*The Content of a Dam as the Supremum of an Infinitely Divisible Process**

J. GANI & R. PYKE

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Summary. This paper discusses the relation between the content of a dam and the supremum of a certain infinitely divisible process.

It is first pointed out that the distribution functions of the supremum over time $0 \leq t \leq T$ of a Poisson process with shift and the content at time T of a dam with Poisson inputs and a constant release rate have identical forms. The backward Kolmogorov equation of the first process is shown to be identical with the forward equation of the second.

A closer relation between the two processes is proved to exist, and a new definition for dam processes in discrete time is given. This is generalized to dam processes in continuous time whose net inputs are from a wide class of infinitely divisible processes.

The distribution function of the total time during which such dams are empty and non-empty is found. Lastly, using a result of KENDALL (1957) in the theory of dams, a simplification of the double Laplace transform due to BAXTER & DONSKER (1957) for the distribution function of the supremum over $0 \leq t \leq T$ of certain infinitely divisible processes with positive jumps is obtained.

1. Introduction. In a recent paper, PYKE (1959a) has obtained the inversion of BAXTER & DONSKER's (1957) double Laplace transform (see Section 8) for the distribution function (d.f.)

$$(1.1) \quad \sigma(x, T) = P \left[\sup_{0 \leq t \leq T} X_t \leq x \right]$$

of the supremum of a separable centered Poisson process with shift $\{X_t : t \geq 0\}$ satisfying $X_0 = 0$, whose characteristic function is

$$(1.2) \quad E(e^{i\omega X_t}) = \exp \{-it\omega\alpha + \lambda t(e^{i\omega} - 1)\},$$

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$\alpha > 0, \lambda > 0$ being given constants. The d.f. is obtained in the form

$$(1.3) \quad \sigma(x, T) = e^{-\lambda T} \sum_{n=0}^{\lfloor \alpha T + x \rfloor} (\alpha T + x - n) \frac{(\lambda/\alpha)^n}{n!} \sum_{r=0}^{\lfloor x \rfloor} \binom{n}{r} (r-x)^r (\alpha T + x - r)^{n-r-1},$$

where it is understood that $\binom{n}{r} = 0$ for $r > n$.

By considering the non-negative content $\{Z_t : t \geq 0\}$ of a dam with Poisson inputs of fixed magnitude h and with constant parameter λ , subject to a release at constant unit rate except when $Z_t = 0$ and with the initial content $Z_0 = z_0 \geq 0$, GANI & PRABHU (1959a) were able to find the explicit solution to the forward Kolmogorov equation for the d.f.

$$(1.4) \quad F(z, t; z_0) = P[Z_t \leq z \mid Z_0 = z_0]$$

of the process. This d.f. satisfies the difference-differential equation

$$(1.5) \quad \frac{\partial F}{\partial t} - \frac{\partial F}{\partial z} = -\lambda \{F(z, t; z_0) - F(z - h, t; z_0)\} \quad (0 \leq z < \infty),$$

a particular case of TAKÁCS's (1955) integro-differential equation for the waiting time in an analogous queueing process.

For $t = T$, this d.f. was found to be

$$(1.6) \quad F(z, T; z_0) = e^{-\lambda T} \sum_{r=0}^{\lfloor z/h \rfloor} \frac{\{\lambda(rh - z)\}^r}{r!} \cdot \sum_{j=0}^{\lfloor (z+T-rh-z_0)/h \rfloor} \frac{\lambda^j}{j!} (z + T - rh - jh)(z + T - rh)^{j-1},$$

a formula closely reminiscent of (1.3).

If h is put equal to $1/\alpha$ in (1.6), the order of summation reversed, the transformations $x = \alpha z$, and $j + r = n$ performed, and the d.f. $F_1(x, T; x_0)$ defined as

$$(1.7) \quad F_1(x, T; x_0) = F(z, T; z_0),$$

then from (1.6)

$$(1.8) \quad F_1(x, T; x_0) = e^{-\lambda T} \sum_{n=0}^{\lfloor x - x_0 + \alpha T \rfloor} \frac{(\lambda/\alpha)^n}{n!} (x + \alpha T - n) \cdot \sum_{r=0}^{\lfloor x \rfloor} \binom{n}{r} (x + \alpha T - r)^{n-r-1} (r - x)^r,$$

where $\binom{n}{r} = 0$ for $r > n$ as before. When $x_0 = 0$, the d.f. $F_1(x, T; 0)$ is precisely the $\sigma(x, T)$ of (1.3); this identity leads naturally to the hypothesis that PYKE's process as defined in (1.1) must satisfy a Kolmogorov equation of the type (1.5).

