On Quasi-Linear Parabolic Equations of the Second Order II

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Introduction. Let D denote a bounded $(n + 1)$ -dimensional domain of real variables $(x, t) = (x_1, \dots, x_n, t)$. We assume that D is bounded by two hyperplanes $t = 0$ and $t = t_0 > 0$, and by a surface S (S is a closed set) lying between the hyperplanes. Denote by $\bar{D}$ the closure of D , by B the base of D (that is, $B = \bar{D} \cap \{t = 0\}$), by D_τ the domain $D \cap \{0 < t < \tau\}$, by S_τ the intersection $S \cap \{0 \leq t \leq \tau\}$, by $\bar{D}_\tau$ the closure of D_τ , and, finally, denote by ∂D_τ (∂D) the set $S_\tau \cup B$ ($S \cup B$).

Consider the linear parabolic operator

$$(0.1) \quad Lu \equiv \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t}$$

with coefficients defined in $\bar{D}$. In [2] we proved existence and uniqueness theorems for the first mixed boundary problem:

$$(0.2) \quad Lu = f(x, t, u, \nabla u) \quad \text{for } (x, t) \text{ in } D,$$

$$(0.3) \quad u = h(x, t) \quad \text{for } (x, t) \text{ on } \partial D$$

where $\nabla u = (u_{x_1}, \dots, u_{x_n})$ and where f is defined in the domain

$$(0.4) \quad (x, t) \in \bar{D}, \quad -\infty < u < \infty, \quad -\infty < u_{x_i} < \infty \quad (i = 1, \dots, n).$$

The proof of the existence of solutions depended upon an *a priori* $(1 + \delta)$ -estimate which was derived in the same paper. In deriving that estimate, however, we employed Green's function and quoted results which (as we noticed later) are not known. So that proof is incomplete.

In the present paper, we not only complete the proof of the $(1 + \delta)$ -estimate, but we also derive it under much weaker assumptions than in [2]. As a consequence, we obtain existence theorems under very weak differentiability assump-

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tions on the coefficients of L . We also prove the existence of solutions for functions $f(x, t, u, \nabla u)$ satisfying weaker growth conditions than in [2].

The derivation of the $(1 + \delta)$ -estimate in this paper employs the fundamental solution constructed by POGORZELSKI [7], [8] for parabolic equations with Hölder continuous coefficients. In [9] POGORZELSKI used his fundamental solutions to prove the existence of solutions in cylindrical domains for the second mixed boundary problem:

$$(0.5) \quad \begin{aligned} Lu &= f(x, t, u, \nabla u) \quad \text{in } B, \\ u &= 0 \quad \text{on } B, \quad \frac{\partial u}{\partial T} + g(x, t, u) = h(x, t, u, \nabla u) \quad \text{on } S \end{aligned}$$

where g, h are given functions and $\partial/\partial T$ is the *transversal* (or *conormal*) derivative. We also mention that in [4] we proved the existence of solutions in cylindrical domains for the first mixed boundary problem:

$$(0.6) \quad Lu = f(x, t, u) \quad \text{in } D, \quad u = h(x, t) \quad \text{on } \partial D,$$

using Green's function.

In Part I we derive the $(1 + \delta)$ -estimate, and in Part II we briefly discuss question of existence of solutions of the system (0.2), (0.3).

PART I: A PRIORI INEQUALITY

1. Statement of the result. Let $v(x, t)$ be defined in a domain G and let $0 < \alpha < 1$. We define

$$(1.1) \quad |v|_0^\alpha = \text{l.u.b.}_{(x, t) \in G} |v(x, t)|, \quad H_\alpha^\alpha[v] = \text{l.u.b.}_{P, Q \in G} \frac{|v(P) - v(Q)|}{[d(P, Q)]^\alpha},$$

$$(1.2) \quad |v|_\alpha^\alpha = |v|_0^\alpha + H_\alpha^\alpha[v]$$

where

$$(1.3) \quad d(P, Q) = (|x - x'|^2 + |t - t'|)^{\frac{1}{2}} \quad \text{if } P = (x, t), \quad Q = (x', t').$$

$H_\alpha^\alpha[v]$ is the Hölder coefficient of v in G . If v is differentiable, we also define

$$(1.4) \quad |v|_{1+\alpha}^\alpha = |v|_\alpha^\alpha + \sum_{i=1}^n \left| \frac{\partial v}{\partial x_i} \right|_\alpha^\alpha,$$

$$(1.5) \quad |v|_{2+\alpha}^\alpha = |v|_{1+\alpha}^\alpha + \sum_{i=1}^n \left| \frac{\partial v}{\partial x_i} \right|_{1+\alpha}^\alpha + \left| \frac{\partial v}{\partial t} \right|_\alpha^\alpha.$$

Finally, we define

$$(1.6) \quad |v|_1^\alpha = |v|_0^\alpha + L^\alpha[v]$$

where

$$L^\alpha[v] = \text{l.u.b.}_{(x, t) \in G, (x', t') \in G} \frac{|v(x, t) - v(x', t')|}{|x - x'| + |t - t'|}.$$

If $|v|_a^q$ exists, then we say that v is of class C^q in G . If $v \in C'$ (note that this does not imply that v is everywhere differentiable) and $\partial v / \partial x_i \in C'$ for all i , then we say that $v \in C^2$. In case there is no confusion, the symbol G will be omitted in the above formulas.

We say that the surface S is of class C^q if S can be locally represented in the form

$$(1.7) \quad x_i = k(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, t) \quad \text{for some } i,$$

and the functions k are of class C^q . We cover S by a finite number of neighborhoods S_i each of which has a fixed global representation of the form (1.7) with k of class C^q . Now let $v(x, t)$ be any function defined on S . Using the local representations (1.7), v becomes, on each S_i , a function of n variables which vary in an n -dimensional domain which we denote by Σ_i . We then define

$$(1.8) \quad |v|_p^s = \max_i |v|_p^{\Sigma_i}.$$

In Theorem 1 stated below, S is assumed to belong to both classes C^2 and $C^{2+\alpha}$. We need also the following assumptions on L :

(A) L is of parabolic type in $\bar{D}$, that is, there exists a positive number H_0 such that, for all $(x, t) \in \bar{D}$ and for any real vector ξ ,

$$(1.9) \quad \sum a_{ii} \xi_i \xi_i \geq H_0 \sum \xi_i^2.$$

(B) a_{ii} are Hölder continuous (exponent α) in $\bar{D}$, b_i and c are continuous in $\bar{D}$ and, in addition, a_{ii} are of class C^1 on the boundary S . Thus

$$(1.10) \quad \sum |a_{ii}|_\alpha^p + \sum |b_i|_0^p + |c|_0^p \leq H_1,$$

$$(1.11) \quad \sum |a_{ii}|_1^s \leq H_2.$$

(B') a_{ii} , b_i and c are Hölder continuous (exponent α) in $\bar{D}$ and, in addition, a_{ii} are of class C^1 . Thus (1.11) holds and

$$(1.10') \quad \sum |a_{ii}|_\alpha^p + \sum |b_i|_\alpha^p + |c|_\alpha^p \leq H_3.$$

We can now state the main result of Part I.

