A Circle Theorem Associated with the Poisson Differential Equation

E. E. JONES

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1. Introduction. Boundary value problems exist where it is required to determine solutions of the Laplace differential equation in the interior and exterior regions of a given closed surface or contour. The solutions possess prescribed singularities in these regions and, together with certain derivatives, satisfy conditional equations on the boundary of separation of the two regions. Such functions can be determined in a general and direct manner. For three-dimensional problems associated with the interior and exterior of a spherical region there exists a sphere theorem [1], which is the culmination of less general theorems due to various authors, providing solutions in terms of prescribed harmonic functions. For two-dimensional problems of a similar nature associated with a circular boundary, a circle theorem analogue [2] provides solutions in terms of complex potential functions having prescribed singularities.

It is proposed here to deduce a circle theorem, which determines solutions to boundary value problems of a similar type to the above, but having controlling equations of the Poisson type.

If the region external to the circle C of radius a , center the origin of coordinates, is referred to as S_e , and the region internal to C as S_i , then it is required to determine functions $\phi_e(x, y)$ and $\phi_i(x, y)$ which satisfy respectively the Poisson type differential equations

$$(1) \quad \nabla^2 \phi_e = F_e(x, y) \quad \text{in } S_e,$$

$$(2) \quad \nabla^2 \phi_i = F_i(x, y) \quad \text{in } S_i,$$

where $F_e(x, y)$ and $F_i(x, y)$ are given continuous functions of x and y in S_e and S_i , respectively, subject to the conditions

$$(3) \quad \phi_e = \phi_i,$$

$$(4) \quad \alpha_e \frac{\partial \phi_e}{\partial r} + \beta_e \phi_e = \alpha_i \frac{\partial \phi_i}{\partial r} + \beta_i \phi_i + \psi,$$

on C , where α_* , β_* , α_i , β_i are given constants and $\psi(x, y)$ is a given continuous function of x and y on C .

The circle theorem provides solutions in terms of particular integrals of the differential equations, and these solutions are shown to be unique.

2. Preliminaries. Any two particular integrals of each differential equation differ by harmonic functions, and thus the complete solutions can be written as $\phi_* = \phi_{0*}^* + \Phi_*^*$ and $\phi_i = \phi_{0i} + \Phi_i$. Here the functions $\phi_{0*}^*(x, y)$ and $\phi_{0i}(x, y)$ are particular integrals of the differential equations (1) and (2) respectively, and, together with their first partial derivatives, are assumed single-valued and continuous in S_* and S_i respectively, except at isolated points in the regions. It is assumed that the singularities of the solutions of the differential equations are reflected wholly in the forms of the expressions for the particular integrals ϕ_{0*} and ϕ_{0i} . The functions $\Phi_*^*(x, y)$ and $\Phi_i(x, y)$ are assumed to be single-valued continuous harmonic functions having continuous partial derivatives up to the second order in S_* and S_i respectively. It is thus possible to write $\Phi_*^* = \operatorname{re} W_*^*(z)$ and $\Phi_i = \operatorname{re} W_i(z)$, with $z = x + iy$. In general, since the region S_* is doubly-connected, the function conjugate to Φ_*^* in W_*^* is multi-valued; hence it follows [4, p. 671] that $W_*^*(z)$ can be expressed in the form

$$W_*^*(z) = A \log z + W_*(z),$$

where A is a real constant and $W_*(z)$ is a single-valued function, regular in the region S_* , including infinity. For convenience it is possible to introduce a function $\phi_{0*} = \phi_{0*}^* + A \log r$, where $r = |z|$, and then $\phi_* = \phi_{0*} + \Phi_*$, where $\Phi_* = \operatorname{re} W_*(z)$. From the definition of Φ_i it is noticed that $W_i(z)$ is also a single-valued function, regular in the region S_i .

