*Non-uniqueness in Cauchy's Problem for Differential Equations of Elliptic Type*¹

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It has been known since 1901 [2] that Cauchy's problem for linear partial differential equations with analytic coefficients has at most one (smooth) solution, provided that the data are given on a non-characteristic manifold. This theorem was extended [1] to the case of Hölder continuously differentiable coefficients under the additional assumption that the characteristics are non-multiple. The example in [3] shows that multiplicity of real characteristics may cause non-uniqueness even in the case of coefficients in C^∞ .

The problem arises whether the multiplicity of essentially complex characteristics can be admitted in the uniqueness theorem. L. HÖRMANDER proved a uniqueness theorem for equations having constant leading coefficients, with at most double multiplicity of essentially complex characteristics.²

In this paper we show that the answer to this problem is in general negative for coefficients in C^m (m finite). The precise statement is as follows.

Theorem. *There exists a linear system of partial differential equations of elliptic type for which the Cauchy problem possesses two distinct solutions of class C^∞ . The leading coefficients are constant and the remaining ones are of class C^m ($0 \leq m < \infty$). The (imaginary) characteristics are $(m + 3)$ -fold. The system consists of $2(m + 3)$ complex equations in two independent (real) variables; it corresponds to $4(m + 3)$ real equations defined on the whole plane.*

(The coefficients and solutions are analytic in x).

The system is ($p = m + 3$)

$$\begin{aligned} u_t^j - iu_x^j + iu_x^{j+1} &= f^j(t, x)v^1 & (j = 1, \dots, p-1), \\ u_t^p - iu_x^p &= \lambda(t)u^1 + f^p(t, x)v^1 \\ v_t^j - 2iv_x^j + 2iv_x^{j+1} &= g^j(t, x)u^1 & (j = 1, \dots, p-1), \\ v_t^p - 2iv_x^p &= \mu(t)v^1 + g^p(t, x)u^1 \end{aligned}$$

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² For further bibliography about uniqueness in Cauchy's problem, see [1].

or in a more concise form

$$(1) \quad \begin{aligned} u_i + iLu_x &= h\lambda(t)u^1 + f(t, x)v^1, \\ v_i + 2iLv_x &= h\mu(t)v^1 + g(t, x)u^1, \end{aligned}$$

where $u = (u^1, \dots, u^p)$, $v = (v^1, \dots, v^p)$; L is a constant $p \times p$ matrix with elements $e_{j,i} = -1$ ($j = 1, \dots, p$), $e_{i,i+1} = 1$ ($j = 1, \dots, p-1$) and the other elements vanishing; h is a p vector $(0, \dots, 0, 1)$; f, g are vector functions in C^∞ ; $\lambda(t)\mu(t)$ are m -times (Hölder) continuously differentiable real functions.

Equations (1) with the initial conditions

$$u(0, x) = 0, \quad v(0, x) = 0$$

possess a non-vanishing solution in C^∞ (besides a trivial one). *This means that solutions of (1) do not possess the unique continuation property.*

P. COHEN has independently constructed an equation of the form $\Delta^3 u = Lu$ without the uniqueness property, where the operator L is of fifth order and possesses continuous complex coefficients (added in proof).

Remark 1. In this example the elementary divisors are $(m+3)$ -fold, but similar examples with all elementary divisors triple (and 3 $(m+1)$ -fold characteristics) can be constructed by an analogous method.

Remark 2. A similar system with non-constant leading coefficients (of class C^m) can be constructed in such a manner that for $t = 0$ the characteristics are $(m+1)$ -fold ($m \geq 0$), the elementary divisors are simple, and for $t \neq 0$ no multiplicity occurs.

Notations. In what follows we denote by K (with or without subscripts) positive constants, not necessarily always the same (even in the same formula).

We denote by $D^q w$ ($q \geq 0$) the sequence of derivatives of the function w up to order q .

