On the Geometrical Significance of the Moments of Order Statistics and of Deviations of Order Statistics from the Mean in Samples from Gaussian Populations

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1. Introduction and summary. Let $\xi \equiv \xi_{r|n}$ denote the r^{th} largest observation in a random sample of n observations from a normal population with zero mean and unit variance. Then the elementary probability law of ξ is given by

$$(1) \quad n \binom{n-1}{r-1} [\Phi(\xi)]^{r-1} [1 - \Phi(\xi)]^{n-r} \phi(\xi),$$

where

$$\begin{aligned} \phi(\xi) &= (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}\xi^2}, \\ \Phi(\xi) &= (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\xi} e^{-\frac{1}{2}\eta^2} d\eta. \end{aligned}$$

(See *e.g.* WILKS [4].)

We shall investigate the distribution of $\xi_{r|n}$, the r^{th} order statistic, and that of $\xi_{r|n} - \bar{\xi}$, the deviation of the r^{th} order statistic from the sample mean ($\bar{\xi} = \sum_{i=1}^n \xi_{i|n}/n$), from the point of view of the moments of these distributions. It will appear that the latter moments are directly related to certain *geometrical* moments of members of a class of simplices, referred to as skew-regular simplices, in $(n-2)$ -dimensional spherical space.

Skew-regular hyperspherical simplices of dimensionality $m-1$ are generated from a regular hyperspherical simplex as follows. Consider a regular simplex constructed on the surface of a sphere (which for convenience may be taken as

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the unit sphere) immersed in m -space with each of its $\frac{1}{2}m(m-1)$ primary bounding angles equal to θ . Then any simplex $S_{l,m}(\theta)$ with $m-l$ vertices common to the generating simplex and the remaining l vertices ($l = 0, 1, \dots, m$) antipodal to the vertices of the latter simplex is a skew-regular simplex with angles either θ or $\pi - \theta$. In particular, the generating simplex $S_{0,m}(\theta)$ is regular with common angle θ , while $S_{m,m}(\theta)$ is the regular simplex antipodal to $S_{0,m}(\theta)$, so that the class of regular simplices is a subclass of the class of skew-regular simplices.

The skew-regular simplices occurring in the present investigation are of dimensionality $n-2$ and have angles either $\frac{2}{3}\pi$ or $\frac{1}{3}\pi$, the generating simplex being $S_{0,n-1}(\frac{2}{3}\pi)$. Generating simplices of this type may be constructed by inscribing a regular (linear) simplex in a unit $(n-1)$ -sphere with the angle between any two flats at a vertex equal to $\arccos \{1/(n-1)\}$. Then n regular $(n-2)$ -dimensional (spherical) simplices result by joining the vertices of the inscribed simplex to the center of the sphere. These n simplices are disjoint and cover the entire surface of the sphere.

In Section 2 the characteristic function of $\xi_{r|n} - \bar{\xi}$, as well as that of $\xi_{r|n}$, will be obtained in a form which will enable the geometrical significance of the moments of the two latter statistics to be assessed. This is achieved in Sections 3 and 4, respectively. As a curious but interesting by-product of our results it will be shown that the s^{th} moment of $\xi_{r|n} - \bar{\xi}$ is equal to the expectation of $n^{-\frac{1}{2}s} H_s(\xi/\sqrt{n})$ in the distribution (1), where $H_s(x)$ is the $(s+1)^{\text{th}}$ Tchebycheff-Hermite polynomial, normalized so that the coefficient of x^s is unity (Eq. 4). Thus the moments of $\xi_{r|n} - \bar{\xi}$ are expressible in terms of rather simple univariate integrals.

