(u) - f_t''(0)] du$$

and

$$f_t^n\left(\frac{v}{\sqrt{n}\sigma}\right) = \left(1 - \frac{v^2}{2n} + \frac{1}{n} \int_0^v (v-u) \frac{1}{\sigma^2} \left[f_t''\left(\frac{u}{\sqrt{n}\sigma}\right) - f_t''(0)\right] du\right)^n.$$

For each fixed t the result follows as usual since f_t'' is continuous at zero. The uniformity in t will follow once it is shown that

$$\frac{1}{\sigma^2} \left[f_t''\left(\frac{a}{\sigma}\right) - f_t''(0) \right]$$

tends to zero uniformly in t for $a \rightarrow 0$. But this can be written

$$\frac{\int_{-\infty}^{\infty} (1 - e^{iaz/\sigma}) x^2 p_t(x) dx}{\int_{-\infty}^{\infty} x^2 p_t(x) dx} = \frac{\int_{-\infty}^{\infty} x^2 (1 - e^{iaz}) p_t(\sigma x) dx}{\int_{-\infty}^{\infty} x^2 p_t(\sigma x) dx}.$$

If we break the range of integration in the numerator at $\pm A$, then clearly the integral over $(-A, A)$ tends to zero uniformly in t for any fixed A . We now show that the tails tend to zero uniformly in t for $A \rightarrow \infty$.

Write

$$\varphi_t(x) = \frac{x^2 p_t(\sigma x)}{\int_{-\infty}^{\infty} x^2 p_t(\sigma x) dx}.$$

By Chebyshev's inequality

$$\int_{|x-m|>A} \varphi_t(x) dx \leq \frac{\rho^2}{A^2},$$

where $m = m(t)$ and $\rho^2 = \rho^2(t)$ are the mean and variance respectively of φ_t . If m and ρ^2 are bounded for all t , the desired result follows.

A tedious computation³ shows that if $p_t(x)$ is a Pólya density with Laplace

³ In this computation the following identities are helpful: If $y = y(x)$ and $x = x(y)$, $y''^2/(y')^3 = (x'')^2/(x')^3$, $y'''/(y')^2 - 3(y'')^2/(y')^3 = -x'''/(x')^2$.

transform of the form (3), then

$$m(t) = \frac{4 \left(\sum \frac{\delta_n^3}{(1 + \delta_n s)^3} \right)^2}{\left(\gamma^2 + \sum \frac{\delta_n^2}{(1 + \delta_n s)^2} \right)^3}$$

and

$$\rho^2(t) = 3 + \frac{6 \sum \frac{\delta_n^4}{(1 + \delta_n s)^4}}{\left(\gamma^2 + \sum \frac{\delta_n^2}{(1 + \delta_n s)^2} \right)^2} - m^2(t)$$

where $s = s(t)$ is the solution of $F'(s)/F(s) = -t$. ρ^2 is bounded by 9, because on multiplying out the denominator of the fraction it is clear that all terms appearing in the numerator also appear in the denominator. For m it will be enough to show boundedness at s_1 and s_2 . If these are both finite, multiplication of numerator and denominator by appropriate powers of $1 - (s/s_1)$ and $1 - (s/s_2)$ removes the unbounded terms. If s_1 or s_2 is infinite then all δ 's are of the same sign as the infinite endpoint. On multiplying out the numerator and denominator, it is clear that the denominator is larger for s near the infinite endpoint. If both s_1 and s_2 are infinite then all δ 's are zero and $m(t) = 0$.

Lemma 2. *Let $p(x)$ be a Pólya density. Then*

$$\sqrt{n}\sigma(t)p_t(\sqrt{n}\sigma(t)x, n) \rightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

uniformly for all x and for all t in the domain of $s(t)$ as $n \rightarrow \infty$.