Theorem 1 Assume that S belongs to both C^2 and $C^{2+\alpha}$ and that L satisfies (A), (B). Let $f(x, t)$ be a continuous function in $\bar{D}$ and let $u(x, t)$ be a solution of the system

$$(1.12) \quad Lu = f(x, t) \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D.$$

Then, for any positive $\delta < 1$, there exists a constant K depending only on δ , H_0 , H_1 , H_2 and on the domain D such that

$$(1.13) \quad |u|_{1+\delta} \leq K |f|_0.$$

The theorem will be proved in several steps. In what follows we denote by K any constant which (unless the contrary is explicitly stated) depends only

on δ, γ (restricted by (3.14) below), D and the H_i . Without loss of generality we may assume that $c(x, t) \leq 0$ (so that the maximum principle holds) since the general case is reduced to that special one by the transformation $\tilde{u} = e^{-\rho t} u$ where $\rho = \max c(x, t)$.

2. Reduction to f, b_i, c Hölder continuous. In this section we shall assume that Theorem 1 has already been proved in case f, b_i and c are Hölder continuous (exponent α) in $\bar{D}$, and we shall prove it in case f, b_i and c are only continuous in $\bar{D}$.

Let $\{f^m\}, \{b_i^m\}, \{c^m\}$ be sequences of functions Hölder continuous (exponent α) in $\bar{D}$ which satisfy

$$(2.1) \quad \epsilon_m \equiv |f^m - f|_0^D \rightarrow 0, \quad |b_i^m - b_i|_0^D \rightarrow 0, \quad |c^m - c|_0^D \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Denote by L_m the operator L when b_i, c are replaced by b_i^m, c^m respectively. By [1] there exists a solution u_m of the system

$$(2.2) \quad L_m u_m = f^m \quad \text{in } D, \quad u_m = 0 \quad \text{on } \partial D.$$

By assumption we have

$$(2.3) \quad |u_m|_{1+\delta} \leq K|f^m|_0 \leq K(|f|_0 + \epsilon_m).$$

From (2.3) it follows that we can extract a subsequence $\{u_{m'}\}$ of $\{u_m\}$ such that

$$(2.4) \quad u_{m'} \rightarrow v, \quad \nabla u_{m'} \rightarrow \nabla v \quad \text{uniformly in } D, \quad \text{as } m \rightarrow \infty.$$

Hence the limit function v satisfies

$$(2.5) \quad |v|_{1+\delta} \leq K \limsup_{m \rightarrow \infty} (|f|_0 + \epsilon_m) \leq K|f|_0.$$

If we prove that

$$(2.6) \quad |u - u_m|_0^D \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

then it would follow that $v \equiv u$ and hence, by (2.5), the proof of Theorem 1 would be completed.

To prove (2.6), we note that

$$(2.7) \quad L(u_m - u) = Lu_m - Lu = (L - L_m)u_m + f^m - f \equiv \tilde{f}_m.$$

From (2.3) it follows that

$$(2.8) \quad |u_m|_{1+\delta} \leq K(|f|_0 + 1) = \text{const.} < \infty.$$

Combining (2.8) with (2.1) we conclude that

$$(2.9) \quad |\tilde{f}_m|_0^D \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Using the maximum principle [6] we then get

$$|u_m - u|_0^D \leq K|\tilde{f}_m|_0^D \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

and the proof of (2.6) is completed.

3. Auxiliary results on fundamental solutions. Let $(a^{ij}(x, t))$ be the matrix inverse to $(a_{ij}(x, t))$ and let $\xi = (\xi_1, \dots, \xi_n)$, $y = (y_1, \dots, y_n)$. Define

$$(3.1) \quad \vartheta^{y, \tau}(x, \xi) = \sum_{i, i=1}^n a^{ii}(y, \tau)(x_i - \xi_i)(x_i - \xi_i),$$

$$(3.2) \quad w^{y, \tau}(x, t; \xi, \tau) = (t - \tau)^{-\frac{1}{2}n} \exp \left\{ -\frac{\vartheta^{y, \tau}(x, \xi)}{4(t - \tau)} \right\}.$$

In any cylinder $\Omega = \{(x, t); x \in E, 0 \leq t \leq \sigma\}$ with base E such that $\bar{E}$ is contained in the interior of B and $\bar{\Omega} \subset D$ we can construct (by [7]) a fundamental solution $\Gamma(x, t; \xi, \tau)$, provided L is parabolic and its coefficients are Hölder continuous. Γ has the form

$$(3.3) \quad \Gamma(x, t; \xi, \tau) = w^{\xi, \tau}(x, t; \xi, \tau) + \int_{\tau}^t \int_E w^{y, \eta}(x, t; y, \eta) \Phi(y, \eta; \xi, \tau) dy d\eta$$

where $dy = dy_1 \dots dy_n$ and Φ is the solution of the Volterra-type integral equation

$$(3.4) \quad \begin{aligned} L[w^{\xi, \tau}(x, t; \xi, \tau)] - \rho(x, t)\Phi(x, t; \xi, \tau) \\ + \int_{\tau}^t \int_E L[w^{y, \eta}(x, t; y, \eta)]\Phi(y, \eta; \xi, \tau) dy d\eta = 0 \end{aligned}$$

where L operates on functions of (x, t) and

$$(3.5) \quad \rho(x, t) = \frac{2^n \pi^{\frac{1}{2}n}}{|\det(a^{ij}(x, t))|^{\frac{1}{2}}}.$$

The inequality

$$(3.6) \quad |\Phi(x, t; \xi, \tau)| \leq \frac{K}{(t - \tau)^{\mu}} \frac{1}{|x - \xi|^{n+2-2\mu-\alpha}}$$

holds for any $1 - \frac{1}{2}\alpha < \mu < 1$.