2. The characteristic functions of $\xi_{r|n}$ and $\xi_{r|n} - \bar{\xi}$. Since the probability density function of $\xi_{r|n}$ is given by (1), the characteristic function of $\xi_{r|n}$ is

$$\begin{aligned}\Psi_{r|n}(t) &= n \binom{n-1}{r-1} \int_{-\infty}^{\infty} e^{it\xi} [\Phi(\xi)]^{r-1} [1 - \Phi(\xi)]^{n-r} \phi(\xi) d\xi \\ &= n \binom{n-1}{r-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\xi} \cdots \int_{-\infty}^{\xi} \int_{\xi}^{\infty} \cdots \int_{\xi}^{\infty} e^{it\xi} \phi(\xi) \phi(u_1) \cdots \phi(u_{n-1}) \\ &\quad d\xi du_1 \cdots du_{n-1}.\end{aligned}$$

On setting

$$\left. \begin{aligned}u_i &= v_i + \xi' \quad (j = 1, 2, \dots, n-1), \\ \xi &= \xi',\end{aligned} \right\}$$

we have

$$\begin{aligned}\Psi_{r|n}(t) &= n \binom{n-1}{r-1} \int_{-\infty}^{\infty} \int_{-\infty}^0 \cdots \int_{-\infty}^0 \int_0^{\infty} \cdots \\ &\quad \cdot \int_0^{\infty} e^{it\xi'} \phi(\xi') \phi(v_1 + \xi') \cdots \phi(v_{n-1} + \xi') d\xi' dv_1 \cdots dv_{n-1}.\end{aligned}$$

Now the integrand

$$\begin{aligned} \exp \{it\xi'\} \phi(\xi') \prod_1^{n-1} \phi(v_i + \xi') &= \exp \{it\xi'\} (2\pi)^{-\frac{1}{2}n} \exp \left\{ -\frac{1}{2} \left[\xi'^2 + \sum_1^{n-1} (v_i + \xi')^2 \right] \right\} \\ &= \exp \{it\xi'\} (2\pi)^{-\frac{1}{2}} \exp \left\{ -\frac{n}{2} \left(\xi' + \frac{1}{n} \sum_1^{n-1} v_i \right)^2 \right\} \\ &\quad \cdot (2\pi)^{-\frac{1}{2}(n-1)} \exp \left\{ -\frac{1}{2} \left[\sum_1^{n-1} v_i^2 - \frac{1}{n} \left(\sum_1^{n-1} v_i \right)^2 \right] \right\}, \end{aligned}$$

and since

$$\begin{aligned} (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp \{it\xi'\} \exp \left\{ -\frac{n}{2} \left(\xi' + \frac{1}{n} \sum_1^{n-1} v_i \right)^2 \right\} d\xi' \\ = n^{-\frac{1}{2}} \exp \left\{ -\frac{t^2}{2n} - \left(\frac{1}{n} \sum_1^{n-1} v_i \right) it \right\}, \end{aligned}$$

integration with respect to ξ' yields

$$\begin{aligned} (2) \quad \Psi_{r|n}(t) &= \exp \left\{ \frac{-t^2}{2n} \right\} \cdot n^{-\frac{1}{2}} n \binom{n-1}{r-1} \int_{-\infty}^0 \cdots \int_{-\infty}^0 \int_0^{\infty} \cdots \int_0^{\infty} (2\pi)^{-\frac{1}{2}(n-1)} \\ &\quad \cdot \exp \left\{ -\frac{1}{2} Q(\mathbf{v}) - \left(\frac{1}{n} \sum_1^{n-1} v_i \right) it \right\} dv_1 \cdots dv_{n-1}, \end{aligned}$$

where $Q(\mathbf{v})$ is a definite positive quadratic form in the v_i ,

$$Q(\mathbf{v}) = \sum_1^{n-1} v_i^2 - \frac{1}{n} \left(\sum_1^{n-1} v_i \right)^2.$$

We note from (2) that the characteristic function of $\xi_{r|n}$ factors into two terms. The first term $\exp \{ -t^2/2n \}$ is the characteristic function of the mean of a random sample of n observations from a normal population with zero mean and unit variance, *i.e.* it represents the characteristic function of $\bar{\xi}$. The second term is the characteristic function of the random variable $-\sum_1^{n-1} v_i/n$, where the v_i are random variables whose joint probability density function is