Two classes of problems present themselves. In the first the particular integrals $\phi_{0*}^*(x, y)$ and $\phi_{0i}(x, y)$ are prescribed functions, and it is required to determine the harmonic functions $\Phi_*(x, y)$ and $\Phi_i(x, y)$ as the real parts of the regular functions $W_*(z)$ and $W_i(z)$ respectively. In the second type of problem it is required firstly to determine the particular integrals $\phi_{0*}^*(x, y)$ and $\phi_{0i}(x, y)$ which have prescribed singularities in the regions in which they are defined. These particular integrals can usually be determined by formal integration of the differential equations in the form

$$(5) \quad 4 \frac{\partial^2 \phi_0}{\partial z \partial \bar{z}} = F(x[z, \bar{z}], y[z, \bar{z}]),$$

where $\bar{z} = x - iy$, with a suitable choice of the arbitrary harmonic functions which occur in the general solutions. It is then required to determine Φ_* and Φ_i as in the first class of problems. The constant A which appears in the expression for ϕ_{0*}^* is uniquely determined either by prescribing the form of the solution at infinite $|z|$ or, when $\beta_i = \beta_*$ in (4), by solving certain constraint relations (*cf.* (12) and (13) below).

In general it is assumed that ϕ_* (or ϕ_*), together with its first derivative in the radial direction, is continuous from S_* (or S_i) on to C . In particular, the functions and the radial derivatives tend to finite single-valued limiting values as the point at which these functions are measured approaches any point on the boundary curve C . These values will subsequently be referred to as the boundary values of the functions.

On C the boundary values of the functions ϕ_{0*} , $\partial\phi_{0*}/\partial r$, and ϕ_{0i} , $\partial\phi_{0i}/\partial r$ are real functions of θ the arc coordinate. Hence by use of the relation $\zeta = ae^{i\theta}$ where ζ is the boundary value of z on C , it is possible to write

$$(6) \quad (\phi_{0*} - \phi_{0i})_C = g(\zeta),$$

$$(7) \quad \left(\alpha_* \frac{\partial\phi_{0*}}{\partial r} + \beta_* \phi_{0*} - \alpha_i \frac{\partial\phi_{0i}}{\partial r} - \beta_i \phi_{0i} - \psi \right)_C = f(\zeta).$$

It will not only be assumed that the real functions $g(\zeta)$ and $f(\zeta)$ are single-valued and continuous, but also that they have continuous derivatives with respect to ζ satisfying the Hölder (or Lipschitz) condition on C [3, Ch. 1]. The functions and their derivatives with respect to ζ are thus Riemann integrable with respect to the arc coordinate θ of C .

It is now required to determine functions of z which are regular in S_i , viz. $g_i(z)$ and $f_i(z)$, which have $g(\zeta)$ and $f(\zeta)$ respectively as real parts on C . The solutions of these Dirichlet type boundary value problems [4, p. 294] are given to within additive arbitrary imaginary constants by

$$(8) \quad g_i(z) = \frac{1}{\pi i} \int_C \frac{g(\zeta) d\zeta}{\zeta - z} - \alpha,$$

where

$$(9) \quad 2\alpha = \frac{1}{\pi i} \int_C \frac{g(\zeta) d\zeta}{\zeta} = \frac{1}{\pi} \int_0^{2\pi} g(ae^{i\theta}) d\theta,$$

with a similar type of expression for $f_i(z)$, replacing $g(\zeta)$ by $f(\zeta)$ and α by γ (say) in (8) and (9). A similar type of analysis was used by LEWIS [5] to obtain solutions to an external-type Dirichlet boundary value problem in terms of boundary values of given harmonic functions.

In the sequel it will be necessary also to determine functions $g_*(z)$ and $f_*(z)$ which are regular in S_* , including infinity, with real parts $g(\zeta)$ and $f(\zeta)$ respectively on C . Consideration of the function $\bar{g}_*(a^2/z) - g_i(z)$, which is regular in S_i and has zero real part on C , leads to the result that to within an additive arbitrary imaginary constant

$$(10) \quad \bar{g}_*\left(\frac{a^2}{z}\right) = g_i(z),$$

and thus from (8) and (9) it is seen that $g_*(\infty) = \alpha$. Similarly to within an additive arbitrary imaginary constant

$$(11) \quad \bar{f}_s\left(\frac{a^2}{z}\right) = f_i(z),$$

and $f_s(\infty) = \gamma$ [4, p. 290].

