On the Boundary Behavior of a Conformal Map of the Open Unit Disk into a Riemann Surface

MAURICE HEINS

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1. The study of the boundary behavior of a meromorphic function whose domain is the open unit disk has received considerable attention. The list of contributions to the field is extensive. There does not appear to be, however, a systematic investigation of the boundary behavior of a conformal map of the open unit disk into an arbitrary Riemann surface, even though the concepts introduced in the classical case of a meromorphic function are not dependent in any essential way on the fact that the image of the map under consideration lies in the extended plane. The object of the present note is to extend two well-known theorems of the theory of the boundary behavior of meromorphic functions to the case of a conformal map of the open unit disk into an arbitrary Riemann surface. The theorems in question are due respectively to COLLINGWOOD & CARTWRIGHT [3] and to CARATHÉODORY [2]. It will be seen that the methods we employ differ considerably from those of the aforementioned authors.

No attempt is made to construct a systematic theory of the boundary behavior of a conformal map of a Riemann surface in the present paper; rather, our attention will be confined to the two theorems to which reference has been made.

2. **The theorem of Collingwood and Cartwright.** We recall the basic terms involved in the theorem of COLLINGWOOD & CARTWRIGHT and its statement. Let f be a meromorphic function whose domain is $\{|z| < 1\}$. A point ζ of the unit circumference is termed a *Fatou point* of f provided that f possesses a sectorial limit at ζ . We let F denote the set of Fatou points of f . By the *range* of f at ζ , $|\zeta| = 1$, is understood the set of points w of the extended plane for which ζ is an accumulation point of $f^{-1}(\{w\})$. A point ζ of the unit circumference is termed a *Picard point* of f provided that the range of f at ζ omits at most two points of the extended plane. We denote the set of Picard points of f by P . The theorem of COLLINGWOOD & CARTWRIGHT asserts

$$(2.1) \quad \{|z| = 1\} = P \cup F'.$$

Here F' denotes the derived set of F .

Let us now consider the corresponding situation for conformal maps. Suppose that φ is a conformal map of $\{|z| < 1\}$ into a Riemann surface S . It is clear that the natural definition of a Fatou point of φ is the following: A point ζ , $|\zeta| = 1$, is termed a *Fatou point* of φ provided that $\varphi(z)$ tends to a point of S as z tends to ζ sectorially or else that $\varphi(z)$ tends to the ideal boundary of S as z tends to ζ sectorially. Here too we shall denote the set of Fatou points of φ by F . When we turn to the notion of a Picard point, we may make our demands more severe in case S is not conformally equivalent to the extended plane. The range of φ at ζ is defined as above. We shall agree to term ζ a *Picard point* of φ provided that the following condition is fulfilled: When S is conformally equivalent to the extended plane, the range of φ at ζ omits at most two points of S ; when S is conformally equivalent to the finite plane, the range of φ at ζ omits at most one point of S ; in all other cases, the range of φ at ζ is S itself. The set of Picard points of φ will be denoted by P as above.

The theorem of COLLINGWOOD & CARTWRIGHT persists for φ . *Proof.* Certain cases may be put aside at once thanks to the fact that the Fatou theorem is in effect. We recall that a conformal map ψ of a hyperbolic Riemann surface G into an unrestricted Riemann surface H is termed *Lindelöfian* provided that

$$(2.2) \quad \sum_{\psi(r)=q} n(r; \psi) \mathfrak{G}_\sigma(p, r) < +\infty$$

for $\psi(p) \neq q$, $\mathfrak{G}_\sigma$ being the Green's function of G and $n(r; \psi)$ the multiplicity of ψ at r [5]. We recall, further, that in case $G = \{|z| < 1\}$ almost all points of $\{|z| = 1\}$ are Fatou points of a Lindelöfian ψ [6]. If H is hyperbolic, any conformal map with image in H is Lindelöfian by virtue of the Lindelöf principle [4]. Hence it is sufficient to establish the theorem of COLLINGWOOD & CARTWRIGHT for the case where S is *parabolic*.