We shall need the following inequalities which can easily be proved:

$$(3.7) \quad |w^{y, \tau}(x, t; \xi, \tau)| \leq \frac{K}{(t - \tau)^{\mu}} \frac{1}{|x - \xi|^{n-2\mu}},$$

$$(3.8) \quad \left| \frac{\partial}{\partial x} w^{y, \tau}(x, t; \xi, \tau) \right| \leq \frac{K}{(t - \tau)^{\mu}} \frac{1}{|x - \xi|^{n+1-2\mu}},$$

$$(3.9) \quad \left| \frac{\partial}{\partial x} w^{\xi, \tau}(x, t; \xi, \tau) - \frac{\partial}{\partial x'} w^{\xi, \tau}(x', t; \xi, \tau) \right| \leq \frac{K}{(t - \tau)^{\mu}} \frac{|x - x'|}{|x - \xi|^{n+2-2\mu}},$$

provided $|x - \xi| \geq 2|x - x'|$,

$$(3.10) \quad \left| \frac{\partial}{\partial x} w^{\xi, \tau}(x, t; \xi, \tau) - \frac{\partial}{\partial x} w^{\xi, \tau}(x, t'; \xi, \tau) \right| \leq \frac{K}{(t - \tau)^{\mu}} \frac{t' - t}{|x - \xi|^{n+3-2\mu}},$$

provided $t' > t > \tau$. In (3.7)–(3.10) μ is any positive number < 1 .

In what follows we shall make use of the elementary inequality

$$(3.11) \quad \int_B \frac{dy}{|x-y|^\beta |y-z|^\gamma} \leq \frac{\text{const.}}{|x-z|^{\beta+\gamma-n}} \quad (0 < \beta < n, \quad 0 < \gamma < n, \quad \beta + \gamma > n).$$

Using (3.11) we obtain from (3.3), (3.6), (3.7)

$$(3.12) \quad |\Gamma(x, t; \xi, \tau)| \leq \frac{K}{(t-\tau)^\mu} \frac{1}{|x-\xi|^{n-2\mu}} \quad (0 < \mu < 1).$$

We shall need the following lemma.

Lemma 1. *For any $0 < \delta < 1$ there exists a constant K such that, for every continuous function $h(x, t)$ in Ω , the function*

$$(3.13) \quad v(x, t) = \int_0^t \int_E \Gamma(x, t; \xi, \tau) h(\xi, \tau) d\xi d\tau$$

satisfies the inequality

$$(3.14) \quad |v|_{1+\delta}^\Omega \leq K \sigma^\gamma |h|_0^\Omega \quad \text{for any } \gamma < \frac{1-\delta}{2}$$

and K depends on E , the H_i in (A), (B), and δ, γ .

Remark. A result similar to Lemma 1 is stated in [8, p. 89] and indication of the proof is given there. Lemma 1, however, differs from that result in that a positive power of the height σ of the cylinder Ω appears on the right side of the inequality. Lemma 1 is used later on to derive interior estimates. To derive boundary estimates we shall need another lemma whose proof is similar to that of Lemma 1.

Proof. In the following proof, K depends on E , the H_i , δ and γ . Using (3.3) we can write

$$(3.15) \quad \begin{aligned} v(x, t) &= \int_0^t \int_E w^{\xi, \tau}(x, t; \xi, \tau) h(\xi, \tau) d\tau \\ &\quad + \int_0^t \int_E w^{\xi, \tau}(x, t; \xi, \tau) h_1(\xi, \tau) d\tau \equiv v_1(x, t) + v_2(x, t) \end{aligned}$$

where v_i denotes the i^{th} integral on the right side of (3.15) and

$$(3.16) \quad h_1(x, t) = \int_0^t \int_E \Phi(x, (t; \xi, \tau) h(\xi, \tau) d\xi d\tau.$$

Using (3.6) one easily obtains

$$(3.17) \quad |h_1|_0 \leq K |h|_0.$$

To prove (3.14) it will be enough to prove that

$$(3.18) \quad I_1 \equiv \left| \frac{\partial}{\partial x_i} v_1(x, t) - \frac{\partial}{\partial x_i} v_1(x', t) \right| \leq K |x - x'|^\delta \sigma^\gamma |h|_0,$$

$$(3.19) \quad I_2 \equiv \left| \frac{\partial}{\partial x_i} v_1(x, t) - \frac{\partial}{\partial x_i} v_1(x, t') \right| \leq K |t - t'|^{\frac{1}{2}\delta} \sigma^\gamma |h|_0.$$

Indeed, using (3.15), (3.17) we then conclude that

$$(3.20) \quad H_\delta \left[\frac{\partial v}{\partial x_i} \right] \leq K |h|_0 \sigma^\gamma$$

for any i . Moreover, the method that is used below to prove (3.18), (3.19) can easily be modified to estimate $\partial v_1 / \partial x_i$, v_1 , and $H_\delta[v_1]$. Using (3.15), (3.17) we then get similar estimates for $\partial v / \partial x_i$, v , $H_\delta[v]$. Combining these inequalities with (3.20), the proof of the lemma is completed.

Proof of (3.18). Let E_1 be the set of points ξ in E which satisfy $|x - \xi| > 2|x - x'|$ and let $E_2 = E - E_1$. Using (3.8), (3.9) we get

$$(3.21) \quad \begin{aligned} I_1 &\leq K |h|_0 \int_0^t \int_{E_1} \frac{1}{(t - \tau)^\mu} \frac{|x - x'|}{|x - \xi|^{n+2-2\mu}} d\xi d\tau \\ &\quad + K |h|_0 \int_0^t \int_{E_2} \frac{1}{(t - \tau)^\mu} \left(\frac{1}{|x - \xi|^{n+1-2\mu}} + \frac{1}{|x' - \xi|^{n+1-2\mu}} \right) d\xi d\tau. \end{aligned}$$

Replacing in the first integral $|x - x'|$ by $|x - \xi|^\epsilon |x - x'|^{1-\epsilon}$ for some positive $\epsilon < 1$, we get

$$(3.22) \quad I_1 \leq K |h|_0 \sigma^{1-\mu} (|x - x'|^{1-\epsilon} + |x - x'|^{2\mu-1}),$$

provided $2\mu + \epsilon > 2$. Taking $\epsilon = 1 - \delta$ and $\mu > \frac{1}{2}(1 + \delta)$ but sufficiently close to $\frac{1}{2}(1 + \delta)$, we conclude that

$$(3.23) \quad I_1 \leq K |h|_0 \sigma^\gamma \quad \text{for any given } \gamma < \frac{1 - \delta}{2}.$$

Proof of (3.19). We may assume that $t' > t$. Let E_3 be the set of points ξ in E which satisfy $|x - \xi| > (t' - t)^{\frac{1}{2}}$ and let $E_4 = E - E_3$. Using (3.8), (3.10) we get, for any $0 < \lambda < 1$, $0 < \mu < 1$,

$$(3.24) \quad \begin{aligned} I_2 &\leq K |h|_0 \int_0^t \int_{E_4} \frac{t' - t}{(t - \tau)^\mu} \frac{1}{|x - \xi|^{n+3-2\mu}} d\xi d\tau \\ &\quad + K |h|_0 \int_t^{t'} \int_{E_3} \frac{1}{(t' - \tau)^\lambda} \frac{1}{|x - \xi|^{n+1-2\lambda}} d\xi d\tau \\ &\quad + K |h|_0 \int_0^{t'} \int_{E_4} \frac{1}{(t' - \tau)^\mu} \frac{1}{|x - \xi|^{n+1-2\mu}} d\xi d\tau. \end{aligned}$$