$$n^{\frac{1}{2}} \binom{n-1}{r-1} (2\pi)^{-\frac{1}{2}(n-1)} \exp \{ -\frac{1}{2} Q(\mathbf{v}) \}, \quad v_i \begin{cases} \leq 0 & (j = 1, 2, \dots, r-1), \\ \geq 0 & (j = r, r+1, \dots, n-1), \end{cases}$$

and zero elsewhere, *i.e.* the v_i have a conditional multivariate normal distribution (with the off-diagonal elements of the correlation matrix for the unconditioned distribution all equal to $\frac{1}{2}$ and with zero expectation vector for the same distribution) in the sense that they are restricted to fall in the orthant defined by $v_j \leq 0$ ($j = 1, 2, \dots, r-1$), $v_j \geq 0$ ($j = r, r+1, \dots, n-1$). Now it follows from a result due to McKAY [1] that $\xi_{r|n} - \bar{\xi}$ and $\bar{\xi}$ are independently distributed. Hence the characteristic function of $\xi_{r|n} - \bar{\xi}$ is given by the second term of the right-hand member in (2). It also follows from (2)

that the distribution of $\xi_{r|n} - \bar{\xi}$ behaves in the limit ($n \rightarrow \infty$) like that of the negative arithmetic mean of the v_i . We note further that the characteristic function of $\xi_{r|n} - \bar{\xi}$ may be expressed in the form

$$(3) \quad \begin{aligned} E \exp \{it(\xi_{r|n} - \bar{\xi})\} &= \exp\left(\frac{t^2}{2n}\right) \Psi_{r|n}(t) \\ &= n \binom{n-1}{r-1} \int_{-\infty}^{\infty} \exp\left(it\xi + \frac{t^2}{2n}\right) \phi(\xi) [\phi(\xi)]^{r-1} [1 - \Phi(\xi)]^{n-r} d\xi. \end{aligned}$$

Differentiate the first and third terms in (3) s times with respect to t , set $t = 0$, and recall that

$$\frac{d^s}{dy^s} \exp\{xy - \tfrac{1}{2}y^2\} \Big|_{y=0} = H_s(x),$$

where $H_s(x)$ is the polynomial of degree s in the family of orthogonal Tchebycheff-Hermite polynomials, normalized so that the coefficient of x^s in $H_s(x)$ is unity. Then

$$(4) \quad \begin{aligned} E (\xi_{r|n} - \bar{\xi})^s &= n \binom{n-1}{r-1} \int_{-\infty}^{\infty} \{n^{-\frac{1}{2}} H_s(\xi \sqrt{n})\} \phi(\xi) [\Phi(\xi)]^{r-1} \\ &\quad \cdot [1 - \Phi(\xi)]^{n-r} d\xi \quad (s = 0, 1, 2, \dots). \end{aligned}$$

Equation (4) may be utilized to express the moments of $\xi_{r|n} - \bar{\xi}$ as linear functions of the moments of $\xi_{r|n}$. This is achieved directly by representing $H_s(\xi \sqrt{n})$ in its explicit polynomial form and by integrating term by term in (4). The desired result is

$$(5) \quad E (\xi_{r|n} - \bar{\xi})^s = \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \frac{s(s-1) \cdots (s-2j+1)}{j! (2n)^j} E \xi_{r|n}^{s-2j},$$

where $[m]$ denotes, as usual, the integral part of m .

3. The moments of $\xi_{r|n} - \bar{\xi}$ as geometrical moments of skew-regular spherical simplices. Denote the characteristic function of $\xi_{r|n} - \bar{\xi}$ by $K_{r|n}(t)$. From (2),

$$\begin{aligned} K_{r|n}(t) &= \exp\left\{\frac{t^2}{2n}\right\} \Psi_{r|n}(t) = n^{\frac{1}{2}} \binom{n-1}{r-1} \int_{-\infty}^0 \cdots \int_{-\infty}^0 \int_0^{\infty} \cdots \int_0^{\infty} (2\pi)^{-\frac{1}{2}(n-1)} \\ &\quad \cdot \exp\left\{-\tfrac{1}{2}Q(\mathbf{v}) - \left(\frac{1}{n} \sum_{i=1}^{n-1} v_i\right) it\right\} dv_1 \cdots dv_{n-1}. \end{aligned}$$