Hence it is sufficient to show that, S being parabolic, if ζ is in the complement of F' with respect to the unit circumference, then $\zeta \in P$. Suppose that this were not the case. There would exist a positive number ρ (< 2) such that there is no Fatou point of φ in $\{0 < |z - \zeta| < \rho\}$ and such that φ^* , the restriction of φ to $\{|z| < 1\} \cap \{|z - \zeta| < \rho\}$, omits at least three points, two points, or one point of S according as S is conformally equivalent to the extended plane, to the finite plane, or to neither the extended plane nor the finite plane. Let E denote in each of the respective cases a set of three points, two points, one point omitted by φ^* . Let σ denote a conformal universal covering of $S - E$ whose domain is $\{|z| < 1\}$. Now let b denote a lifting of φ^* with respect to σ , so that b is an analytic function with domain $\{|z| < 1\} \cap \{|z - \zeta| < \rho\}$ which satisfies

$$\varphi^* = \sigma \circ b.$$

Clearly b possesses a sectorial limit at almost all points of

$$\{|z| = 1\} \cap \{|z - \zeta| < \rho\}.$$

The sectorial limits of b at points of this set other than ζ are all of modulus one. This fact, which is the crux of the proof, is a consequence of our choice of ρ which guarantees the absence of Fatou points of φ in $\{|z| = 1\} \cap \{0 < |z - \zeta| < \rho\}$.

As a matter of convenience we introduce the univalent conformal map ψ of $\{|z| < 1\}$ onto $\{|z| < 1\} \cap \{|z - \zeta| < \rho\}$ which carries the right unit semi-circumference onto $\{|z| = 1\} \cap \{|z - \zeta| < \rho\}$ and 1 into ζ and we let $b_1 = b \circ \psi$. Further, $K(z, \alpha)$ will denote the Poisson kernel

$$\operatorname{Re} \left\{ \frac{\alpha + z}{\alpha - z} \right\}, \quad |z| < 1, \quad |\alpha| = 1.$$

Suppose that b_1 is analytic at each point of $\{e^{i\theta} \mid |\theta| < \frac{1}{2}\pi\}$. Then for each θ satisfying $|\theta| < \frac{1}{2}\pi$, $K(b_1(z), b_1(e^{i\theta}))$ possesses the sectorial limit $+\infty$ at $e^{i\theta}$. The set $\{b_1(e^{i\theta}) \mid |\theta| < \frac{1}{2}\pi\}$ is an open subset of the unit circumference.

Suppose that b_1 is not analytic at each point of $\{e^{i\theta} \mid |\theta| < \frac{1}{2}\pi\}$. We consider for a given α , $|\alpha| = 1$, the positive harmonic function $h_\alpha(z) = K(b_1(z), \alpha)$ and observe that h_α possesses the sectorial limit $+\infty$ at the points of a set clustering at some point of $\{e^{i\theta} \mid |\theta| < \frac{1}{2}\pi\}$. This assertion is a very slight variation of the theorem of CALDERÓN, GONZÁLEZ-DOMÍNGUEZ & ZYGMUND [1] which states that α is the radial limit of b_1 at infinitely many points of $\{e^{i\theta} \mid |\theta| < \frac{1}{2}\pi\}$. However, if one uses the Poisson-Stieltjes representation for h_α as do CALDERÓN, GONZÁLEZ-DOMÍNGUEZ & ZYGMUND, and the theorem of F. & M. RIESZ, it is easy to show that if our assertion were false, then

$$\frac{\alpha + b_1}{\alpha - b_1}$$

would be analytic at each point of $\{e^{i\theta} \mid |\theta| < \frac{1}{2}\pi\}$, save possibly for poles, and consequently b_1 would be analytic at each point of $\{e^{i\theta} \mid |\theta| < \frac{1}{2}\pi\}$.