It is convenient here to introduce a lemma to the main theorem, a result which imposes a conditional restraint on $g(\zeta)$ and $f(\zeta)$.

Lemma. *If $\beta_i = \beta_s$, then to within an additive arbitrary imaginary constant*

$$(12) \quad (a) \quad \beta_i g_i(0) = f_i(0),$$

$$(13) \quad (b) \quad \beta_i g_s(\infty) = f_s(\infty).$$

Since $\Phi_i = \operatorname{re} W_i(z)$, where $W_i(z)$ is regular in S_i , and $\Phi_s = \operatorname{re} W_s(z)$, where $W_s(z)$ is regular in S_s including infinity, then

$$(14) \quad \int_0^{2\pi} \left(\frac{\partial \Phi_i}{\partial r}\right)_c d\theta = \int_0^{2\pi} \left(\frac{\partial \Phi_s}{\partial r}\right)_c d\theta = 0.$$

Hence from (3), (4), (6), and (7), with $\beta_i = \beta_s$,

$$(15) \quad \beta_i \int_0^{2\pi} g(ae^{i\theta}) d\theta = \int_0^{2\pi} f(ae^{i\theta}) d\theta.$$

But from (8) and (9) to within additive arbitrary imaginary constants

$$g_i(0) = \alpha = \frac{1}{2\pi} \int_0^{2\pi} g(ae^{i\theta}) d\theta,$$

and similarly

$$f_i(0) = \gamma = \frac{1}{2\pi} \int_0^{2\pi} f(ae^{i\theta}) d\theta.$$

Hence (15) reduces to (12), and the result (13) readily follows by application of (10) and (11).

3. The Circle Theorem. The solutions of the differential equations (1) and (2), subject to the boundary conditions (3) and (4), are derived from the functions $\Phi_s = \operatorname{re} W_s(z)$, $\Phi_i = \operatorname{re} W_i(z)$ such that, to within additive arbitrary imaginary constants,

$$(16) \quad (\alpha_i + \alpha_s)W_s(z) = -\alpha_i g_s(z) + \int_1^\infty \{af_s(zt) - \nu g_s(zt)\} t^{-\mu-1} dt$$

and

$$(17) \quad (\alpha_i + \alpha_s)W_i(z) = \alpha_s g_i(z) + \int_0^1 \{af_i(zt) - \nu g_i(zt)\} t^{\mu-1} dt,$$

where

$$(18) \quad \nu = a(\beta_s \alpha_s + \beta_s \alpha_i) / (\alpha_i + \alpha_s),$$

$$(19) \quad \mu = a(\beta_i - \beta_s) / (\alpha_i + \alpha_s).$$

It is necessary that $\mu \geq 0$, and when $\mu = 0$ the solutions must be considered in conjunction with (12) and (13).

It is noted here that both functions $W_*(z)$ and $W_i(z)$ can be expressed entirely in terms of $f_*(z)$ and $g_*(z)$, or entirely in terms of $f_i(z)$ and $g_i(z)$, as readily follows from the relations (10) and (11).

Verification. The Cauchy integral in (8) is a regular function of z in S_i , and thus so is $g_i(z)$. Consideration of this integral shows that

$$g_i(z) = g_i(0) + O(|z|)$$

for $|z| \leq R$, where R is any fixed positive constant less than a . Similar results hold for $f_i(z)$.

On expressing (17) in the form

$$(21) \quad (\alpha_i + \alpha_*)W_i(z) = \alpha_*g_i(z) + I_i(z),$$

where

$$(22) \quad I_i(z) = \int_0^1 \{af_i(zt) - \nu g_i(zt)\} t^{\mu-1} dt,$$

it is noticed that the integrand of $I_i(z)$ is a continuous function of the real variable t in the range $0 < t \leq 1$ and of the complex variable z for all points in S_i . The integrand is also a regular function of z in S_i for every value of t in the range of integration. By use of (20) and the analogous expression for $f_i(z)$ and (12), it can be shown that $I_i(z)$ is uniformly convergent for $\mu \geq 0$. Hence $I_i(z)$ and thus, from (21), the functions $W_i(z)$ are regular functions of z in S_i ; moreover, $W'_i(z)$ is determined in part by differentiating the integral $I_i(z)$ under the integral sign [6, p. 100].