Substituting in the first integral $t' - t \leq (t' - t)^{\frac{1}{2}\epsilon} |x - \xi|^{2-\epsilon}$ and in the third integral $|x - \xi|^\rho \leq (t' - t)^{\frac{1}{2}\rho}$, we obtain

$$(3.25) \quad I_2 \leq K |h|_0 [\sigma^{1-\mu}(t' - t)^{\frac{1}{2}\epsilon} + (t' - t)^{1-\lambda} + \sigma^{1-\mu}(t' - t)^{\frac{1}{2}\rho}],$$

provided $2\mu - 1 - \epsilon > 0$, $2\lambda > 1$, $2\mu - 1 - \rho > 0$. Taking $\rho = \epsilon = \delta$, λ close to $\frac{1}{2}$ and μ close to $\frac{1}{2}(1 + \delta)$, we conclude that

$$(3.26) \quad I_2 \leq K |h|_0 (t' - t)^{\frac{1}{2}\delta} \sigma^\gamma,$$

and the proof of Lemma 1 is thereby completed.

Consider now functions $w(x, t)$ of the form

$$(3.27) \quad w(x, t) = \int_0^t \int_{\partial E} \Gamma(x, t; \xi, \tau) k(\xi, \tau) d\Sigma d\tau$$

where ∂E is the boundary of E , $d\Sigma = d\Sigma(\xi)$ is the area element on ∂E at the point ξ , and $k(\xi, \tau)$ is a continuous function. We shall assume that ∂E is of class $C^{1+\beta}$ for some $0 < \beta < 1$. We define (see [8])

$$(3.28) \quad \frac{d}{dT_y} w(x, t) = \sum_{i,j} a_{ij}(x, t) \cos(N_y, x_i) \frac{\partial}{\partial x_i} w(x, t) \quad (y \in \partial E)$$

to be the *transversal* derivative of w at the point (x, t) and with respect to the point (y, t) on ∂E . Here N_y is the interior normal to ∂E at the point y . It was proved in [8, p. 68] that

$$(3.29) \quad \lim_{x \rightarrow y} \frac{dw(x, t)}{dT_y} = -\frac{1}{2}\rho(y, t)k(y, t) + \int_0^t \int_{\partial E} \frac{d\Gamma(y, t; \xi, \tau)}{dT_y} k(\xi, \tau) d\Sigma d\tau$$

where ρ is defined in (3.5). Furthermore,

$$(3.30) \quad \left| \frac{d\Gamma(y, t; \xi, \tau)}{dT_y} \right| \leq \frac{K}{(t - \tau)^\mu} \frac{1}{|y - \xi|^{n+1-2\mu-\nu}} \quad (\xi \in \partial E, y \in \partial E)$$

where $\nu = \min(\alpha, \beta)$ and μ is any number satisfying $1 - \frac{1}{2}\nu < \mu < 1$.

The calculations in [8, p. 69] easily show that if we define

$$(3.31) \quad \frac{d^*}{dT_y} w(x, t) = \sum_{i,j} a_{ij}(y, t) \cos(N_y, x_i) \frac{\partial}{\partial x_i} w(x, t),$$

then (3.29) holds also with d/dT_y replaced by d^*/dT_y .

Using (3.3) and Lemma 1, it easily follows that if ξ varies on ∂E and x is restricted to a closed domain E_0 lying in the interior of E , then as a function of (x, t) (we write $\Gamma = \Gamma(\cdot, \cdot; \xi, \tau)$) Γ satisfies

$$(3.32) \quad |\Gamma(\cdot, \cdot; \xi, \tau)|_{1+\delta}^{\Omega_0} \leq K\sigma^\gamma \quad \left(0 < \delta < 1, \gamma < \frac{1-\delta}{2} \right)$$

where Ω_0 is the cylinder with base E_0 and height $0 < t < \sigma$ and K depends only on E_0 , E , δ , γ and the H_i (in (A), (B)).

4. Interior estimates for small t . We proceed to estimate the $(1 + \delta)$ -norm of u in the interior of D_σ . In the next section we shall estimate it near the bound-

ary. We recall that by §2 we may assume that b_i , c and $f(x, t)$ are Hölder continuous in $\bar{D}$. Hence $u \in C^{2+\alpha}$ in $\bar{D}$. Let E be a fixed subdomain of B , $\bar{E}$ lying in the interior of B , and let G be a fixed subdomain of E , $\bar{G} \subset E$. If σ is sufficiently small then the cylinder Ω with base E and height $0 < t < \sigma$ is contained in D . Denote by $|u|_{q, \sigma, A}$ the q -norm of u in the cylinder with base A and height σ , and by $|u|_{q, \sigma}$ the q -norm of u in D_σ . In this section we shall estimate $|u|_{1+\varepsilon, \sigma, G}$.

Consider in Ω the function

$$(4.1) \quad \begin{aligned} z(x, t) = & \int_0^t \int_E \frac{\Gamma(x, t; \xi, \tau)}{\rho(\xi, \tau)} f(\xi, \tau) d\xi d\tau \\ & + \int_0^t \int_{\partial E} \Gamma(x, t; \xi, \tau) k(\xi, \tau) d\Sigma d\tau \end{aligned}$$

and assume that ∂E is of class C^2 . Since f is Hölder continuous, we conclude, by [7, pp. 50–52], that $Lz = f$. We now choose k so that it satisfies

$$(4.2) \quad \begin{aligned} -\frac{1}{2}\rho(y, t)k(y, t) + \int_0^t \int_{\partial E} \frac{d\Gamma(y, t; \xi, \tau)}{dT_y} k(\xi, \tau) d\Sigma d\tau \\ = - \int_0^t \int_E \frac{1}{\rho(\xi, \tau)} \frac{d\Gamma(y, t; \xi, \tau)}{dT_y} f(\xi, \tau) d\xi d\tau + \frac{du(y, t)}{dT_y}. \end{aligned}$$

By (3.30), the kernel of the Volterra-type integral equation (4.2) is integrable. Hence $k(y, t)$ is uniquely determined from (4.2) and it satisfies the inequality

$$(4.3) \quad |k|_0 \leq K \left(|f|_{0, \sigma, E} + \sum_i \left| \frac{\partial u}{\partial x_i} \right|_{0, \sigma, E} \right).$$

Using a remark made at the end of §3 (in connection with (3.31)) and (4.2), we conclude that the function $w = z - u$ satisfies

$$(4.4) \quad \lim_{x \rightarrow y} \frac{d^*}{dT_y} w(x, t) = 0.$$

Taking, in particular, $y \rightarrow x$ along the transversal direction l (whose direction cosines are $\sum_i a_{ij}(y, t) \cos(N_j, x_i)$), we get $\partial w / \partial l \rightarrow 0$ as $(x, t) \rightarrow (y, t)$. In particular, we conclude that for any sequence $P_m = (x_m, t)$ lying on l and tending to $P = (y, t)$ the difference quotient (along l) $\Delta w / \Delta P_m$ tends to zero as $P_m \rightarrow P$.