We prove in the next section that this is in fact true.

2. Derivation of the backward Kolmogorov equation. Let $\sigma(x, T; x_0)$ be

a slight generalisation of the d.f. considered by PΥΚΕ (1959a) which is defined by

$$(2.1) \quad \sigma(x, t; x_0) = P \left[\sup_{0 \leq t \leq T} X_t \leq x, X_T \leq x - x_0 \right] \quad (x_0 \geq 0).$$

Clearly $\sigma(x, T) = \sigma(x, T; 0)$. Then for an interval of time $0 \leq t \leq T + \delta T$ we have, by considering the events occurring in $(0, \delta T)$, that

$$(2.2) \quad \begin{aligned} \sigma(x, T + \delta T; x_0) &= \sigma(x + \alpha \delta T, T; x_0) \{1 - \lambda \delta T - o(\delta T)\} \\ &\quad + \sigma(x - 1 + \alpha \delta T, T; x_0) \{\lambda \delta T + o(\delta T)\}. \end{aligned}$$

For $\delta T \rightarrow 0$, it follows that the backward Kolomogorov equation is obtained as

$$(2.3) \quad \frac{\partial \sigma}{\partial T} - \alpha \frac{\partial \sigma}{\partial x} = -\lambda \{\sigma(x, T; x_0) - \sigma(x - 1, T; x_0)\} \quad (0 \leq x < \infty).$$

Now, putting $x = \alpha z$ and $h = 1/\alpha$, (2.3) may be rewritten as

$$(2.4) \quad \left(\frac{\partial}{\partial T} - \frac{\partial}{\partial z} \right) \sigma(\alpha z, T; \alpha z_0) = -\lambda \{\sigma(\alpha z, T; \alpha z_0) - \sigma(\alpha(z - h), T; \alpha z_0)\} \quad (0 \leq z < \infty)$$

which is identical in form with the forward Kolmogorov equation (1.5) for $F(z, T; z_0)$. Since it is known that this equation has a unique solution (cf. ΤΑΚÁCS (1955)), it is clear from (1.7) that $\sigma(x, T; x_0) \equiv F_1(x, T; z_0)$, so that (1.8) will be the general solution of (2.3).

It is shown in the following sections that a stronger relation exists between the two processes; first a lemma, which is the basis of this relationship, will be proved.

3. A preliminary lemma. For arbitrary random variables (r.v.) $U_1, U_2, \dots, U_n$, set $U_0 = 0$ and

$$X_k = \sum_{i=0}^k U_i, \quad M_k = \max_{0 \leq i \leq k} X_i \quad (k = 0, 1, \dots, n).$$

For any real number z , define the r.v.'s $T_k(z)$ recursively by

$$(3.1) \quad T_0(z) = z^+ \quad T_k(z) = [U_k + T_{k-1}(z)]^+ \quad (k = 1, 2, \dots, n)$$

where $z^+ = \max(0, z)$. It is known that if the r.v.'s $U_1, U_2, \dots, U_n$ are independent and identically distributed, then, for each k , M_k and $T_k(0)$ have the same d.f. This fact has been obtained and used for example by LINDLEY (1952) and by SPITZER (1957). This result is generalized in the following way.

Lemma 1. For all real z and for every $k = 1, 2, \dots, n$

$$(3.2) \quad T_k(z) = \max \{ \max_{0 \leq i \leq k} (X_k - X_i), X_k + z \}.$$

Moreover, if $U_1, U_2, \dots, U_n$ are *reversible*, i.e., if $(U_1, U_2, \dots, U_n)$ and $(U_n, U_{n-1}, \dots, U_1)$ have the same joint d.f., then for all real x

$$(3.3) \quad P[T_k(z) \leq x] = P[M_k \leq x, X_k \leq x - z].$$

Proof. Clearly it suffices to prove (3.2) and (3.3) for $z \geq 0$ and $x \geq 0$. From (3.2) it follows that

$$P[T_k(z) \leq x] = P[\max_{0 \leq i \leq k} (X_k - X_i) \leq x, X_k \leq x - z].$$