Proof.

$$p_t(x, n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivx} f_t^n(v) dv$$

and

$$\sqrt{n}\sigma p_t(\sqrt{n}\sigma x, n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivx} f_t^n\left(\frac{v}{\sqrt{n}\sigma}\right) dv.$$

Since

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixv - \frac{1}{2}v^2} dv$$

we need to show

$$\int_{-\infty}^{\infty} e^{-ivx} \left[f_t^n\left(\frac{v}{\sqrt{n}\sigma}\right) - e^{-\frac{1}{2}v^2} \right] dv \rightarrow 0$$

uniformly in t and x as $n \rightarrow \infty$.

Breaking the range of integration at $\pm A$, the tail integrals, by the following Lemma 3, are less than or equal to

$$\int_{|x|>A} [(1 + \tfrac{1}{4}v^4)^{-\frac{1}{2}} + e^{-\frac{1}{2}v^2}] dv.$$

Taking A large enough to make this small, A is then held fixed. By Lemma 1 the remaining integral tends to zero uniformly in t as $n \rightarrow \infty$.

Lemma 3. *Let f_t be as in Lemma 1. Then*

$$\left| f_t \left(\frac{v}{\sqrt{n\sigma}} \right) \right| \leq (1 + \tfrac{1}{4}v^4)^{-\frac{1}{2}}$$

for all real v and $n \geq 2$.

Proof. We have, by direct computation, letting $\eta_k(t) = \delta_k/(1 + \delta_k s(t))$

$$\begin{aligned} \left| f_t \left(\frac{v}{\sqrt{n\sigma}} \right) \right| &= \left[e^{-(\gamma^2 v^2 / \sigma^2 n)} \prod \left(1 + \frac{\eta_k^2 v^2}{n\sigma^2} \right) \right]^{-\frac{1}{2}n} \\ &= \lim_{N \rightarrow \infty} \left[e^{(\gamma^2 v^2 / n\sigma^2)} \prod_1^N \left(1 + \frac{\eta_k^2 v^2}{n\sigma^2} \right) \right]^{-\frac{1}{2}n} \\ &\leq \lim_{N \rightarrow \infty} \left[e^{(\gamma^2 v^2 / n\sigma^2)} \left(1 + \frac{1}{n} \sum_1^N \frac{\eta_k^2 v^2}{\sigma^2} \right) \right]^{-\frac{1}{2}n} \\ &= \left[e^{(\gamma^2 v^2 / n\sigma^2)} \left(1 + \frac{1}{n\sigma^2} \sum_1^\infty \eta_k^2 v^2 \right) \right]^{-\frac{1}{2}n} \\ &\leq \left[\left(1 + \frac{\gamma^2 v^2}{n\sigma^2} \right) \left(1 + \frac{1}{n\sigma^2} \sum_1^\infty \eta_k^2 v^2 \right) \right]^{-\frac{1}{2}n} \\ &\leq \left[1 + \frac{1}{n\sigma^2} \left(\gamma^2 + \sum_1^\infty \eta_k^2 \right) v^2 \right]^{-\frac{1}{2}n} \\ &= \left(1 + \frac{1}{n} v^2 \right)^{-\frac{1}{2}n} \leq (1 + \tfrac{1}{4}v^4)^{-\frac{1}{2}}. \end{aligned}$$

We use here the fact that $\sigma^2(t) = \gamma^2 + \sum_1^\infty \eta_k^2(t)$.

To finish the proof of Theorem 1 we continue to assume $s_1 < s(t) < s_2$. By the definition of $s(t)$ the density p_t has mean zero. Convolution gives

$$p_t(x, n) = \frac{e^{-s(t)x} p(nt + x, n)}{\left[\int_{-\infty}^{\infty} e^{-s(t)y} p(y + t) dy \right]^n}.$$

Evidently p_t has a bilateral Laplace transform in $(s_1 - s(t), s_2 - s(t))$ so that it has all moments.