We may now complete the proof as follows. It is to be recalled that S is parabolic and hence that $S - E$ is parabolic also. Let $a \in E$. With a there is associated a class of conjugate parabolic transformations τ which leave σ invariant and have the following property:

Let α denote the fixed point of a transformation τ . Let U denote a neighborhood of a . Then for c sufficiently large,

$$(2.3) \quad \sigma(\{K(z, \alpha) > c\}) \subset U.$$

Since $S - E$ is parabolic, the set of the fixed points of the transformations τ is dense on the unit circumference. We now fix a τ in the following manner. If b_1 is analytic at each point of $\{e^{i\theta} \mid |\theta| < \frac{1}{2}\pi\}$, we take a τ whose fixed point is of the form $b_1(e^{i\theta})$ for some θ satisfying $0 < |\theta| < \frac{1}{2}\pi$. If b_1 is not analytic at each point of $\{e^{i\theta} \mid |\theta| < \frac{1}{2}\pi\}$, we select an arbitrary τ . In either case h_α possesses the sectorial limit $+\infty$ at some point of $\{e^{i\theta} \mid 0 < |\theta| < \frac{1}{2}\pi\}$, α being the fixed point of τ . Consequently by (2.3) $\varphi^* = \sigma \circ b$ possesses the sectorial limit α at some point of $\{|z| = 1\} \cap \{0 < |z - \zeta| < \rho\}$. Hence the assumption that $\zeta \notin P$ is untenable. The theorem follows.

3. Under certain circumstances it is possible to assert that $F' = \{|z| = 1\}$. This is the case when S is hyperbolic. If S is conformally equivalent to the finite plane or the punctured finite plane, or if S is compact, then it is easy to give examples of φ for which $F' = \emptyset$.

There remains to be considered the case where S is parabolic but is not compact nor conformally equivalent to the finite plane or the punctured finite plane. If, in addition, we are assured that S is *continuable*, i.e. that S is conformally equivalent to a proper subregion of some Riemann surface, then $F' = \{|z| = 1\}$. This may be seen as follows. We may suppose that S is a proper subregion of a Riemann surface T which is such that if $a \in T - S$, the conformal universal covering of $T - \{a\}$ has a hyperbolic domain. On paraphrasing the argument of §2, we see that if ζ , $|\zeta| = 1$, is not an accumulation point of the set of points at which φ possesses a sectorial limit ($\in S$), then ζ is an accumulation point of the set of points at which φ possesses the sectorial limit a .

If, however, S is *not continuable* but meets the other restrictions of the preceding paragraph, the question whether $F' = \{|z| = 1\}$ remains unsettled. The difficulty lies in the fact that information concerning the boundary behavior of a conformal universal covering is fairly limited.

4. It is to be observed that in formulating the notion of a Picard point for a conformal map it is possible to dispense with the reservation concerning exceptional points when S is conformally equivalent to neither the extended plane nor the finite plane. That the reservation is indispensable for the two special cases as far as the theorem of COLLINGWOOD & CARTWRIGHT is concerned may be seen by observing that there exists a nonvanishing analytic function in $\{|z| < 1\}$ for which $F' = \emptyset$. It is therefore of interest to determine conditions on φ which guarantee that the range of φ at each point of the unit circumference is S , when S is conformally equivalent to either the extended plane or the finite plane. Given $a \in S$, let $H(a)$ denote the condition:

There exists a region Ω of S containing a such that $\varphi^{-1}(\Omega) \neq \emptyset$, each component of $\varphi^{-1}(\Omega)$ is relatively compact,¹ and finally

$$(4.1) \quad \sup_{\omega} v_{\omega} < +\infty,$$

where v_{ω} is the number of times that the restriction of φ to a component ω of $\varphi^{-1}(\Omega)$ attains a point of Ω .

The following theorem holds.

Theorem. *If for each $a \in S$ the condition $H(a)$ is fulfilled, then the range of φ at each point of the unit circumference is S .*

Proof. We take S , as we may, to be the extended plane or the finite plane. It is a consequence of the hypothesis that φ has no asymptotic points belonging to S . Each point of the unit circumference is a Picard point, for if some point

¹ In the sense of the topology of the open unit disk.

of the unit circumference were not a Picard point, there would exist a sectorial limit value of φ belonging to S (cf. [3], p. 140; §2 of the present paper yields this result readily).