That this implies the truth of (3.3) under the given assumption is immediate since it is equivalent to assuming that $(X_0, X_1, \dots, X_n)$ and $(X_n - X_n, X_n - X_{n-1}, \dots, X_n - X_0)$ have the same d.f. The proof is by induction. Clearly (3.2) is satisfied for $k = 1$. Assume that (3.2) holds for $k = m \geq 1$. Then

$$\begin{aligned} T_m(z) &= \max \{U_m + T_{m-1}(z), 0\} \\ &= \max \{U_m + \max_{0 \leq i \leq m-1} [\max_{0 \leq i \leq m-1} (X_{m-1} - X_i), X_{m-1} + z], 0\} \\ &= \max \{ \max_{0 \leq i \leq m-1} [\max_{0 \leq i \leq m-1} (X_m - X_i), X_m + z], 0 \} \end{aligned}$$

which is equivalent to (3.2) as required. *Q.e.d.*

For what follows later, it is important to note that expressions analogous to (3.3) may be given for the joint d.f. of any finite set of the r.v.'s $T_1(z), \dots, T_k(z)$.

It is easily seen that (3.2) holds whenever $U_1, U_2, \dots, U_n$ are symmetrically dependent, i.e., if the joint distribution function of $(U_1, U_2, \dots, U_n)$ is invariant under all permutations of its arguments. Of greater importance for the following discussion is the fact that (3.3) is true when $U_1, U_2, \dots, U_n$ are independent and identically distributed r.v.'s. In this case (3.3) becomes, for $z = 0$, the result used by LINDLEY (1952) and SPITZER (1957).

4. Dam processes in discrete time. Dam (or, equivalently, provisioning or waiting-time) processes over discrete time are defined as follows. Let $U_1, U_2, \dots$ be a sequence of r.v.'s and interpret U_n to be the net input (input minus the amount demanded for consumption) of water into a dam at the n^{th} unit of time. For a fixed $z_0 \geq 0$, define the stochastic process $\{Z_n : n \geq 0\}$ by $Z_0 = z_0$ and

$$(4.1) \quad Z_n = \max \{U_n + Z_{n-1}, 0\} = (U_n + Z_{n-1})^+ \quad (n > 0).$$

It is easily seen from the definition of MORAN (1954) and subsequent workers (cf. GANI (1957)) that if Z_0 denotes the initial content of a dam, then Z_n represents the dam's content at the n^{th} unit of time, since when the dam is empty no water can be released for consumption. By comparing (4.1) with (3.1) of the preceding section, Z_n is seen to be equivalent to $T_n(z_0)$ as defined for the r.v.'s $(U_1, U_2, \dots, U_n)$. Thus, by (3.2) of Lemma 1, one may define

$$Z_n = \max \{ \max_{0 \leq i \leq n} (X_n - X_i), X_n + z_0 \},$$

where $X_k = \sum_{i=1}^k U_i$ as before for every $k = 1, 2, \dots$. Again, by Lemma 1, it follows that if, for every n , $U_1, U_2, \dots, U_n$ are reversible r.v.'s, then

$$(4.2) \quad P[Z_n \leq x] = P\left[\max_{0 \leq i \leq n} X_i \leq x, X_n \leq x - z_0\right].$$

Of particular interest is the special case of a dam process in which the net input r.v.'s are independent and identically distributed. For this case, (4.2) states that to know the probability that the dam's content at time n does not exceed x is equivalent to knowing the probability that the first $n - 1$ partial sums of independent and identically distributed r.v.'s do not exceed x and the n^{th} partial sum does not exceed $x - z_0$. A study of the latter probability when $z_0 = 0$ has been made by SPITZER (1956).

5. Dam processes in continuous time: $L(0) > -\infty$. In Section 1, a dam process was described in which the net input into the dam during the time interval $(0, t)$ was the r.v. X_t of a Poisson process with shift $\{X_t : t \geq 0\}$. In the present section, we shall consider dam processes whose net inputs are representable by members of a wide class of infinitely divisible (I.D.) processes.