Hence

$$\begin{aligned} E (\xi_{r|n} - \bar{\xi})^s &= (-i)^s \frac{d^s}{dt^s} K_{r|n}(t) \Big|_{t=0} = (-1)^s n^{-s} n^{\frac{1}{2}} \binom{n-1}{r-1} \int_{-\infty}^0 \cdots \int_{-\infty}^0 \int_0^{\infty} \cdots \\ &\quad \cdot \int_0^{\infty} (2\pi)^{-\frac{1}{2}(n-1)} \exp\left\{-\tfrac{1}{2}Q(\mathbf{v})\right\} \left(\sum_{i=1}^{n-1} v_i\right)^s dv_1 \cdots dv_{n-1}. \end{aligned}$$

On using the orthogonal transformation

$$\left. \begin{aligned} w_i &= \sum_{j=1}^{n-1} a_{ij} v_j \quad (i = 1, 2, \dots, n-2), \\ w_{n-1} &= (n-1)^{-\frac{1}{2}} \sum_{j=1}^{n-1} v_j, \end{aligned} \right\}$$

the above equation transforms into

$$(6) \quad \begin{aligned} E(\xi_{r|n} - \bar{\xi})^s &= (-1)^s n^{-s} n^{\frac{s}{2}} \binom{n-1}{r-1} (n-1)^{\frac{s}{2}} \int \cdots \int_R (2\pi)^{-\frac{1}{2}(n-1)} \\ &\quad \cdot \exp \left\{ -\frac{1}{2} \left[\sum_1^{n-2} w_i^2 + \frac{1}{n} w_{n-1}^2 \right] \right\} w_{n-1}^s dw_1 \cdots dw_{n-1}. \end{aligned}$$

Apply now the simple scaling transformation

$$\left. \begin{aligned} w_i^* &= w_i \quad (i = 1, 2, \dots, n-2), \\ w_{n-1}^* &= n^{-\frac{1}{2}} w_{n-1}. \end{aligned} \right\}$$

Then

$$(7) \quad \begin{aligned} E(\xi_{r|n} - \bar{\xi})^s &= (-1)^s n^{-s} n^{\frac{s}{2}} \binom{n-1}{r-1} \{n(n-1)\}^{\frac{s}{2}} \int \cdots \int_{R^*} (2\pi)^{-\frac{1}{2}(n-1)} \\ &\quad \cdot \exp \left\{ -\frac{1}{2} \sum_1^{n-1} w_i^{*2} \right\} w_{n-1}^* n^{\frac{s}{2}} dw_1^* \cdots dw_{n-1}^*. \end{aligned}$$

The domains of integration R and R^* in (6) and (7), respectively, are easily obtained on noting that

$$\begin{aligned} v_i &= \sum_{l=1}^{n-2} a_{li} w_l + (n-1)^{-\frac{1}{2}} w_{n-1} \quad (i = 1, 2, \dots, n-1), \\ v_i &= \sum_{l=1}^{n-2} a_{li} w_l^* + \left\{ \frac{n}{n-1} \right\}^{\frac{1}{2}} w_{n-1}^* \quad (i = 1, 2, \dots, n-1). \end{aligned}$$

The regions R and R^* are thus defined by

$$\begin{aligned} R: & \left\{ \begin{aligned} \sum_{l=1}^{n-2} a_{li} w_l + (n-1)^{-\frac{1}{2}} w_{n-1} &\leq 0 & (i = 1, 2, \dots, r-1), \\ &\geq 0 & (i = r, r+1, \dots, n-1), \end{aligned} \right\} \\ R^*: & \left\{ \begin{aligned} L_i(\mathbf{w}^*) \equiv \sum_{l=1}^{n-2} a_{li} w_l^* + \left(\frac{n}{n-1} \right)^{\frac{1}{2}} w_{n-1}^* &\leq 0 & (i = 1, 2, \dots, r-1), \\ &\geq 0 & (i = r, r+1, \dots, n-1). \end{aligned} \right\} \end{aligned}$$