It is easy to see that it suffices to show that the following statement is true: Suppose that ζ , $|\zeta| = 1$, is an accumulation point of $\varphi^{-1}(\{0\})$ and that $H(0)$ is fulfilled by virtue of the fact that $\{|w| < 1\}$ satisfies the requirements imposed on Ω . Then ζ is an accumulation point of $\varphi^{-1}(\{w\})$ for each w , $|w| < 1$.

To establish this reduced statement we proceed as follows. Let ω denote a component of $\varphi^{-1}(\{|w| < 1\})$; let α be a zero of φ lying in ω . We let ω_1 denote the image of ω with respect to

$$z \rightarrow \frac{z - \alpha}{1 - \bar{\alpha}z},$$

and we define g with domain ω_1 by

$$g(z) = \varphi\left(\frac{z + \alpha}{1 + \bar{\alpha}z}\right).$$

Thereupon we define an analytic function f in the open unit disk by

$$f(w) = \prod_{g(z)=w} z^{n(z;g)},$$

$n(z; g)$ being the multiplicity of g at z . It is immediate that $f(0) = 0$ and that f is of modulus less than one. We fix w_0 , $|w_0| < 1$. By Schwarz's lemma we have

$$|f(w_0)| \leq |w_0|.$$

Let β denote an antecedent of w_0 with respect to g which is of least modulus. Then

$$|\beta|^{r_\omega} \leq |w_0|.$$

The proof of the reduced statement may now be completed as follows. Let $(\alpha_k)_1^\infty$ denote a sequence of antecedents of zero with respect to φ satisfying $\lim \alpha_k = \zeta$. We take ω of the preceding paragraph as the component of $\varphi^{-1}(\{|w| < 1\})$ containing α_k and $\alpha = \alpha_k$. Suppose that β_k is the β selected for the present ω and α . From the fact that

$$|\beta_k| \leq |w_0|^{1/m}$$

where m is a positive integer independent of k , it follows that

$$\lim_{k \rightarrow \infty} \frac{\alpha_k + \beta_k}{1 + \bar{\alpha}_k \beta_k} = \zeta.$$

Since

$$\varphi\left(\frac{\alpha_k + \beta_k}{1 + \bar{\alpha}_k \beta_k}\right) = w_0,$$

the reduced statement follows.

5. Carathéodory's cluster set theorem. We shall formulate directly the generalization of CARATHÉODORY's cluster set theorem [2] for a conformal map φ of $\{|z| < 1\}$ into an arbitrary Riemann surface S . It is to be recalled that the cluster set C of φ at ζ , $|\zeta| = 1$, is simply

$$\bigcap_{r>0} \overline{\varphi(\Delta_r)}$$

where $\Delta_r = \{|z| < 1\} \cap \{|z - \zeta| < \rho\}$. Let O denote the set of points $s \in S$ satisfying the following condition: There exist an open set Ω , $s \in \Omega$, and $\rho > 0$ such that for almost all η satisfying $|\eta| = 1$, $|\eta - \zeta| < \rho$, either $\varphi(r\eta)$ tends to a point of $S - \Omega$ as $r \rightarrow 1$ or else $\varphi(r\eta)$ tends to the ideal boundary of S as $r \rightarrow 1$. Clearly O is open. If S is compact, then $S - O \neq \emptyset$. Now let O_1 denote the union of the components of O which have points in common with C . Then CARATHÉODORY's cluster set theorem takes the form

$$(5.1) \quad C = (S - O) \cup O_1.$$

It is to be remarked that actually CARATHÉODORY formulated the statement of his theorem in terms of a countable basis of open circular disks; however, it is readily verified that the basis free formulation (5.1) is equivalent to CARATHÉODORY's when S is the extended plane. Let R denote the range of φ at ζ . Then we have

Theorem. *The relation (5.1) holds for a conformal map φ of $\{|z| < 1\}$ into an arbitrary Riemann surface S . Further, $O_1 - R$ is a set of zero capacity.*

Two remarks should be made. First, the statement concerning $O_1 - R$ is new even in the classical case of a meromorphic function. Second, the proof we give for the present theorem differs markedly from the proof given by CARATHÉODORY for his cluster set theorem. The latter appeals to properties of bounded analytic functions and of plane convex sets in an elegant but special way. Our treatment of the present theorem will be based on the notion of a map of *type-B1* which we have given in [4]. The basic facts wanted for the present argument will be recalled below.