Let $\{X_t : t \geq 0\}$ be a separable I.D. process which is representable as the difference of two independent non-negative I.D. processes (thus excluding the Brownian process). This representation insures (*cf.* Lemma 1 and Theorem 2 of PYKE (1959b)) that the process is centered and hence has a characteristic function of the form

$$E\{e^{i\omega X_t}\} = e^{t\phi(\omega)}$$

where ϕ can be expressed as

$$(5.1) \quad \phi(\omega) = i\omega m + \left\{ \int_{-\infty}^{0-} + \int_0^{\infty} \right\} \left(e^{i\omega x} - 1 - \frac{i\omega x}{1+x^2} \right) dL(x),$$

where m is real and L is a right continuous function which is non-decreasing on both $(-\infty, 0)$ and $[0, \infty)$ and satisfies $L(\pm\infty) = 0$ (*cf.* Chap. VIII of DOOB (1953)). Let us assume that the X_t -process has a finite expectation. Under this assumption (5.1) may be rewritten as

$$(5.2) \quad \phi(\omega) = i\omega\alpha + \left\{ \int_{-\infty}^{0-} + \int_0^{\infty} \right\} (e^{i\omega x} - 1) dL(x)$$

where α is real and

$$E(X_1) = \alpha + \int_{-\infty}^{\infty} x dL(x).$$

It may be shown from known properties of processes with independent increments (*cf.* Chap. VIII of DOOB (1953)) that almost all sample functions of the process $\{X_t - \alpha t : t \geq 0\}$ decrease and increase by jumps only, and have right and left hand limits at every t . Furthermore, it is known that the number of

jumps in $(0, t]$ of magnitude greater than x for $x \geq 0$ (not exceeding x for $x < 0$) is a Poisson r.v. with expectation $-tL(x)$ [$tL(x)$].

In this section we shall assume that $\alpha \leq 0$ and that $L(0) > -\infty$. Under these assumptions it follows from the above remark that

$$(5.3) \quad \begin{aligned} &P[X_t \text{ is non-increasing in } (0, T]] \\ &= P[\text{no jumps of positive magnitude in } (0, T]] = e^{-\lambda T} \end{aligned}$$

where $\lambda = -L(0)$. One may then define a stochastic process $\{Z_t : t \geq 0\}$ to represent the content of the dam over time in the following natural way. Set $Z_0 = z_0 \geq 0$. Define the r.v.'s

$$\begin{aligned} \tau_1 &= \inf \{t \geq 0 : X_t + z_0 \leq 0\}, \\ T_1 &= \sup \{t \geq \tau_1 : X_t \text{ is non-increasing in } (\tau_1, t)\}, \end{aligned}$$

the measurability of which is a consequence of the separability of the X_t -process (a complete probability space is assumed). Set

$$(5.4) \quad Z_t = \begin{cases} X_t + z_0 & \text{if } 0 \leq t < \tau_1 \\ 0 & \text{if } \tau_1 \leq t < T_1 \\ X_{T_1+} - X_{T_1-} & \text{if } t = T_1. \end{cases}$$

Now define recursively, for each $k > 1$, the r.v.'s

$$\tau_k = \inf \{t \geq T_{k-1} : X_t + Z_{T_{k-1}} \leq 0\}$$

and

$$T_k = \sup \{u \geq \tau_k : X_u \text{ is non-increasing in } (\tau_k, u)\}$$

and set

$$(5.5) \quad Z_t = \begin{cases} X_t - X_{T_{k-1}-} & \text{if } T_{k-1} < t < \tau_k \\ 0 & \text{if } \tau_k \leq t < T_k \\ X_{T_k+} - X_{T_k-} & \text{if } t = T_k. \end{cases}$$

Since, in particular, the r.v.'s $\{T_k - \tau_k : k \geq 1\}$ are independent and identically distributed, each having by (5.3) an exponential d.f. with parameter $0 \leq \lambda \equiv -L(0) < \infty$, it follows that with probability one $T_k \rightarrow +\infty$ as $k \rightarrow +\infty$. Therefore $\{Z_t : t \geq 0\}$ is a well-defined stochastic process for all $t \geq 0$. The interpretation of this process as a dam process is clear, since if initially the content of a dam is z_0 and if the net input during $(0, t]$ is X_t , then the content of the dam will equal $X_t + z_0$ until this quantity first becomes non-positive, namely, at time τ_1 . When this happens, the content is set equal to zero and remains there until the net input process has a positive jump, namely, at time T_1 , the magnitude of which denotes a new "initial" content, and the procedure repeats itself.