Now the function w vanishes on E and satisfies in Ω the equation $Lw = 0$. We can therefore apply the maximum principle [6] and [4, Theorem 2] and thus conclude that $w \equiv 0$. Hence, by (4.1),

$$(4.5) \quad \begin{aligned} u(x, t) = & \int_0^t \int_E \frac{\Gamma(x, t; \xi, \tau)}{\rho(\xi, \tau)} f(\xi, \tau) d\xi d\tau \\ & + \int_0^t \int_{\partial E} \Gamma(x, t; \xi, \tau) k(\xi, \tau) d\Sigma d\tau \end{aligned}$$

and k is a continuous function satisfying (4.3).

Applying Lemma 1, (3.32) and (4.3), we get

$$(4.6) \quad |u|_{1+\delta, \sigma, G} \leq K\sigma^\gamma \left(|f|_{0, \sigma, E} + \sum_i \left| \frac{\partial u}{\partial x_i} \right|_{0, \sigma, E} \right) \quad \left(\gamma < \frac{1-\delta}{2}; \quad \tilde{G} \subset E \right),$$

which is the desired *a priori* inequality in the interior of D_σ .

5. Boundary estimates for small t . We proceed to estimate the $(1 + \delta)$ -norm of u in $D_\sigma - \Omega$. If σ is sufficiently small, then $\bar{D}_\sigma - \Omega$ is covered by a finite number of $(n + 1)$ -dimensional spheres V_j ($j = 1, \dots, j_0$) such that each portion $S_\sigma \cap V_j$, call it S_j , has a global representation of the form (1.7) and the functions k belong to the class $C^{2+\alpha}$. Since j_0 does not depend on σ (provided σ is smaller than a constant depending only on S), it is sufficient to estimate the $(1 + \delta)$ -norm of u in $D_i = D_\sigma \cap V_i$; the bound that will be obtained can then serve also as a bound (with a different K) on the $(1 + \delta)$ -norm of u in $D_\sigma - \Omega$.

In S_i we have a representation of the form (1.7). We assume without loss of generality that $i = n$. We perform in D_i the transformation

$$(5.1) \quad \begin{aligned} y_m &= x_m \quad \text{for } m = 1, \dots, n-1, \\ y_n &= x_n - k(x_1, \dots, x_{n-1}, t) \end{aligned}$$

which, as we may assume, maps D_i one-to-one onto a domain Δ lying in the half space $y_n < 0$. Denote by Σ the closed portion of the boundary of Δ lying on $y_n = 0$. Without loss of generality we assume that Σ contains the origin. Putting

$$(5.2) \quad U(y, t) = u(x, t), \quad F(y, t) = f(x, t), \quad C(y, t) = c(x, t),$$

the equation $Lu = f$ becomes

$$(5.3) \quad \Delta U \equiv \sum_{i,j} A_{ij}(y, t) \frac{\partial^2 U}{\partial y_i \partial y_j} + \sum_i B_i(y, t) \frac{\partial U}{\partial y_i} + C(y, t)U - \frac{\partial U}{\partial t} = F(y, t)$$

where

$$(5.4) \quad A_{ij} = \sum_{\lambda, \mu} a_{\lambda\mu} \frac{\partial y_i}{\partial x_\lambda} \frac{\partial y_j}{\partial x_\mu}, \quad B_i = \sum_{\lambda, \mu} a_{\lambda\mu} \frac{\partial^2 y_i}{\partial x_\lambda \partial x_\mu} + \sum_\lambda b_\lambda \frac{\partial y_i}{\partial x_\lambda} - \frac{\partial y_i}{\partial t}.$$

By our assumptions it follows that A_{ij} , B_i , C are Hölder continuous (exponent α) in $\bar{\Delta}$ and that A_{ij} are of class C^1 on Σ .

We wish to construct a "local" Green's function for Δ in Δ , that is, a fundamental solution of Δ in Δ which vanishes on Σ . The method is essentially by reflection with respect to $y_n = 0$. To accomplish it, however, the coefficients of $\partial^2 / \partial y_i \partial y_n$ in Δ have to vanish on Σ for $i < n$. Since this need not be the case, we perform another transformation such that the above required property will hold for the transform of Δ . We try the transformation

$$(5.5) \quad \begin{aligned} z_i &= \varphi_i(\bar{y}, t) + y_n \psi_i(y, t) \quad (i = 1, \dots, n-1), \\ z_n &= y_n \psi_n(\bar{y}, t) \quad \text{where } \bar{y} = (y_1, \dots, y_{n-1}). \end{aligned}$$

Denote by Δ' , Σ' the images of Δ , Σ respectively. Putting

$$(5.6) \quad U'(z, t) = U(y, t), \quad F'(z, t) = F(y, t), \quad C'(z, t) = C(y, t),$$

we obtain for U' the equation

$$(5.7) \quad \begin{aligned} \Delta' U' &\equiv \sum_{i,i} A'_{ii}(z, t) \frac{\partial^2 U'}{\partial z_i \partial z_i} \\ &+ \sum_i B'_i(z, t) \frac{\partial U'}{\partial z_i} + C'(z, t) U' - \frac{\partial U'}{\partial t} = F'(z, t) \end{aligned}$$

where

$$(5.8) \quad A'_{ii} = \sum_{\lambda, \mu} A_{\lambda\mu} \frac{\partial z_i}{\partial y_\lambda} \frac{\partial z_i}{\partial y_\mu}, \quad B'_i = \sum_{\lambda, \mu} A_{\lambda\mu} \frac{\partial^2 z_i}{\partial y_\lambda \partial y_\mu} + \sum_\lambda B_\lambda \frac{\partial z_i}{\partial y_\lambda} - \frac{\partial z_i}{\partial t}.$$