The inclusion relation

$$C \subset (S - O) \cup O_1$$

is obvious. The essential part of the proof is to verify that inclusion in the opposite direction also holds, as well as that $O_1 - R$ has zero capacity.

(i) We show first that $S - O \subset C$. If $C = S$, the assertion is trivial. Suppose that C is a proper part of S . Let $s \in S - O$. Then for each open set Ω containing s and each positive number ρ there exists η satisfying: $|\eta| = 1$, $|\eta - \zeta| < \rho$, $\lim_{r \rightarrow 1} \varphi(r\eta)$ exists and lies in Ω . Here use is made of the fact that for ρ sufficiently small $\varphi(\Delta_\rho)$ is not dense in S . Consequently $\lim_{r \rightarrow 1} \varphi(r\eta)$ exists or $\varphi(r\eta)$ tends to the ideal boundary of S as $r \rightarrow 1$ p.p. for $|\eta| = 1$, $|\eta - \zeta| < \rho$ provided that ρ is sufficiently small [6]. We now conclude that $s \in C$. The assertion $S - O \subset C$ is thereby established.

(ii). Let Ω now denote a component of O_1 . We shall show

$$(5.2) \quad \Omega \cap C \subset \Omega \cap (\text{int } \bar{R}).$$

From (5.2) we conclude immediately that

$$(5.3) \quad \Omega \cap C \subset \Omega \cap (\text{int } \bar{R}) \subset \Omega \cap \bar{R} \subset \Omega \cap C.$$

From (5.3) it follows that each member of (5.3) is equal to Ω . Consequently $O_1 \subset C$. We also show that for each $s \in O_1$ there exists an open neighborhood $V(s)$ of s such that $V(s) - R$ has zero capacity. It will then follow that $O_1 - R$ has zero capacity.

Suppose that $s \in \Omega \cap C$. There exist a Jordan arc α lying in $\{|z| < 1\}$, save for endpoints on the unit circumference which are distinct from ζ , and an open subset Δ of S which contains s such that (1) s is exterior to the image of α less its endpoints with respect to φ , (2) for each η of the subarc of the unit circumference whose endpoints are those of α and which contains ζ , save for a set of zero measure, $\varphi(r\eta)$ tends to a point of $S - \Delta$ as $r \rightarrow 1$ or $\varphi(r\eta)$ tends to the ideal boundary as $r \rightarrow 1$. Now let Δ_1 denote an open disk satisfying (1) $s \in \Delta_1 \subset \bar{\Delta}_1 \subset \Delta$, (2) the frontier of Δ_1 is a regular analytic closed Jordan curve γ , (3) $\bar{\Delta}_1$ is contained in the complement of the image with respect to φ of α less its endpoints. Let D denote the component of $\{|z| < 1\} - \alpha$ satisfying $\zeta \in \text{fr } D$.

We now consider $\varphi^{-1}(\Delta_1) \cap D$ and note that it is not empty since $s \in C$. Let δ denote a component of $\varphi^{-1}(\Delta_1) \cap D$. We shall now show that φ_δ , the restriction of φ to δ , is a map² of type-B1 of δ into Δ_1 . This is, of course, immediate if δ is relatively compact,³ the greatest harmonic minorant of $z \rightarrow \mathfrak{G}_{\Delta_1}(\varphi_\delta(z), q)$, $q \in \Delta_1$, being zero. Suppose then that δ is not relatively compact³ and that b is a non-negative bounded harmonic function with domain δ which is dominated by the greatest harmonic minorant of $\mathfrak{G}_{\Delta_1}(\varphi_\delta(z), q)$. Let b^* denote the function with domain $\{|z| < 1\}$ whose restriction to δ is b and which vanishes elsewhere. Clearly, b^* is bounded and subharmonic in $\{|z| < 1\}$. Let $\bar{b}$ denote the least harmonic majorant of b^* . From standard results of the theory of subharmonic functions, in particular, Littlewood's theorem [8, §7.22], we know that b^* possesses radial limits p.p. and that they coincide p.p. with those of $\bar{b}$. If b^* possessed positive radial limits for a set of positive measure, then there would exist a