By use of (10) and (11) it follows that $g_*(z)$ and $f_*(z)$ are regular functions of z in S_* . Since (16) can be written as

$$(23) \quad (\alpha_i + \alpha_*)W_*(z) = -\alpha_i\bar{g}_i(a^2/z) + \bar{I}_i(a^2/z),$$

then $W_*(z)$ is a regular function of z in S_* , and from the development of $\bar{g}_i(a^2/z)$ and $\bar{I}_i(a^2/z)$ for $|z| > a$ it is seen that $W_*(z)$ is also regular at infinity. An application of Cauchy's formula for the derivative of a regular function leads to the result that $W'_*(z) = O(1/|z|^2)$ for large $|z|$, which is consistent with the assumption assumed in deducing (14).

To complete the verification it is necessary to show that the limiting boundary values of $W_i(z)$ and $W_*(z)$ and their radial derivatives satisfy the conditional equations

$$(24) \quad \operatorname{re} \{W_*(z) - W_i(z)\}_c = -g(\zeta),$$

$$(25) \quad \operatorname{re} \left\{ \alpha_* \frac{\partial W_*}{\partial r} + \beta_* W_* - \alpha_i \frac{\partial W_i}{\partial r} - \beta_i W_i \right\}_c = -f(\zeta).$$

It is known [3, p. 53] that if $g(\zeta)$ satisfies the Hölder condition on C , then so does $g_i(z)$ in $S_i + C$. Then $g_i(z)$ is continuous on to the boundary C with

a limiting value $g_i(\zeta)$ given by the Plemelj formula associated with the Cauchy integral, the real part of which is $g(\zeta)$ [4, p. 296]. A similar statement holds for $f_i(z)$. Due to the restrictions on $g_i(z)$ and $f_i(z)$ it can be shown that $I_i(z)$ of (22) and thus $W_i(z)$ are continuous on to C from S_i . The added restriction that $g'_i(\zeta)$ and $f'_i(\zeta)$ satisfy the Hölder condition on C leads to the result that $g'_i(z)$ and $f'_i(z)$ are continuous on to C from S_i [4, p. 265]. On integration of $I'_i(z)$ by parts, it is seen for $\mu \geq 0$ that

$$(26) \quad (\alpha_i + \alpha_e) \frac{\partial W_i}{\partial r} = \frac{\alpha_e z}{a} g'_i(z) + \frac{1}{a} \{af_i(z) - \nu g_i(z)\} - \frac{\mu}{a} I_i(z).$$

Hence $\partial W_i/\partial r$ is continuous on to C from S_i . By use of (10) and (11) it follows that $W_e(z)$ and $\partial W_e/\partial r$ are continuous on to C from S_e .

Having established the continuity of the functions concerned on to C , then from (21) and (23)

$$\operatorname{re} \{\bar{W}_e(\bar{z}) - W_i(z)\}_C = -\operatorname{re} \{g_i(z)\}_C,$$

and thus (24) is verified. Also, from the expression for $\partial W_i/\partial r$ in (26) and the expression for $\partial \bar{W}_e/\partial r$ deduced from (23), after some reduction there results

$$\operatorname{re} \left\{ \alpha_e \frac{\partial \bar{W}_e}{\partial r} + \beta_e \bar{W}_e - \alpha_i \frac{\partial W_i}{\partial r} - \beta_i W_i \right\}_C = -\operatorname{re} \{f_i(z)\}_C.$$

Hence (25) is verified.

It is noticed that if $\beta_i = \beta_e$ with $\alpha_e \neq 0$, then, on using the conditional relation (12) in conjunction with (6) and (7) and substituting $\phi_{0e} = \phi_{0e}^* + A \log r$, it readily follows that

$$A = \frac{a}{2\pi\alpha_e} \int_0^{2\pi} \left(\alpha_i \frac{\partial \phi_{0i}}{\partial r} - \alpha_e \frac{\partial \phi_{0e}^*}{\partial r} + \psi \right) d\theta.$$

This equation determines A uniquely in terms of the boundary values on C of prescribed functions. For all other values of the constants α_e , β_e , α_i , β_i , the value of A is uniquely determined by prescribing the form of the solution $\phi_e(x, y)$ at infinity.