We shall now impose the conditions that $A'_{in} = 0$ on Σ' (where $z_n = 0$) for $i = 1, \dots, n-1$. Noting that $\partial z_n / \partial y_\mu = 0$ on Σ' if $\mu < n$, we obtain the equations

$$(5.9) \quad \sum_{\lambda=1}^{n-1} A_{\lambda n}(\bar{y}, 0, t) \frac{\partial \varphi_i(\bar{y}, t)}{\partial y_\lambda} + A_{nn}(\bar{y}, 0, t) \psi_i(\bar{y}, 0, t) = 0$$

$$(i = 1, \dots, n-1),$$

provided $\partial z_n / \partial y_n \neq 0$ on Σ' . These equations are satisfied if we take

$$(5.10) \quad \begin{cases} \psi_n = 1, & \varphi_i = y_i & (i = 1, \dots, n-1), \\ \psi_i(\bar{y}, 0, t) = -\frac{A_{in}(\bar{y}, 0, t)}{A_{nn}(\bar{y}, 0, t)} \equiv \gamma_i(\bar{y}, t) & (i = 1, \dots, n-1). \end{cases}$$

Following a method of FRIEDRICH [5] we take

$$(5.11) \quad \psi_i(y, t) = \int_{\Sigma'} \gamma_i(\bar{y} + y_n \xi, t + y_n \tau) j(\xi, \tau) d\xi_1 \cdots d\xi_{n-1} d\tau$$

where $\xi = (\xi_1, \dots, \xi_{n-1})$ and where $j(\xi, \tau)$ has second continuous derivatives in (ξ, τ) , vanishes outside Σ' and

$$\int_{\Sigma'} j(\xi, \tau) d\xi d\tau = 1.$$

Noting that $\gamma_i(\xi, \tau)$ satisfies a Lipschitz condition in (ξ, τ) , we easily conclude that the $z_i(y, t)$ have Hölder continuous (exponent α) second y -derivatives and first t -derivatives in closed subsets of Δ' , and $\partial^2 z / \partial y_i \partial y_i$, $\partial z / \partial t$ are bounded functions in $\bar{\Delta}'$ (the closure of Δ') *independently* of σ . Furthermore, $\partial z / \partial y_i$ are Lipschitz continuous in $\bar{\Delta}'$. Hence A'_{ii} and C' are Hölder continuous (exponent α) in $\bar{\Delta}'$, and the B'_i are Hölder continuous (exponent α) in closed subsets of Δ' and they are bounded in Δ' independently of σ .

It is immediately seen that the Jacobian of the transformation (5.5) is equal

to 1 on $y_n = 0$. Clearly, we may take σ sufficiently small so that each of the domains D_i which cover $D_\sigma - \Omega$ can be taken arbitrarily small. Hence we assume, without loss of generality, that the y_i ($i = 1, \dots, n$) are sufficiently small so that the transformation (5.5) with φ_i, ψ_i defined by (5.10), (5.11) is one-to-one from $\bar{\Delta}$ onto $\bar{\Delta}'$ and has a nonvanishing Jacobian. In particular, it follows that Δ' is of parabolic type.

Denote by Δ'_* the reflection of Δ' with respect to the hyperplane $z_n = 0$ and put $\Delta'' = \Delta' \cup \Sigma' \cup \Delta'_*$. Without loss of generality we may assume that Δ'' is a cylinder with base E'' and that $\partial E''$ is of class C^2 . Let E' be the base of Δ' . Denoting by z^* the point $(z_1, \dots, z_{n-1}, -z_n)$ which is the reflection of z with respect to the hyperplane $z_n = 0$, we extend the definition of the coefficients of Δ' into Δ'' as follows:

$$\begin{aligned}
 (5.12) \quad & A'_{ij}(z^*, t) = A'_{ij}(z) \text{ if } 1 \leq i \leq n-1, \quad 1 \leq j \leq n-1 \text{ or if } i = j = n, \\
 & A'_{in}(z^*, t) = -A'_{in}(z) \quad \text{if } 1 \leq i \leq n-1, \\
 & B'_i(z^*, t) = B'_i(z, t) \quad \text{if } 1 \leq i \leq n-1, \quad B'_n(z^*, t) = -B'_n(z, t), \\
 & C'(z^*, t) = C'(z, t).
 \end{aligned}$$

Note that, with the exception of the B'_i , all the extended coefficients are Hölder continuous (exponent α) in $\bar{\Delta}''$, and B'_i are bounded in $\Delta' \cup \Delta''$ and Hölder continuous (exponent α) in closed subsets of $\Delta' \cup \Delta''$.

We can now proceed by the method of POGORZELSKI (see §3) to construct a fundamental solution $\Gamma'(z, t; \xi, \tau)$ for the operator Δ' in Δ'' . We easily find that $\Gamma'(z, t; \xi, \tau)$ is a continuous function for $(z, t) \in \bar{\Delta}''$, $(\xi, \tau) \in \bar{\Delta}''$, provided $(z, t) \neq (\xi, \tau)$, and that for every fixed $(\xi, \tau) \in \bar{\Delta}''$ it satisfies the equation $\Delta' \Gamma' = 0$ for $(z, t) \in \Delta' \cup \Delta'_*$.

Using the definitions (5.12) we conclude that $\Gamma'(z^*, t; \xi, \tau)$ is a solution of $\Delta' \Gamma' = 0$ as a function of (z, t) where (z, t) varies in $\Delta' \cup \Delta'_*$ and (ξ, τ) is a fixed point in $\bar{\Delta}''$. Hence the function

$$(5.13) \quad G'(z, t; \xi, \tau) = \Gamma'(z, t; \xi, \tau) - \Gamma'(z^*, t; \xi, \tau)$$

is a solution of $\Delta' G' = 0$ for $(z, t) \in \Delta'$, (ξ, τ) being any fixed point in $\bar{\Delta}'$, and $G'(z, t; \xi, \tau) = 0$ when $z_n = 0$. Thus G' is the desired "local" Green's function of Δ' . We remark that G' has first z -derivatives which are uniformly continuous in $(z, t) \in \Delta'$.

It is now easy to see that Lemma 1 remains true if we replace Γ by G' and E by E' (the base of Δ') and that the proof follows by slightly modifying the proof of Lemma 1. We also note that (3.30) holds with Γ replaced by G' and $y \in \partial E$ replaced by $y \in \partial \Delta' - \Sigma'$ (the proof follows by slightly modifying the proof of (3.30) as given in [8]). We finally note that (3.32) holds with Γ replaced by G' , $(\xi, \tau) \in \partial \Delta' - \Sigma'$ and Ω_0 any cylinder having its closure in $\Delta' \cup \Sigma'$.