The above constructive definition of the dam process is equivalent to a more direct one, as is seen in the following

Lemma 2. *With probability one,*

$$(5.6) \quad Z_t = \max \left\{ \sup_{0 \leq \tau \leq t} (X_t - X_{\tau-}), X_t + z_0 \right\}$$

for all $t \geq 0$.

Proof. For each $t > 0$, define k_t by $T_{k_t} \leq t < T_{k_t+1}$ where one sets $T_0 = 0 = \tau_0$.

Since, with probability one, $T_k \rightarrow +\infty$, the right-hand side of (5.6) can with probability one be rewritten for every t as

$$(5.7) \quad \max \left\{ \max_{1 \leq k \leq k_t} \sup_{T_{k-1} < \tau \leq T_k} (X_t - X_{\tau-}), \sup_{T_{k_t} < \tau \leq t} (X_t - X_{\tau-}), X_t + z_0 \right\}.$$

However, by definition, it is clear that with probability one

$$\begin{aligned} \inf_{T_{k-1} < \tau \leq T_k} X_{\tau-} &= \inf_{\tau_k < \tau \leq T_k} X_{\tau-} = X_{T_k-}, \\ \inf_{T_{k_t} < \tau \leq t} X_{\tau-} &= X_{T_{k_t}-}, \quad X_{T_k-} \leq X_{T_{k-1}-}, \quad z_0 \leq X_{T_1} \end{aligned}$$

for every k and t . Hence (5.7) is equal, with probability one for every t , to

$$(5.8) \quad \begin{aligned} &\max \left\{ \max_{1 \leq k \leq k_t} (X_t - X_{T_k-}), X_t - X_{T_{k_t}-}, X_t + z_0 \right\} \\ &= \max \{ X_t - X_{T_{k_t}-}, X_t + z_0 \} = \begin{cases} X_t + z_0 & \text{if } k_t = 0 \\ X_t - X_{T_{k_t}-} & \text{if } k_t > 0 \end{cases} \end{aligned}$$

which is equal to Z_t as defined in (5.4) and (5.5). *Q.e.d.* (If one were to make the convention that $X_{0-} = -z$, then by (5.8) one may write $Z_t = X_t - X_{T_{k_t}-}$ for all t .)

REICH (1958) has essentially proven Lemma 2 for the case $\alpha = -1$ and $L(0-) = 0$. However, the integral appearing in the statement of Lemma 5.1 of REICH (1958) must be over $[t-x, t]$ instead of $(t-x, t]$ for the statement to be correct. The case $z_0 = 0$ of Lemma 2 was known earlier to Dr. FRANK SPITZER as indicated in an oral communication.

6. Dam processes in continuous time: General case. In the foregoing section, the assumptions that $\alpha \leq 0$ and $L(0) > -\infty$ were sufficient to enable a constructive definition to be made. More precisely, under these assumptions, the duration of the successive time periods during which the dam is empty, namely $\{T_k - \tau_k : k > 0\}$, are well-defined, independent and identically distributed r.v.'s, each having an exponential distribution with parameter $\lambda = -L(0)$. When the assumptions are not satisfied, these r.v.'s are zero with probability one, and hence an analogous constructive definition of a dam process as attempted by GANI & PRABHU (1959b) is impossible. For this case, in view of Lemma 2, the following definition is proposed.

Definition. Let $\{X_t : t \geq 1\}$ be a separable centered I.D. process having a characteristic function of the form given by (5.2). The dam process corresponding to this (net-input) process and having initial content z_0 is defined by

$$Z_t = \max \left\{ \sup_{0 \leq \tau \leq t} (X_t - X_{\tau-}), X_t + z_0 \right\} \quad (t \geq 0).$$

In this section it will be shown that not only is this process a natural generalization of the preceding ones, but it is also a limit (with probability one at each t) of a corresponding sequence of dam processes in discrete time. To this end, define for each $n \geq 1$ a sequence of r.v.'s $\{U_k^{(n)} : k \geq 1\}$ by $U_k = X_{k2^{-n}} - X_{(k-1)2^{-n}}$. Since the X_t -process is I.D., these r.v.'s are independent and identically distributed for each n . Let $\{Z_k^{(n)} : k \geq 0\}$ be the discrete dam process, as defined in Section 4, having net inputs $U_1^{(n)}, U_2^{(n)}, \dots$ and satisfying $Z_0^{(n)} = z_0$, but where the discrete unit of time is equal to 2^{-n} . As was stated in Section 4, it is a consequence of Lemma 1 that

$$(6.1) \quad Z_k^{(n)} = \max \left\{ \max_{0 \leq i \leq k} (X_k^{(n)} - X_i^{(n)}), X_k^{(n)} + z \right\}$$

where

$$U_0^{(n)} = 0 \quad \text{and} \quad X_i^{(n)} = \sum_{j=0}^i U_j^{(n)} = X_{i2^{-n}}.$$