4. The Uniqueness Theorem. In order to investigate the uniqueness of the solutions $\phi_e(x, y)$ and $\phi_i(x, y)$ it will be assumed that two such sets of solutions exist, *viz.* $\phi_e(x, y)$ and $\phi'_e(x, y)$ in S_e and $\phi_i(x, y)$ and $\phi'_i(x, y)$ in S_i . The difference functions $\psi_e = \phi_e - \phi'_e$, $\psi_i = \phi_i - \phi'_i$ are introduced, where both ψ_e and ψ_i are harmonic functions when $F_e(x, y)$ and $F_i(x, y)$ of (1) and (2) are prescribed functions.

Since ϕ'_e and ϕ'_i have the same singularities as have ϕ_e and ϕ_i respectively, then $\psi_e = \operatorname{re} \Omega_e(z)$, $\psi_i = \operatorname{re} \Omega_i(z)$ are such that $\Omega_e(z)$ is a regular function of z in S_e including infinity and $\Omega_i(z)$ is a regular function of z in S_i . From the definitions of ψ_e and ψ_i the continuity of $\Omega_e(z)$, $\Omega_i(z)$ and their derivatives $\Omega'_e(z)$, $\Omega'_i(z)$ on to C is established. The boundary conditions satisfied by the limiting

values of these functions are from (3) and (4) given by

$$(27) \quad \Omega_i(\zeta) + \bar{\Omega}_i\left(\frac{a^2}{\zeta}\right) = \Omega_e(\zeta) + \bar{\Omega}_e\left(\frac{a^2}{\zeta}\right),$$

$$(28) \quad \alpha_i \left\{ \frac{\zeta}{a} \Omega'_i(\zeta) + \frac{a}{\zeta} \bar{\Omega}'_i\left(\frac{a^2}{\zeta}\right) \right\} + \beta_i \left\{ \Omega_i(\zeta) + \bar{\Omega}_i\left(\frac{a^2}{\zeta}\right) \right\} \\ = \alpha_e \left\{ \frac{\zeta}{a} \Omega'_e(\zeta) + \frac{a}{\zeta} \bar{\Omega}'_e\left(\frac{a^2}{\zeta}\right) \right\} + \beta_e \left\{ \Omega_e(\zeta) + \bar{\Omega}_e\left(\frac{a^2}{\zeta}\right) \right\}$$

if $\psi(x, y)$ is a prescribed function, substituting $\zeta\bar{\zeta} = a^2$ on C .

From (27), on differentiation with respect to the arc coordinate θ on C , it is seen that

$$(29) \quad \frac{\zeta}{a} \Omega'_i(\zeta) - \frac{a}{\zeta} \bar{\Omega}'_i\left(\frac{a^2}{\zeta}\right) = \frac{\zeta}{a} \Omega'_e(\zeta) - \frac{a}{\zeta} \bar{\Omega}'_e\left(\frac{a^2}{\zeta}\right).$$

Assuming in the first instance that $\alpha_i \neq 0$, it is possible to eliminate $\bar{\Omega}'_i(a^2/\zeta)$ from (28) and (29), and with the aid of (27) there results

$$(30) \quad 2\alpha_i \frac{\zeta}{a} \Omega'_i(\zeta) - (\alpha_e - \alpha_i) \frac{a}{\zeta} \bar{\Omega}'_e\left(\frac{a^2}{\zeta}\right) - (\beta_e - \beta_i) \bar{\Omega}_e\left(\frac{a^2}{\zeta}\right) \\ = (\alpha_e + \alpha_i) \frac{\zeta}{a} \Omega'_e(\zeta) + (\beta_e - \beta_i) \Omega_e(\zeta).$$