Using the method of §4 we can now represent the solution U' of (5.7), which vanishes on Σ' , in the form (compare (4.5))

$$(5.14) \quad \begin{aligned} U'(z, t) = & \int_0^t \int_{E'} \frac{G'(z, t; \xi, \tau)}{\rho'(\xi, \tau)} F'(\xi, \tau) d\xi d\tau \\ & + \int_0^t \int_{S'} G'(z, t; \xi, \tau) k'(\xi, \tau) d\Sigma' d\tau \end{aligned}$$

where ρ' is defined analogously to (3.5) and S' is the part of the boundary of E' which does not lie on $z_n = 0$. k' satisfies an inequality analogous to (4.3).

Let R'_i be a closed cylinder lying in $\Delta' \cup \Sigma'$. We then obtain (compare (4.6))

$$(5.15) \quad |U'|_{1+\delta, \sigma, R'_i} \leq K\sigma^\gamma \left(|F'|_{0, \sigma, \Delta'} + \sum_i \left| \frac{\partial U'}{\partial z_i} \right|_{0, \sigma, \Delta'} \right).$$

Reversing the transformations (5.5), (5.1) and using the definitions (5.6), (5.2), we get

$$(5.16) \quad |u|_{1+\delta, \sigma, R_i} \leq K\sigma^\gamma \left(|f|_{0, \sigma, D_i} + \sum_i \left| \frac{\partial u}{\partial x_i} \right|_{0, \sigma, D_i} \right)$$

where R'_i is the image of R_i under the product of the transformations (5.1), (5.5). Here we made use of the Lipschitz continuity of $\partial z_i / \partial y_j$.

If σ is sufficiently small, D_i and R_i can be taken in such a manner that the domains R_i cover $D_\sigma - \Omega$. Hence, combining (5.16) with (4.6), we get

$$(5.17) \quad |u|_{1+\delta, \sigma} \leq K\sigma^\gamma \left(|f|_{0, \sigma} + \sum_i \left| \frac{\partial u}{\partial x_i} \right|_{0, \sigma} \right)$$

Taking σ such that $2K\sigma^\gamma < 1$, we obtain the inequality

$$(5.18) \quad |u|_{1+\delta, \sigma} \leq K\sigma^\gamma |f|_{0, \sigma} \quad \text{for any } \gamma < \frac{1-\delta}{2}$$

and K depends on D , the H_i and δ, γ .

6. Completion of the proof. Having estimated (in (5.18)) the $(1 + \delta)$ -norm of u in D_σ , we now proceed to estimate it step by step in D . We note that, for fixed $\delta < 1$ and $\gamma < \frac{1}{2}(1 - \delta)$, the σ in (5.18) is sufficiently small depending only on D and the H_i . Furthermore, we can choose σ so that (5.18) is valid in any domain $D_{\rho+\sigma} - D_\rho$ ($0 \leq \rho \leq t_0 - \sigma$), provided $Lu = f$ and provided u vanishes on the lower base of this domain. From now on σ will be a fixed small number such that the last property holds. We introduce the domains

$$(6.1) \quad D^i = D \cap \left\{ i \frac{\sigma}{2} < t < (i+2) \frac{\sigma}{2} \right\},$$

and denote by $|v|_{q, i}$ the q -norm of v in D^i .

Let $\zeta(t)$ be a twice continuously differentiable function of t which vanishes for $t \leq \frac{1}{2}\sigma$ and is equal to 1 for $t \geq 1$. Consider the function

$$(6.2) \quad u^*(x, t) = \zeta\left(\frac{t}{\sigma}\right)u(x, t).$$

It vanishes on the lower base of D^1 and is equal to $u(x, t)$ for $t > \sigma$. Using (5.18) we get

$$(6.3) \quad |u^*|_{1+\delta, 1} \leq K\sigma^\gamma |Lu^*|_{0, 1}.$$

We proceed to estimate Lu^* . We have

$$(6.4) \quad |Lu^*| = |f\zeta| + \left| u \frac{d\zeta}{dt} \right| \leq N |f| + \frac{N}{\sigma} |u|$$

where N is a universal constant. To estimate $|u|$, we introduce the function $v = |f|_0^p t$. It satisfies

$$(6.5) \quad Lv = -|f|_0^p \text{ in } D, \quad v \geq 0 \text{ on } \partial D.$$

Applying the maximum principle [6] to $v \pm u$ we get

$$(6.6) \quad |u|_{0, \rho} \leq \rho |f|_0^p \text{ for any } 0 < \rho \leq t_0.$$

Substituting (6.6) with $\rho = \frac{2}{3}\sigma$ into (6.4) we get

$$(6.7) \quad |Lu^*| \leq (N + \frac{2}{3}) |f|_0^p.$$

Substituting (6.7) into (6.3) and recalling that $u^*(x, t) = u(x, t)$ if $\sigma < t < \frac{3}{2}\sigma$, we get (making use of (5.18))

$$(6.8) \quad |u|_{1+\delta, \frac{3}{2}\sigma} \leq [K\sigma^\gamma + K\sigma^\gamma(N + \frac{2}{3})] |f|_0^p \equiv K_1 |f|_0^p.$$

The next step is to estimate the $(1 + \delta)$ -norm of u in $D_{2\sigma}$. We use the previous method with D^1 replaced by D^2 and $\zeta(t/\sigma)$ replaced by $\zeta((t + \frac{1}{2})/\sigma)$. We then obtain

$$(6.9) \quad |u|_{1+\delta, 2\sigma} \leq K_2 |f|_0^p \quad (K_2 \text{ constant}).$$

Note that the constants K_i depend on σ , but since σ is fixed and depends only on D and the H_i , the K_i depend only on D and the H_i .

Proceeding now step by step in the above manner, the proof of Theorem 1 is completed.

7. Further results. Using Theorem 1 and its proof we can obtain some further results.

Remark 1. Consider the system

$$(7.1) \quad Lu = f \text{ in } D, \quad u = h \text{ on } \partial D,$$

and assume, in addition to the assumptions made in Theorem 1, that there exists a function $\Psi(x, t)$ such that $\partial\Psi/\partial x_i$, $\partial^2\Psi/\partial x_i \partial x_i$, $\partial\Psi/\partial t$ are continuous in $\bar{D}$ and which coincides with h on ∂D . If u is a solution of (7.1), then

$$(7.2) \quad |u|_{1+\delta} \leq K(|f|_0 + |\Psi|_2) \text{ for any } 0 < \delta < 1.$$

The proof follows by applying Theorem 1 to $u - \Psi$.

Remark 2. In the course of the proof of Theorem 1 we proved the inequality

$$(7.3) \quad |u|_{1+\delta, \sigma} \leq K \sigma^\gamma |f|_{0, \sigma} \quad \text{for any } \gamma < \frac{1-\delta}{2}$$

This inequality will be used later on.