Denote the angle interior to R^* between the i^{th} and j^{th} flats, $L_i(\mathbf{w}^*) = 0$, $L_j(\mathbf{w}^*) = 0$, which bound R by (ij) ($j \neq i$). Then

$$\cos(ij) = \pm \frac{\sum_{l=1}^{n-2} a_{li} a_{lj} + \frac{n}{n-1}}{\sum_{l=1}^{n-2} a_{li}^2 + \frac{n}{n-1}} = \pm \frac{-\frac{1}{n-1} + \frac{n}{n-1}}{\left(1 - \frac{1}{n-1}\right) + \frac{n}{n-1}} = \pm \frac{1}{2} \quad (j \neq i),$$

where the plus sign holds if i and j are in opposite blocks $\{1, 2, \dots, r-1\}$, $\{r, r+1, \dots, n-1\}$, and the negative sign if i and j are in the same block. (In the evaluation of $\cos(ij)$ the orthogonality of the matrix (a_{ij}) has been exploited.) Thus

$$(8) \quad (ij) = \begin{cases} \frac{2}{3}\pi; i, j \text{ in the same set of the two} \\ \text{sets } \{1, 2, \dots, r-1\}, \{r, r+1, \dots, n-1\}, & j \neq i, \\ \frac{1}{3}\pi; i, j \text{ in different sets.} \end{cases}$$

Finally, transform to polar coordinates with

$$w_{n-1}^* = r \cos \phi.$$

Then (7) reduces to

$$(9) \quad \begin{aligned} E(\xi_{r|n} - \bar{\xi})^s &= (-)^s n^{-\frac{1}{2}(s-2)} (n-1)^{\frac{1}{2}s} \binom{n-1}{r-1} \int_{R^*} \dots \int (2\pi)^{-\frac{1}{2}(n-1)} \\ &\quad \cdot e^{-\frac{1}{2}r^2} r^{n-2+s} \cos^s \phi \, dr \, d\omega \\ &= (-)^s n^{-\frac{1}{2}(s-2)} (n-1)^{\frac{1}{2}s} \binom{n-1}{r-1} (2\pi)^{-\frac{1}{2}(n-1)} \int_0^\infty e^{-\frac{1}{2}r^2} r^{n-2+s} \, dr \\ &\quad \cdot \int_{S_{r-1, n-1}(\frac{2}{3}\pi)} \cos^s \phi \, d\omega \\ &= (-)^s n^{-\frac{1}{2}(s-2)} (n-1)^{\frac{1}{2}s} \binom{n-1}{r-1} (2\pi)^{-\frac{1}{2}(n-1)} 2^{\frac{1}{2}(n-3+s)} \\ &\quad \cdot \Gamma\left(\frac{n-1+s}{2}\right) \int_{S_{r-1, n-1}(\frac{2}{3}\pi)} \cos^s \phi \, d\omega \quad (s = 0, 1, 2, \dots), \end{aligned}$$

where $d\omega$ is the content of an infinitesimal element of surface on the unit sphere imbedded in $(n-1)$ -space and $S_{r-1, n-1}(\frac{2}{3}\pi)$ denotes the skew-regular simplex on the surface of the latter sphere which is demarcated by the $n-1$ flats $L_1(\mathbf{w}^*) = 0, \dots, L_{n-1}(\mathbf{w}^*) = 0$, *i.e.*

$$S_{r-1, n-1}(\frac{2}{3}\pi) = \left\{ \mathbf{w}^* \mid \mathbf{w}^* \cdot \mathbf{w} = 1, \quad \begin{array}{ll} L_i(\mathbf{w}^*) \leq 0 & (i = 1, 2, \dots, r-1), \\ L_i(\mathbf{w}^*) \geq 0 & (i = r, r+1, \dots, n-1). \end{array} \right\}$$