The function defined by

$$(31) \quad F_i(z) = 2\alpha_i \frac{z}{a} \Omega'_i(z) - (\alpha_e - \alpha_i) \frac{a}{z} \bar{\Omega}'_e\left(\frac{a^2}{z}\right) - (\beta_e - \beta_i) \bar{\Omega}_e\left(\frac{a^2}{z}\right)$$

is regular in the region S_i and continuous on to C . Similarly the function defined by

$$(32) \quad F_e(z) = (\alpha_e + \alpha_i) \frac{z}{a} \Omega'_e(z) + (\beta_e - \beta_i) \Omega_e(z)$$

is regular in S_e including infinity and continuous on to C . From (30) it is seen that the limiting values of $F_i(z)$ and $F_e(z)$ on C are equal; then $F_i(z)$ of (31) and $F_e(z)$ of (32) continue each other analytically [4, pp. 129, 429]. Thus the function defined by

$$F(z) = \begin{cases} F_i(z) & \text{in } S_i, \\ F_i(\zeta) = F_e(\zeta) & \text{on } C, \\ F_e(z) & \text{in } S_e, \end{cases}$$

is regular in the entire z -plane. Since $F_e(z)$ is bounded at infinity, then by Liouville's theorem the function $F(z) = B$, a constant, in the entire z -plane.

On integration of (32) it can readily be deduced that in general

$$\begin{aligned}\Omega_e(z) &= \frac{B}{\beta_e - \beta_i} + Dz^\mu \quad \text{when } \mu \neq 0, \\ &= \frac{aB}{\alpha_e + \alpha_i} \log z + E \quad \text{when } \mu = 0,\end{aligned}$$

where μ is defined as in (19) and D, E are arbitrary constants. On substitution for $\Omega_e(z)$ into (31) it can be shown that

$$\begin{aligned}\Omega_i(z) &= \frac{a(B + \bar{B})}{2\alpha_i} \log z + \bar{D}a^{2\mu}z^{-\mu} + F \quad \text{when } \mu \neq 0, \\ &= \frac{a}{2\alpha_i} \left\{ B + \frac{\alpha_e - \alpha_i}{\alpha_e + \alpha_i} \bar{B} \right\} \log z + G \quad \text{when } \mu = 0,\end{aligned}$$

where F, G are arbitrary constants.

Since the existence of the solutions given in Section 3 depended on the inequality $\mu \geq 0$, this condition will be adhered to in the derivation of uniqueness. If $\mu > 0$, then $D = 0$ and B is purely imaginary in order that $\Omega_e(z)$ and $\Omega_i(z)$ are regular in S_e and S_i respectively. From (27) and (28) it follows that F is also an imaginary constant. If $\mu = 0$, then $B = 0$ in order that the functions $\Omega_e(z)$ and $\Omega_i(z)$ are regular, and from (27) and (28) it can be proved that the real parts of the constants E and G are equal.

It has thus been proved that when $\mu > 0$ the solutions of the differential equations (1) and (2) are uniquely determined, and when $\mu = 0$ they are determined to within equal arbitrary constants (*cf.* [7]).

If $\alpha_i = 0$ and $\alpha_e \neq 0$, then $\bar{\Omega}'_e(z)$ is eliminated from (28) and (29) and the theory is developed as above with the same conclusions. Again the method establishes the uniqueness of solutions for the internal or external type boundary value problem, *i.e.* when (4) is the sole boundary condition, including the cases where certain of the constants and/or the function ψ are zero.

5. Further Considerations. (i) A certain amount of arbitrariness is allowed in the choice of the particular integrals ϕ_{0e} and ϕ_{0i} . If ϕ_{0e} is replaced by $\phi_{0e} + \text{re } G_e(z)$ and ϕ_{0i} by $\phi_{0i} + \text{re } G_i(z)$, where $G_e(z)$ is an arbitrary regular function of z in S_e including infinity and $G_i(z)$ is an arbitrary regular function of z in S_i , then the evaluation of $g_i(z)$ and $f_i(z)$ from equations of the type (8), and subsequently $W_i(z)$ from (17), leads to the result that ϕ_i is unaltered when $\mu > 0$ and changes by a constant when $\mu = 0$. A similar result holds for ϕ_e , and, further, the constant change when $\mu = 0$ is equal to that occurring in ϕ_i . The results are consistent with the uniqueness theorem of Section 4.