It is more convenient, and not too confusing, to treat these discrete dam processes as if they were defined for continuous time. This can be done most simply by defining $Z_t^{(n)} = Z_{k2^{-n}}^{(n)}$ whenever $k2^{-n} \leq t < (k+1)2^{-n}$. With this notation, it is possible to prove

Lemma 3. *The stochastic processes $\{Z_t^{(n)} : t \geq 0\}$ converge to the process $\{Z_t : t \geq 0\}$ in such a way that for each $t \geq 0$*

$$P[Z_t = \lim_{n \rightarrow \infty} Z_t^{(n)}] = 1.$$

Proof. Because of the separability of the X_t -process it is known that

$$P \left[\sup_{0 \leq \tau \leq t} (X_t - X_{\tau-}) = \sup_{n \geq 1} \max_{0 \leq k \leq [2^n t]} (X_t - X_{k2^{-n}}) \right] = 1,$$

and since the process is centered, it is known that $X_{[2^n t]2^{-n}} \rightarrow X_t$ with probability one. By the definition of $Z_t^{(n)}$ it follows from (6.1) that

$$Z_t^{(n)} = \max \left\{ \max_{0 \leq k \leq [2^n t]} (X_{[2^n t]2^{-n}} - X_{k2^{-n}}), X_{[2^n t]2^{-n}} + z_0 \right\}$$

which by the above remarks must converge to Z_t with probability one as desired.

A dam process in continuous time for which the assumptions of Section 5 are not satisfied has been considered by MORAN (1956), namely that for which $\alpha = -1$ and

$$L(x) = - \int_x^\infty \frac{e^{-\rho u}}{u} du$$

for $\rho > 0$. This corresponds to the case in which the input during time $(0, t]$ has the Gamma density function

$$k(x, t) = \frac{(x/\rho)^{t-1} e^{-x/\rho}}{\rho \Gamma(t)} \quad (x \geq 0).$$

MORAN's process is well-defined by him as the limit of a sequence of dam processes in discrete time. The n^{th} process of MORAN's approximating sequence has a time unit of n^{-1} and is such that the input during one such unit of time is a r.v. whose characteristic function is $[1 + \rho n(1 - e^{i\omega/n})]^{-1/n}$, the n^{th} root of a geometric characteristic function. This approximating sequence is not the same as that considered in Lemma 3, but the general ideas are certainly equivalent.

Let us remark here that in Sections 4, 5, 6 the assumption of independent and identically distributed inputs (incremental) is not necessary, in view of Lemma 1 in which only reversibility is needed. Consider for example the dam process of Section 1 which has a Poisson input. If the Poisson input were non-homogeneous with parameter $\lambda(t)$, then using reversibility the relationship between the dam process and the supremum of a process with independent increments (not identically distributed) would still hold for a fixed t , provided only that $\lambda(u) = \lambda(t - u)$ for all $0 \leq u \leq t$.

7. Distribution of times in the empty and non-empty states. In this section we shall consider a dam process $\{Z_t : t \geq 0\}$ for which $-L(0) < \infty$ as defined in Section 5. Let us define for each $t \geq 0$, as in TAKÁCS' paper (1957), a r.v.

$$(7.1) \quad \chi(t) = \begin{cases} 1 & \text{if } Z_t > 0 \\ 0 & \text{if } Z_t = 0. \end{cases}$$

The r.v.

$$(7.2) \quad \beta(t) = \int_0^t \chi(u) du = \sum_{i=0}^{k_t} [\min(\tau_{i+1}, t) - T_i]$$

will then denote the total time in $(0, t]$ during which the dam is non-empty, while $\alpha(t) = t - \beta(t)$ is the time during which the dam is empty. Set $W_i = T_i - \tau_i$, $Y_i = \tau_{i+1} - T_i$ ($i = 0, 1, 2, \dots$) and $T_0 = \tau_0 = 0$. W_i and Y_i are respectively the lengths of the i^{th} empty and non-empty periods. It is known (cf. (5.3)) that the r.v.'s W_i are independent and identically distributed with common d.f.