² We recall the facts that we shall want concerning maps of type-B1 [4]. A non-negative harmonic function u on a Riemann surface is termed *singular* [7] provided that the only non-negative bounded harmonic function on the surface which is dominated by u is zero. A conformal map f of a Riemann surface F into a hyperbolic Riemann surface G is termed of *type-B1* provided that for each $q \in G$ the greatest harmonic minorant of $p \rightarrow \mathfrak{G}_G(f(p), q)$ is singular, $\mathfrak{G}_G$ being the Green's function for G . Essential use will be made of the following properties of a map f of type-B1: (1) If ν_f denotes the valence function for f , i.e. if $\nu_f(q) = \sum_{f(p)=q} n(p; f)$, $n(p; f)$ being the multiplicity of f at p , then $\{\nu_f < \sup \nu_f\}$ is a set of zero capacity. (2) The greatest harmonic minorant of $p \rightarrow \mathfrak{G}_G(f(p), q)$ vanishes, save for a set of q which is an F_σ of zero capacity.

³ In the sense of the topology of the open unit disk.

subset of positive measure of $\text{fr } D - \alpha$ for each point η of which $\varphi(r\eta)$ would fail either to tend to the ideal boundary of S as $r \rightarrow 1$ or to tend to a point of $S - \Delta$ as $r \rightarrow 1$. This is impossible. Since $\bar{b}$ is bounded and the radial limits of $\bar{b}$ vanish p.p., $\bar{b} = 0$ and hence, *a fortiori*, $b = 0$. We conclude that φ_δ is a map of type-B1 of δ into Δ_1 .

We recall that $s \in C$. Let (z_k) denote a sequence of points in D satisfying: $\lim z_k = \zeta$; $\lim \varphi(z_k) = s$; $\varphi(z_k) \in \Delta_1$ for all k . Let δ_k denote the δ which contains z_k . There are two cases to consider.

Case 1. *The set $\{\delta_k\}_1^\infty$ is infinite.* We may as well assume that $k \rightarrow \delta_k$ is univalent. Given $\sigma > 0$, as a consequence of the boundary behavior of φ in D ,

$$\delta_k \subset \{|z - \zeta| < \sigma\}$$

for k sufficiently large. Since φ accounts for each point of Δ_1 , save for a set of zero capacity, on each δ_k , it is readily seen that $\Delta_1 - R$ has zero capacity. Hence $\Delta_1 \subset \bar{R}$, so that $s \in \Omega \cap (\text{int } \bar{R})$.

Case 2. *The set $\{\delta_k\}_1^\infty$ is finite.* Here we may assume that the δ_k are all equal. We designate the common set by δ . It will be shown that the set of points $q \in \Delta_1$ for which ζ is not an accumulation point of $\varphi_\delta^{-1}(\{q\})$ has zero capacity, φ_δ being the restriction of φ to δ . This is a consequence of the following assertion: For each $q \in \Delta_1$, for which the greatest harmonic minorant of $z \rightarrow \mathfrak{G}_{\Delta_1}(\varphi_\delta(z), q)$ vanishes, ζ is an accumulation point of $\varphi_\delta^{-1}(\{q\})$.

Suppose this latter assertion false. Then for some $q \in \Delta_1$, say q_0 ,

$$(5.4) \quad \mathfrak{G}_{\Delta_1}(\varphi_\delta(z), q_0) = \sum_{\varphi_\delta(t) = q_0} n(t; \varphi) \mathfrak{G}_\delta(z, t)$$

and ζ would not be an accumulation point of $\varphi_\delta^{-1}(\{q_0\})$. Let $\sigma_1, \sigma_2, \sigma_3$ be so chosen that (1) $\sigma_1 > \sigma_2 > \sigma_3 > 0$, (2) $\{|z| < 1\} \cap \{|z - \zeta| < \sigma_1\} \subset D$, (3) no point of $\varphi_\delta^{-1}(\{q_0\})$ lies in $\{|z - \zeta| < \sigma_1\}$. The number of components of $\delta \cap \{|z - \zeta| < \sigma_2\}$ which contain points of $\{|z - \zeta| < \sigma_3\}$ is finite. This is a consequence of the boundary behavior of φ in D . It may now be concluded that $\mathfrak{G}_{\Delta_1}(\varphi_\delta(z), q_0)$ tends to zero when z tends to ζ .