Lemma. *Let*

$$(2) \quad \delta, a, b, d, \zeta, \sigma, \gamma$$

be constants such that $-1 \leq \delta \leq a < b < d$, $\zeta > 1$, $1 > \sigma > 0$, $\gamma > 0$. Let $A(t)$ be a function of class C^∞ for all t and such that $A(t) = 1$ for $t \leq a$, $A(t) = 0$ for $t \geq b$, and

$$(3) \quad A'(t) \leq 0.$$

Then the solution $w(t)$, $w = (w^1, \dots, w^p)$, of the system of ordinary differential equations

$$(4) \quad w' = \zeta Lw + hA(t)\zeta(1 - \sigma)^p w^1,$$

with the initial condition

$$(5) \quad w(\delta) = \gamma s, \quad \text{where } s = (1, 1 - \sigma, \dots, (1 - \sigma)^{p-1}),$$

has the following properties:

$$(6) \quad w(t) = \gamma s e^{-\sigma \zeta(t-\delta)} \quad \text{for } t \leq a,$$

$$(7) \quad |D^q w(t)| \leq K_a \zeta^q (|D^q A(t)|^q + 1) |w(t)|, \quad q = 1, 2, \dots, \\ \text{where } |w| = \max(|w^1|, \dots, |w^p|),$$

$$(8) \quad w^1(t) > w^j(t) > 0 \quad (j = 2, \dots, p),$$

$$(9) \quad w^1(t) \text{ is a decreasing function,}$$

$$(10) \quad w^1(t) \geq K \zeta^{-K} e^{-\zeta(t-d)} w^1(d) \quad \text{for } b \leq t \leq d,$$

$$(11) \quad w^1(t) \leq K \zeta^K e^{-\zeta(t-d)} w^1(d) \quad \text{for } d \leq t \leq 0,$$

where constants K are independent of constants (2).

Formula (6) can be verified directly.

Formula (7) for $q = 1$ results from (4); for $q > 1$ it can be obtained by differentiating (4).

For the proof of the other properties we shall apply the theorem on ordinary differential inequalities. The right side of the j^{th} equation (4) is a non-decreasing function of the variables $w^1, \dots, w^{j-1}, w^{j+1}, \dots, w^p$, and therefore this theorem can be applied in the positive direction of the t -axis.

It follows from (6) that

$$(12) \quad w(t) > 0,$$

where the inequality sign applies to each component, for $t \leq a$. Comparing $w(t)$ with the trivial solution $w = 0$ of (4) we deduce that (12) is satisfied for all t . By differentiating (4) we obtain by (3) and (12) that $W = w'$ satisfies the inequality

$$W' \leq \zeta L W + h A(t) \zeta (1 - \sigma)^p W^1$$

and by (6)

$$(13) \quad W(t) < 0$$

for $t \leq a$. Comparing $W(t)$ with the trivial solution of the differential equation (4) we obtain (13) for all t . From (4) and (13) it follows that

$$w^j > w^{j+1} \quad (j = 1, \dots, p-1)$$

and by (12) we have (8). Property (9) results from (13).

The functions

$$W^i(t) = w^1(d) p! e^{-\zeta(t-d)} \frac{[\zeta(t-d+1)]^{p-i}}{(p-i)!}$$

satisfy (4) for $t \geq d$ and we have $W(d) > w(d)$. Hence $W(t) > w(t)$ for $t \geq d$ and (11) follows.

The functions

$$(14) \quad W^i(t) = w^i(d)e^{-\zeta(t-d)}2^{-p}\zeta^{-p} \frac{[\zeta(t-d+2)]^{p-i}}{(p-i)!}$$

satisfy (4) for $t \geq b$. Suppose $W(\tau) > w(\tau)$ for certain τ ($b < \tau < d$). Then by the theory of differential inequalities we have $W(d) > w(d)$ which contradicts (14). Therefore

$$\min [W^1(t), \dots, W^p(t)] \leq w^1(t)$$

for $b < t < d$ and (10) results from (14). The proof of the lemma is thus complete.

Now let

$$(15) \quad a_n, b_n, c_n, d_n, e_n, \sigma_n, \zeta_n$$

be sequences such that $-1 \leq a_n \nearrow 0$, $a_n < b_n < c_n < d_n < e_n < a_{n+1}$, $1 > \sigma_n > 0$, $\zeta_n > 1$.