$$(7.3) \quad B(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

where $\lambda = -L(0)$. The r.v.'s Y_i ($i > 0$) are also independent and identically distributed. It is easily seen that the common d.f. for these r.v.'s is given by

$$(7.4) \quad H(x) = \int_0^\infty G(z, x) d\Lambda(z)$$

where $\Lambda(x) = 1 + \lambda^{-1}L(x)$ is the d.f. of the magnitude of a positive jump in the net-input process, and where $G(z, x)$ is the d.f. of the first passage time $Y_0 = \tau_1$, that is, the time of first emptiness of the dam given an initial content z . Note that $Y_0 = 0$ when $z = 0$. Upon taking Laplace-Stieltjes transforms in (7.4), we obtain directly, for $s \geq 0$, that

$$(7.5) \quad h(s) = \int_0^\infty e^{-sz} dH(x) = \int_0^\infty g(z, s) d\Lambda(z)$$

where

$$(7.6) \quad g(z, s) = \int_{0-}^\infty e^{-sz} d_x G(z, x) = e^{-z\eta(s)},$$

the last equality being a consequence of the fact that the first passage time r.v. is infinitely divisible. If one defines

$$\xi(s) = \int_0^\infty (1 - e^{-sz}) dL(x),$$

the 'positive part' of (5.2) with $\omega = is$, it follows from (7.5) and (7.6) that

$$(7.7) \quad h(s) = \int_0^\infty e^{-sz\eta(s)} d\Lambda(z) = 1 - \lambda^{-1} \int_0^\infty (1 - e^{-sz\eta(s)}) dL(z) = 1 - \lambda^{-1}\xi[\eta(s)].$$

Define, as in TAKÁCS (1957), the d.f.

$$(7.8) \quad \Omega(t, x) = P[\beta(t) \leq x].$$

TAKÁCS obtains $\Omega(t, x)$ explicitly for the case $Z_0 = 0$. Only a slight modification of this is needed to obtain $\Omega(t, x)$ for the case $Z_0 = z_0 \geq 0$, which, when performed, yields

$$\Omega(t, x) = \sum_{n=0}^\infty \int_{0-}^x H_n(x) e^{-\lambda(t-x)} \frac{\lambda^n (t-x)^n}{n!} d_u G(z_0, u)$$

for $x < t$ and $\Omega(t, x) = 1$ for $x \geq t$. It then follows that for fixed $a \geq 0$

$$(7.9) \quad \omega(s) = \int_0^\infty e^{-sz} d_x \Omega(a+x, x) = g^*(z_0, s) e^{-\lambda a[1-h(s)]} = e^{-z_0 \eta(s) - a\xi[\eta(s)]}.$$

In the particular case where in (5.2) $\alpha = -1$ and $L(0-) = 0$, so that

$$(7.10) \quad \phi(\omega) = -i\omega - \xi(-i\omega),$$

it has been shown by TAKÁCS (1955) and KENDALL (1957) that $\eta(s)$ must satisfy

$$(7.11) \quad \eta(s) = s + \xi[\eta(s)].$$

In this case, therefore, $\omega(s)$ may be rewritten as

$$(7.12) \quad \omega(s) = e^{-(z_0+a)\eta(s)+as}.$$

It is interesting to note from (7.6) that

$$P [\beta(a + x) \leq x] = P [\tau_1(a + z_0) \leq a + x]$$

where $\tau_1(a + z_0)$ denotes the first time of emptiness for the dam process similar to the given one, but having initial content $Z_0 = a + z_0$.

It should be pointed out that the inverse $G(z, t)$ of (7.6) has been obtained explicitly by KENDALL (1957), for the special case determined by (7.10), as

$$(7.13) \quad d_t G(z, t) = \frac{z}{t} [d_x K(x, t)]_{x=t-z}$$

where $K(x, t) = P[X_t + t \leq x]$ is the d.f. of the dam input. More correctly KENDALL obtained (7.13) when the density function $k(x, t) = dK(x, t)/dx$ exists.