By Lemma 2,

$$\sqrt{n\sigma(t)} p_t(\sqrt{n\sigma(t)}x, n) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

tends to zero uniformly for all x as $n \rightarrow \infty$, for all t . In particular,

$$\sqrt{n}\sigma p_t(0, n) - \frac{1}{\sqrt{2\pi}} \rightarrow 0$$

and

$$\sqrt{n-k}\sigma p_t(kt-x, n-k) - \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(kt-x)^2}{2(n-k)\sigma^2}\right\} \rightarrow 0.$$

Then

$$\sqrt{\frac{n-k}{n}} \frac{p_t(kt-x, n-k)}{p_t(0, n)} - \exp\left\{-\frac{(kt-x)^2}{2(n-k)\sigma^2}\right\} = \epsilon_n$$

where $\epsilon_n \rightarrow 0$ uniformly in x and t . By the definition of p , this becomes

$$\sqrt{\frac{n-k}{n}} \frac{e^{s(t)x}}{[F(s(t))]^k} \frac{p(nt-x, n-k)}{p(nt, n)} - \exp\left\{\frac{(kt-x)^2}{2(n-k)\sigma^2}\right\} = \epsilon_n$$

whence

$$\begin{aligned} & \left| \frac{p(nt-x, n-k)}{p(nt, n)} - \frac{e^{-sx}}{[F(s)]^k} \right| \\ & \leq \frac{e^{-sx}}{[F(s)]^k} \left[\left| 1 - \sqrt{\frac{n}{n-k}} \exp\left\{-\frac{(kt-x)^2}{2(n-k)\sigma^2}\right\} \right| + \epsilon_n \sqrt{\frac{n}{n-k}} \right]. \end{aligned}$$

The terms in square brackets are uniformly bounded and tend to zero uniformly for t in any closed sub-interval of (c, d) with x fixed. We now show that $e^{-s(t)x}/[F(s(t))]^k$ tends to zero as t tends to the ends of (c, d) , or, equivalently, as s tends to the ends of (s_1, s_2) . From this it will follow that the convergence is uniform for all t in (c, d) . Furthermore, it is clear that $p(nt-x, n-k)/p(nt, n)$ tends to zero at the ends of (c, d) .

Because s and t are related by $s = s(t)$ we can show $e^{-sz}/F(s) \rightarrow 0$ as $s \rightarrow s_2 \{s_1\}$ with $c < z < d$. If s_2 is finite the result is clear because of the fact that $F(s) \rightarrow \infty$ as $s \rightarrow s_2$. If $s_2 = +\infty$ consider

$$e^{sz}F(s) = \int_{-\infty}^{\infty} e^{-sz}p(x+z) dx \geq \int_{-\infty}^{\Delta} e^{-sz}p(x+z) dx$$

where $c-z < \Delta < 0$. But this is $\geq e^{-\Delta s} \int_{-\infty}^{\Delta} p(x+z) dx$ which tends to ∞ as s tends to ∞ since

$$\int_{-\infty}^{\Delta} p(x+z) dx = \int_c^{\Delta+z} p(x) dx > 0.$$

This completes the proof of Theorem 1. A similar theorem for lattice valued random variables was stated by KEMENY [4].

Proof of Theorem 2. Changing the variable of integration in

$$u(x, k) = \int u(x+t, k+n)p(t, n) dt$$

gives

$$u(x, k) = \int_a^b \frac{p(nt - x, n - k)}{p(nt, n)} d\mu_n,$$

where

$$\mu_n(t) = n \int_c^t p(ny, n) u(ny, n) dy, \quad a = \frac{(m - k)c + x}{n}, \quad b = \frac{(n - k)d + x}{n}.$$

It is assumed that n is large enough so that $p(nt, n) > 0$ for t in (c, d) . Since $c < x/k < d$ it follows that $c < a < b < d$. We have

$$\mu_n(t) = n \int_c^t p(ny, n) u(ny, n) dy \leq \int_{na}^{nd} p(y, n) u(y, n) dy = u(0, 0).$$