Note that $\cos \phi$ is the distance of a point on the surface of the unit sphere from the flat Λ orthogonal to the line joining the center of the sphere and the centroid of the generating regular simplex $S_{0, n-1}(\frac{2}{3}\pi)$, this distance being taken as positive if the point in question lies on the same side of Λ as $S_{0, n-1}(\frac{2}{3}\pi)$, and negative otherwise. (Λ is the flat of symmetry which is equally inclined to the bounding flats $L_i(\mathbf{w}^*) = 0$ ($i = 1, 2, \dots, n-1$) of $S_{0, n-1}(\frac{2}{3}\pi)$.) Thus, according to (9), the moments of $\xi_{r|n} - \bar{\xi}$ are simple multiples of the geometrical moments of $S_{r-1, n-1}(\frac{2}{3}\pi)$ with respect to Λ . In terms of the normalized geometrical moments $h_{r-1, n-1; s}$, defined by

$$h_{r-1, n-1; s} = \int_{S_{r-1, n-1}(\frac{2}{3}\pi)} \cos^s \phi \, d\omega / \int_{S_{r-1, n-1}(\frac{2}{3}\pi)} d\omega,$$

equation (9) reduces to

$$(10) \quad E (\xi_{r|n} - \bar{\xi})^s = (-)^s \left\{ \frac{2(n-1)}{n} \right\}^{\frac{1}{2}s} \frac{\Gamma\left(\frac{n-1+s}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} h_{r-1, n-1; s} \\ (s = 0, 1, 2, \dots).$$

These are also the asymptotic moments ($n \rightarrow \infty$) of $\xi_{r|n}$.

4. The moments of $\xi_{r|n}$ as geometrical moments of skew-regular spherical simplices. It has been shown elsewhere (RUBEN [2]) that the moments of $\xi_{r|n}$ may be expressed as linear functions of contents of simplices of various dimensionalities, the angles of the corresponding generating regular simplices being arc cos $(-\frac{1}{3})$, arc cos $(-\frac{1}{5})$, $\dots$ or arc cos $(-\frac{1}{2})$, arc cos $(-\frac{1}{4})$, $\dots$, according as the moments are odd or even. We now establish the interesting result that these moments are expressible as linear combinations of the geometrical moments of a skew-regular simplex with angles $\frac{2}{3}\pi$ and $\frac{1}{3}\pi$. In particular when $r = 1$ or $r = n$ (extreme order statistics) the latter simplex is regular.

From (2), the characteristic functions of $\xi_{r|n}$ and $\xi_{r|n} - \bar{\xi}$, $\Psi_{r|n}(\cdot)$ and $K_{r|n}(\cdot)$, respectively, are related by the equation

$$\Psi_{r|n}(t) = e^{-t^2/2n} K_{r|n}(t).$$

On differentiating around $t = 0$ s times, we obtain, after using (10),

$$(11) \quad E \xi_{r|n}^s = \sum_{j=0}^s \binom{s}{j} (-i)^{s-j} \left. \frac{d^{s-j}}{dt^{s-j}} e^{-t^2/2n} \right|_{t=0} E (\xi_{r|n} - \bar{\xi})^j \\ = \sum_j' (-)^j \frac{(s-j)!}{\left(\frac{s-j}{2}\right)! (2n)^{\frac{1}{2}(s-j)}} \left\{ \frac{2(n-1)}{n} \right\}^{\frac{1}{2}j} \frac{\Gamma\left(\frac{n-1+j}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} h_{r-1, n-1; j},$$

where $\sum_j'$ indicates summation over all integral j from $j = 0$ to $j = s$ such that $s - j$ is even. Equation (11) is the required result.

A comparison of equation (11) and the result referred to at the beginning of this section indicates the existence of linear relationships between the geometrical moments of $S_{r-1, n-1}(\frac{2}{3}\pi)$ with respect to the flat of symmetry Λ and the contents of such simplices as $S_{p,q}(\theta)$, where $\theta = \text{arc cos } (-\frac{1}{3})$, arc cos $(-\frac{1}{4})$, etc. (cf. Eq (5)). This is, in fact, a particular case of a more general result (RUBEN [3]) which relates the geometrical moments of $S_{l,m}(\theta)$ relative to the flat of symmetry of $S_{0,m}(\theta)$, for arbitrary permissible θ , to the contents of the edges (of various dimensionality) of $S_{l,m}(\theta)$, these edges being themselves skew-regular simplices.

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