(ii) In particular, when $F_e(x, y) = F_i(x, y) = 0$ in (1) and (2), the integrals ϕ_{0e} and ϕ_{0i} are harmonic functions. In this case it is possible to express ϕ_{0e} and ϕ_{0i} as the real parts of complex regular functions having prescribed singularities. Thus $\phi_{0e} = \text{re } H_e(z)$, $\phi_{0i} = \text{re } H_i(z)$, where $H_e(z)$ is regular in S_e having singu-

larities only in S_* , and *vice versa* for $H_i(z)$. Application of the theorem of Section 3 leads to results which are identical with those deduced by POWER & JACKSON [2] (allowing for a misprint in the sign of β in the index of t in functions $H(t)$ and $G(t)$ of that paper).

(iii) When the boundary functions $g(\zeta)$ and $f(\zeta)$ of (6) and (7) respectively are rational functions of ζ , then the Cauchy integral in (8) for $g_i(z)$ and that in $f_i(z)$, in conjunction with (10) and (11), are readily evaluated by the theory of residues, yielding functions $g_i(z)$, $f_i(z)$ regular in S_i and $g_*(z)$, $f_*(z)$ regular in S_* including infinity.

These functions can, however, be determined by a simpler process if $g(\zeta)$ and $f(\zeta)$ are separable into two sets of functions distinguished by the location of their poles. Thus, by the method of partial fractions or otherwise, it will be possible to write

$$(33) \quad 2g(\zeta) = g_*(\zeta) + g_i(\zeta),$$

where $g_*(z)$ has singularities only in S_i and $g_i(z)$ has singularities only in S_* . Since $g_*(\zeta)$ is a real function when $\zeta\bar{\zeta} = a^2$, then also

$$(34) \quad 2g(\zeta) = \bar{g}_*(a^2/\zeta) + \bar{g}_i(a^2/\zeta).$$

It follows that the function $G(z)$ defined by

$$(35) \quad G(z) = \begin{cases} g_*(z) - \bar{g}_i(a^2/z) & \text{in } S_* \\ \bar{g}_*(a^2/z) - g_i(z) & \text{in } S_i, \end{cases}$$

and suitably chosen on C , is a sectionally regular function continued analytically across C and is bounded at infinity. Hence it is a constant in the entire z -plane—in fact a real constant, as can be seen from (35). Thus, to within an arbitrary real constant, $\bar{g}_*(a^2/z) = g_i(z)$, and consequently $g(\zeta)$ is the real part of $g_*(z)$ on C , where $g_*(z)$ is a regular function in S_* including infinity, or the real part of $g_i(z)$, a function regular in S_i . Hence $g_*(z)$ and $g_i(z)$ can be interpreted to within real constants as the functions required in (16) and (17). Similarly for $f_*(z)$ and $f_i(z)$.

In some cases it is possible to apply the above results when $g(\zeta)$ and $f(\zeta)$ contain logarithmic singularities. Power series expansions for $g_i(z)$ and $f_i(z)$ can also be used to facilitate the evaluation of $W_*(z)$ and $W_i(z)$ if required.

(iv) There are several elementary applications of the circle theorem. (a) For external-internal type problems with $\beta_* = \beta_i = 0$. Examples are: electrostatic dielectric problems with prescribed volume charge, the magnetostatic analogue with prescribed permanent magnetism, current flow magnetostatic problems with prescribed current density, and gravitational problems. (b) For internal type problems involving boundary condition (4) only with ϕ_* redundant and $\alpha_* = \beta_* = 0$. These include heat conduction problems with prescribed distribution of heat generation [8, p. 9]. External type problems also exist. (c) Internal type problems as in (b) with α_i also zero and $\beta_i = 1$. In this case $W_i(z) = f_i(z)$,

and the application when $f(\zeta)$ is a rational function is particularly simple. Examples are: torsion problems in terms of the stress function [4, p. 573] and flexure problems [4, p. 590]. External type problems also exist, and here correspondingly $W_*(z) = -f_*(z)$, examples being electrostatic problems involving conductors with prescribed volume charge in the surrounding dielectric medium and perfect fluid flow problems with constant fluid vorticity in conjunction with a cylinder moving in a general manner [9]. These results can be considerably extended by the use of conformal mapping.

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The University of Nottingham