By HELLY's theorem there is a subsequence of μ_n converging to a monotone μ on (c, d) . Consider

$$\begin{aligned} \left| u(x, k) - \int_c^d e^{-s(x)} [F(s(t))]^{-k} d\mu \right| &\leq \left| \int_c^d \frac{p(nt - x, n - k)}{p(nt, n)} d[\mu_n - \mu] \right| \\ &+ \left| \int_c^d \left(\frac{p(nt - x, n - k)}{p(nt, n)} - e^{-s(x)} \right) [F(s(t))]^{-k} d\mu \right|. \end{aligned}$$

The second term tends to zero by the uniform convergence of the integrand to zero. For the first term we have, if c or d is infinite, tails that are dominated by $\epsilon_n u(0, 0)$ where ϵ_n tends to zero as A tends to infinity. This leaves a term over a finite interval

$$\int_{-A}^A \frac{p(nt - x, n - k)}{p(nt, n)} d(\mu_n - \mu)$$

which clearly tends to zero since μ_n tends to μ and the integrand tends uniformly to a limit. Finally the variable s can be introduced in place of t by $s = s(t)$.

If we interpret a path of the form (nt, n) as a radial path, then the content of the following theorem is the analogue of non-tangential approach to the boundary in the Poisson integral of a continuous function. A non-tangential path will be of the form (α_n, n) with $\alpha_n = nt + O(\sqrt{n})$.

Theorem 3. Let u be given by

$$u(x, k) = \int_{s_1}^{s_2} \frac{e^{-sx}}{[F(s)]^k} d\mu(s)$$

where μ is of the form $\mu = \int f d\lambda$, with f bounded and continuous, λ a measure. If

$$u_1(x, k) = \int_{s_1}^{s_2} \frac{e^{-sx}}{[F(s)]^k} d\lambda,$$

then

$$\lim_{n \rightarrow \infty} \frac{u(\alpha_n, n)}{u_1(\alpha_n, n)} = f(s(t))$$

for all t such that $s(t)$ is in the interior of the support of λ .

The proof makes use of the fact that $e^{-sx}/[F(s)]^k$ has a unique maximum. For the details the reader is referred to [3] in which the theorem is proved for a kernel with the same properties, though arising from a discrete distribution.

EXAMPLES

1. Gaussian:

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

The kernel is $e^{-\frac{1}{2}ks^2 + xs}$, $-\infty < t < \infty$.

2. Truncated exponential:

$$p(x) = \begin{cases} e^{-(x+1)}, & x \geq -1, \\ 0, & x < -1. \end{cases}$$

The kernel is $(s+1)^k e^{-s(x+k)}$, $-1 \leq s < \infty$. These two densities are generators for the class of Pólya densities.

3. The Cauchy distribution with density

$$p(x) = \frac{1}{\pi(1+x^2)}$$

is not covered by our treatment, because $p(x)$ has no Laplace transform. However,

$$\lim_{n \rightarrow \infty} \frac{p(mt - x, n - k)}{p(nt, n)} = 1$$

in this case. In addition, it is easy to show that the process has only trivial non-negative martingales.

REFERENCES

- [1] J. L. DOOB, *First Boundary Value Problem*, Third Berkeley Symposium, University of California Press, Berkeley, Vol. II, 1956, pp. 49–80.
- [2] J. L. DOOB, Discrete potential theory and boundaries, *Journal of Math. and Mech.*, **8**, 1959.
- [3] J. L. DOOB, J. L. SNELL & R. E. WILLIAMSON, Applications of boundary theory to sums of independent random variables, *to appear*.
- [4] J. G. KEMENY, A probability limit theorem assuming no moments, *to appear*.
- [5] A. I. KHINCHIN, *Mathematical Foundations of Statistical Mechanics*, Dover, New York, 1949.
- [6] I. J. SCHOENBERG, On Pólya frequency functions, I, *Journal d'Analyse Math.*, **6**, 1951, pp. 331–374.
- [7] J. VILLE, *Etude Critique de la Notion de Collectif*, Borel Monographies des Probabilités, Paris, 1934.

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