We will specify the sequences later. Let $J(y)$ be a function of class C^∞ such that $J(y) = 1$ for $y \leq 0$, $J(y) = 0$ for $y \geq 1$, $J'(y) \leq 0$ for all y . We define

$$(16) \quad \begin{aligned} A_n(t) &= J\left(\frac{t-a_n}{b_n-a_n}\right), \\ B_n(t) &= J\left(\frac{t-b_n}{c_n-b_n}\right), \\ E_n(t) &= J\left(\frac{t-e_n}{a_{n+1}-e_n}\right). \end{aligned}$$

Now let us define a sequence of vector functions $w_n(t)$. Here $w_1(t)$ is the solution of (4), (5) with $A = A_1(t)$, $\zeta = \zeta_1$, $\sigma = \sigma_1$, $\delta = -1$, $\gamma = 1$. The function $w_{n+1}(t)$ ($n \geq 1$) is the solution of (4), (5) with $A = A_{n+1}(t)$, $\zeta = \zeta_{n+1}$, $\sigma = \sigma_{n+1}$, $\delta = d_n$,

$$(17) \quad \gamma = w_n^1(d_n).$$

From (6) ($w = w_{n+1}$, $\delta = d_n$), (10) ($w = w_n$), and (17) it follows that

$$(18) \quad w_{n+1}^1(t)/w_n^1(t) \leq K \zeta_n^K e^{\zeta_n(t-d_n)} \quad \text{for } b_n \leq t \leq d_n,$$

where

$$(19) \quad z_n = \zeta_n - \sigma_{n+1}\zeta_{n+1}.$$

From (6), (11), and (17) it follows that

$$(20) \quad w_n^1(t)/w_{n+1}^1(t) \leq K \zeta_n^K e^{-z_n(t-d_n)} \quad \text{for } d_n \leq t \leq a_{n+1}.$$

(Constants K in (18), (20) are independent of n .)

Now we define the solution and the coefficients of (1) for $t \leq a_2$ and in strips $a_n \leq t \leq a_{n+1}$ ($n = 2, 3, \dots$):

$$(21) \quad \begin{aligned} u &= E_n(t)w_n(t) \exp \{i\zeta_n x\}, \\ v &= [1 - B_n(t)]w_{n+1}(t) \exp \{\tfrac{1}{2}i\zeta_{n+1}x\}, \end{aligned}$$

$$\begin{aligned}
(22) \quad & \lambda(t) = A_n(t)(1 - \sigma_n)^p \zeta_n, \\
& \mu(t) = [1 - A_n(t)](1 - \sigma_{n+1})^p \zeta_{n+1}, \\
(23) \quad & f(t, x) = E'_n(t) \exp \{i(\zeta_n - \tfrac{1}{2}\zeta_{n+1})x\} w_n(t)/w_{n+1}^1(t), \\
& g(t, x) = -B'_n(t) \exp \{i(\tfrac{1}{2}\zeta_{n+1} - \zeta_n)x\} w_{n+1}(t)/w_n^1(t)
\end{aligned}$$

for $-\infty < t \leq a_2$ ($n = 1$) and $a_n \leq t \leq a_{n+1}$ (n odd).

For n even the roles of u and v are interchanged:

$$\begin{aligned}
(24) \quad & v = E_n(t)w_n(t) \exp \{\tfrac{1}{2}i\zeta_n x\}, \\
& u = [1 - B_n(t)]w_{n+1}(t) \exp \{i\zeta_{n+1}x\}, \\
(25) \quad & \mu(t) = A_n(t)(1 - \sigma_n)^p \zeta_n, \\
& \lambda(t) = [1 - A_n(t)](1 - \sigma_{n+1})^p \zeta_{n+1}, \\
(26) \quad & g(t, x) = E'_n(t) \exp \{i(\tfrac{1}{2}\zeta_n - \zeta_{n+1})x\} w_n(t)/w_{n+1}^1(t), \\
& f(t, x) = -B'_n(t) \exp \{i(\zeta_{n+1} - \tfrac{1}{2}\zeta_n)x\} w_{n+1}(t)/w_n^1(t)
\end{aligned}$$

for $a_n \leq t \leq a_{n+1}$ (n even).

Now let

$$(27) \quad u(t, x) = v(t, x) = f(t, x) = g(t, x) = \lambda(t) = \mu(t) = 0$$

for $t \geq 0$. Formulas (21)–(27) define u, v, f, g, λ, μ on the whole plane. It is not difficult to see that they are of class C^∞ for $t \neq 0$. The smoothness of these functions at $t = 0$ depends on their behavior for $t \nearrow 0$. It is easy to verify that system (1) is satisfied (it is evident that for our functions (n odd) $iLu_x = -\zeta_n Lu$, $2iLv_x = -\zeta_{n+1}Lv$).