8. On the formula of Baxter and Donsker (1957). In the preceding sections the close relationship between the content of a dam in continuous time (in discrete time) and the supremum of an I.D. process (the maximum of partial sums of independent and identically distributed r.v.'s) is emphasized. Furthermore, reference is made to the papers of SPITZER (1956) and BAXTER & DONSKER (1957) in which, respectively, the maximum of partial sums and the supremum of an I.D. process are studied. It is obvious that there is also a close relationship between first passage times and suprema. In this section we shall use this relationship to give a simplification of a result of BAXTER & DONSKER.

Let $\{V_t : t \geq 0\}$ denote an arbitrary separable centered I.D. process and set

$$(8.1) \quad \sigma(z, t) = P \left[\sup_{0 \leq \tau \leq t} V_\tau \leq z \right].$$

Define the double transform

$$(8.2) \quad \Phi(s, \theta) = s \int_0^\infty \int_0^\infty e^{-st - \theta z} d_z \sigma(z, t) dt$$

for $s, \theta \geq 0$ and set

$$(8.3) \quad \psi(\theta, t) = \int_0^\infty e^{-\theta v} d_v P[V_t^+ < v] = 1 + \int_0^\infty (e^{-\theta v} - 1) d_v P[V_t < v].$$

BAXTER & DONSKER (1957, equation (3.6)) show that

$$(8.4) \quad \log \Phi(s, \theta) = \int_s^\infty \int_0^\infty e^{-ut} [\psi(\theta, t) - 1] dt du$$

by generalizing the corresponding result for partial sums of independent and identically distributed r.v.'s obtained by SPITZER (1956).

It is clear that $G(z, t)$, as defined for (7.4), satisfies

$$(8.5) \quad \begin{aligned} G(z, t) &= 1 - P[Z_t > 0, 0 \leq \tau \leq t] \\ &= 1 - P \left[\sup_{0 \leq \tau \leq t} (-X_\tau) \leq z \right] = 1 - \sigma(z, t) \end{aligned}$$

when $\sigma(z, t)$ is defined by (8.1) with $V_t = -X_t$. Upon substitution of (8.5) into (8.2), straightforward computation yields

$$\begin{aligned} \Phi(s, \theta) &= s \int_0^\infty e^{-st} [1 - \theta \int_0^\infty e^{-\theta z} G(z, t) dz] dt \\ (8.6) \qquad &= 1 - \theta \int_0^\infty e^{-\theta z} g^*(z, s) dz = \frac{\eta(s)}{\theta + \eta(s)}, \end{aligned}$$

the last equality following from (7.6.)

Consider, in particular, the important special case of a dam process mentioned at the end of Section 7, namely that in which the net input process is determined by (5.2) with $\alpha = -1$ and $L(0-) = 0$. For this case, it is known that $\eta(s)$ is a solution of the functional equation (7.11), and hence (8.6) together with this solution gives a simple expression for the double generating function Φ . Under these assumptions a direct evaluation of (8.2) is also possible. Assume for simplicity that $K(x, t)$ has a derivative with respect to x , namely $k(x, t)$ where $K(x, t) = P[X_t + t \leq x]$, as defined for (7.13). Then from (8.3)

$$\psi(\theta, t) = 1 + \int_0^t (e^{-\theta v} - 1)k(t - v, t) dv,$$

and hence (8.4) becomes

$$\begin{aligned} \log \Phi(s, \theta) &= - \int_0^\infty \int_0^t \frac{e^{-st}}{t} (1 - e^{-\theta v})k(t - v, t) dv dt \\ (8.7) \qquad &= - \int_0^\infty \frac{1 - e^{-\theta v}}{v} \int_v^\infty \frac{ve^{-st}}{t} k(t - v, t) dt dv \\ &= - \int_0^\infty \frac{1 - e^{-\theta v}}{v} e^{-v\eta(s)} dv, \end{aligned}$$

the last step being an immediate consequence of (7.6) and (7.13). By writing

$$\frac{1 - e^{-\theta v}}{v} = \int_0^\theta e^{-vx} dx$$

and interchanging integrations (8.7) becomes

$$\log \Phi(s, \theta) = - \int_0^\theta \frac{1}{x + \eta(s)} dx = \log \frac{\eta(s)}{\theta + \eta(s)}$$

which checks with (8.6). A parallel proof for the case of arbitrary $K(x, t)$ is possible, provided care is exercised in interpreting and working with (5.13).

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Columbia University
New York, N. Y.