In order to see this, we let ω denote one of the distinguished components of $\delta \cap \{|z - \zeta| < \sigma_2\}$ which contains a point of $\{|z - \zeta| < \sigma_3\}$. There exists a Jordan arc β lying in $\{|z| < 1\}$ save for endpoints on $\{|z| = 1\}$ whose points z satisfy

$$\sigma_3 < |z - \zeta| < \sigma_2$$

and which, in addition, is such that $\overline{\omega \cap \beta}$ lies in the open unit disk and $\omega \cap \beta$ consists of a finite number of mutually disjoint open arcs. This is possible, again thanks to the boundary behavior of φ in D . Now $\mathfrak{G}_{\Delta_1}(\varphi_\delta(z), q_0)$ is bounded on $\omega \cap \beta$. It is now seen on using (5.4) that $\mathfrak{G}_{\Delta_1}(\varphi_\delta(z), q_0)$ yields the solution of the Dirichlet problem for the intersection of ω with that component of $\{|z| < 1\} - \beta$ for which ζ is a frontier point, the boundary values being given by $\mathfrak{G}_{\Delta_1}(\varphi_\delta(z), q_0)$ on $\omega \cap \beta$ and zero elsewhere. We are led to the conclusion

that $\mathfrak{G}_{\Delta_1}(\varphi_s(z), q_0)$ tends to zero when z tends to ζ . This contradicts, however, the fact that $s = \lim \varphi(z_k)$, $z_k(\epsilon\delta) \rightarrow \zeta$. Hence for each $q \in \Delta_1$, for which the greatest harmonic minorant of $z \rightarrow \mathfrak{G}_{\Delta_1}(\varphi_s(z), q)$ vanishes, ζ is an accumulation point of $\varphi_s^{-1}(\{q\})$. Hence $\Delta_1 - R$ has zero capacity and, as in case 1, $s \in \Omega \cap (\text{int } \bar{R})$. The proof of the theorem is now readily completed.

It is possible to make a stronger statement concerning $O_1 - R$ than that $O_1 - R \subset C$. In fact, on using results of [4] we obtain at once the following proposition:

For each point $q \in O_1 - R$, ζ adheres to the set of points on $\{|z| = 1\}$ which are endpoints of asymptotic paths on which φ tends to q .

BIBLIOGRAPHY

- [1] CALDERÓN, A. P., GONZÁLEZ-DOMÍNGUEZ, A., ZYGMUND, A., Nota sobre los valores limites de funciones analíticas, *Revista Union Mat. Argentina* **14**, 16–19 (1949).
- [2] CARATHÉODORY, C., Zum Schwarzschen Spiegelungsprinzip, *Comment. Math. Helv.* **19**, 263–278 (1946).
- [3] COLLINGWOOD, E. F. & CARTWRIGHT, M. L., Boundary theorems for a function meromorphic in the unit circle, *Acta Math.* **87**, 83–146 (1952).
- [4] HEINS, M., On the Lindelöf principle, *Ann. of Math.* **61**, 440–473 (1955).
- [5] HEINS, M., Lindelöfian maps, *Ann. of Math.* **62**, 418–446 (1955).
- [6] HEINS, M., *Functions of bounded characteristic and Lindelöfian maps*, International Congress of Mathematicians, Edinburgh, 1958.
- [7] PARREAU, M., Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann, *Ann. Inst. Fourier Grenoble* (1951), 103–197.
- [8] RADÓ, T., *Subharmonic functions*, Berlin, 1937.

University of Illinois
Urbana, Illinois