Now we shall prove that u, v, f, g, λ, μ are smooth at $t = 0$, provided the sequences (15) have the following properties:

$$(28) \quad [\max(\zeta_n, \zeta_{n+1})]^K \exp \{-z_n r_n\} [\min(r_n, r_{n+1})]^{-K} \rightarrow 0$$

when $n \rightarrow \infty$, for each (fixed) K , where

$$r_n = \min(b_n - a_n, c_n - b_n, d_n - c_n, e_n - d_n, a_{n+1} - e_n)$$

and z_n is defined by (19);

$$(29) \quad r_n^{-m} \max[(1 - \sigma_n)^p \zeta_n, (1 - \sigma_{n+1})^p \zeta_{n+1}] \rightarrow 0 \quad \text{when } n \rightarrow \infty.$$

Proof. Formula (28) implies that $z_n > 0$ for $n \geq N$, where N is a positive integer. From (18) we deduce that

$$(30) \quad \frac{w_{n+1}^1(t)}{w_n^1(t)} \leq K \zeta_n^K e^{-z_n r_n} \quad \text{for } b_n \leq t \leq c_n \quad (n \geq N);$$

similarly from (20) we obtain

$$(31) \quad \frac{w_n^1(t)}{w_{n+1}^1(t)} \leq K \zeta_n^K e^{-z_n r_n} \quad \text{for } e_n < t < a_{n+1} \quad (n \geq N).$$

From (30) for $t = b_n$, (9), and (17) it follows that

$$(32) \quad w_{n+1}^1(t) \leq K \zeta_n^K w_N^1(b_N) e^{-z_n r_n} \quad \text{for } t \geq b_n \quad (n = N, N+1, \dots).$$

From (21), (24), (7), (16), (32) we see that

$$(33) \quad |D^q u|, |D^q v| \leq K r_n^{-K} [\max(\zeta_i, \zeta_{i+1})]^K e^{-z_i r_i} \quad \text{for } a_n < t < a_{n+1}$$

($j = n-1$ or n ; $0 \leq q < \infty$). It follows from (7), (8) that

$$|D^q[w_n^1(t)]^{-1}| \leq K \zeta_n^K (|D^q A(t)|^K + 1) [w_n^1(t)]^{-1}.$$

Hence from (23), (26), (7), (16), (8), (30) or (31) we deduce that

$$(34) \quad |D^q f|, |D^q g| \leq K r_n^{-K} [\max(\zeta_n, \zeta_{n+1})]^K e^{-z_n r_n} \quad \text{for } a_n \leq t \leq a_{n+1}.$$

From (33), (34), and (28) it follows that

$$D^q u, D^q v, D^q f, D^q g \rightarrow 0 \quad \text{when } t \nearrow 0 \quad (q = 0, 1, \dots);$$

therefore by (27) the functions u, v, f, g are of class C^∞ on the whole plane. From (22), (25), (16) we see that

$$(35) \quad |D^m \lambda|, |D^m \mu| \leq K r_n^{-m} \max[(1 - \sigma_n)^p \zeta_n, (1 - \sigma_{n+1})^p \zeta_{n+1}]$$

for $a_n \leq t \leq a_{n+1}$. From (35) and (29) it follows that $D^m \lambda, D^m \mu \rightarrow 0$ when $t \nearrow 0$; therefore by (27) the functions λ, μ are of class C^m on the whole plane.

For completing the proof it is now sufficient to show the existence of sequences (15) having properties (28), (29). It is easy to verify that the following sequences have these properties for $1/p \geq \alpha > 0$:

$$\begin{aligned} a_n &= -n^{-\alpha} & r_n &= \frac{1}{5}(a_{n+1} - a_n), \\ b_n &= a_n + r_n, & c_n &= b_n + r_n, \\ d_n &= c_n + r_n, & e_n &= d_n + r_n, \\ 1 - \sigma_n &= 4/(n+5), & \zeta_n &= n^{2+2\alpha}. \end{aligned}$$

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