Remark 3. Theorem 1 and Remarks 1, 2 remain true also under the assumption that the boundary S belongs to both classes C^2 and $C^{1+\alpha}$ (instead of C^2 and $C^{2+\alpha}$). The proof is by approximation of D by larger domains.

PART II: EXISTENCE THEOREMS

With the aid of Theorem 1 we can establish existence theorems for the system (0.2), (0.3) by a fixed-point-theorem technique. The method was carried out in detail in [2], but we shall present here a few new ideas and derive stronger results. We shall first consider the special case $h \equiv 0$, $c \leq 0$. We shall need two remarks.

Remark 1. Suppose we prove that any solution in D must be bounded, say $|u| \leq K$, and then we change the definition of $f(x, t, u, w)$ ($w = (u_{x_1}, \dots, u_{x_n})$) for $|u| > K$, for instance by defining

$$(1) \quad \tilde{f}(x, t, u, w) = \begin{cases} f(x, t, u, w) & \text{for } |u| \leq K \\ f(x, t, \pm K, w) & \text{for } \pm u > K. \end{cases}$$

If we then prove the existence of a solution u for the system

$$(2) \quad Lu = \tilde{f} \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D,$$

then u is also a solution of the original system (0.2), (0.3). Note that if f is Hölder continuous in all its variables, the same is true for the function $\tilde{f}$ as defined in (1).

Remark 2. Suppose next that we can find functions f_1, f_2, ϕ_1, ϕ_2 which satisfy the inequalities

$$(3) \quad \begin{aligned} L\phi_1 &< f_1(x, t, \phi_1, \nabla\phi_1) \quad \text{in } D, & Lu &\geq f_1(x, t, u, \nabla u) \quad \text{in } D, \\ & & \phi_1 &> u \quad \text{on } \partial D, \end{aligned}$$

$$(4) \quad \begin{aligned} L\phi_2 &> f_2(x, t, \phi_2, \nabla\phi_2) \quad \text{in } D, & Lu &\leq f_2(x, t, u, \nabla u) \quad \text{in } D, \\ & & \phi_2 &< u \quad \text{on } \partial D. \end{aligned}$$

Then, as was proved in [2],

$$(5) \quad \phi_2(x, t) < u(x, t) < \phi_1(x, t) \quad \text{in } D.$$

In (3), (4) f_i is assumed only to be a continuous function.

We shall use the above remarks in order to prove the existence of a solution

in case f satisfies the inequality

$$(6) \quad |f(x, t, u, \nabla u)| \leq A + B |u| + C |\nabla u| \quad (|\nabla u| = \sum |u_{x_i}|),$$

for all $(x, t, u, \nabla u)$ in the cylinder (0.4), and C is sufficiently small.

Let u be a solution of (0.2), (0.3) with $h \equiv 0$. Then the function $v = e^{Bt}u$ satisfies

$$(7) \quad Lv = Bv + e^{Bt}f(x, t, e^{-Bt}v, e^{-Bt}\nabla v) \equiv f_0(x, t, v, \nabla v).$$

Using (6) we conclude that

$$(8) \quad Lv \geq Bv - B |v| - A' - C |\nabla v|$$

where A' is any constant larger than Ae^{Bt} for all $t \leq t_0$.

We shall compare v with the function $\tilde{\phi} = M\phi$ where

$$\phi = e^{\lambda R} - e^{\lambda x_1}, \quad M > 0.$$

We take R such that, for all $(x, t) = (x_1, \dots, x_n, t)$ in D , $2x_1 \leq R$, and then take λ sufficiently large so that

$$L\phi < -2\delta \quad \text{for some } \delta > 0.$$

We finally take M sufficiently large so that $M\delta \geq A'$. We then have

$$(9) \quad L\tilde{\phi} < -A' - \delta M \leq B\tilde{\phi} - B |\tilde{\phi}| - A' - C |\nabla \tilde{\phi}|,$$

provided C is sufficiently small, that is, provided

$$C |\nabla \phi| < \delta.$$

Since $\tilde{\phi} > v$ on ∂D , we can apply Remark 2 to conclude that

$$(10) \quad u(x, t) = e^{-Bt}v(x, t) < M\phi(x)e^{-Bt} \quad \text{in } D.$$

In a similar way we have, by (7),

$$(11) \quad Lv \leq Bv + B |v| + A' + C |\nabla v|.$$

Multiplying both side of (11) by -1 we get

$$(12) \quad L(-v) \geq B(-v) - B |-v| - A' - C |\nabla(-v)|.$$

Comparing $\tilde{\phi}$ with $-v$ we conclude that

$$(13) \quad -u(x, t) = -e^{-Bt}v(x, t) < M\phi(x)e^{-Bt} \quad \text{in } D.$$

Combining (10), (13) we get

$$(14) \quad |u(x, t)| \leq M_0$$

where M_0 is an appropriate constant.

Using Remark 1 and (6) we conclude that if we prove the existence of a solution for the system (0.2), (0.3) with $h \equiv 0$, $c \leq 0$ and with f locally Hölder continuous in all its variables and satisfying the inequality

$$(15) \quad |f(x, t, u, \nabla u)| \leq A'' + C |\nabla u|$$

for some constants A'' , C where C is sufficiently small (so that (9) holds), then existence follows also in the more general case where f satisfies (6) and C is small.

Now, existence of a solution in case that (15) holds and C is sufficiently small can be established by the fixed-point-theorem technique of [2], using Theorem 1.

We now state the main result of Part II.

Theorem 2. *Assume that the lateral boundary S of D is of both classes C^2 and $C^{2+\alpha}$, that L satisfies the assumptions (A), (B'), that $f(x, t, u, w)$ ($(w = u_{x_1}, \dots, u_{x_n})$) is locally Hölder continuous in the domain (0.4) and that it satisfies (6) where A, B are any positive numbers and C is sufficiently small. Assume finally that there exists a function Ψ of class $C^{2+\alpha}$ in $\bar{D}$ which coincides with h on ∂D . Then there exists a solution of the system (0.2), (0.3). Furthermore, the solution is of class $C^{1+\beta}$ in $\bar{D}$ for any $0 < \beta < 1$ and of class $C^{2+\gamma}$ in $\bar{D}$ for some $\gamma > 0$.*

Remark. C depends only on upper bounds on $A, B, |\Psi|_{2+\alpha}$, the H_i and the domain D .

The theorem has been proved above in the special case $h \equiv 0$ and under the assumption $c \leq 0$. The assumption $c \leq 0$ can easily be removed by applying the result already proved to the system obtained from the given system by the transformation $v = e^{-\rho t}u$ where $\rho = \max c(x, t)$.

To remove the restriction $h \equiv 0$ we consider, in the general case, the function $w = u - \Psi$. It satisfies the equation

$$(16) \quad Lw = -L\Psi + f(x, t, w + \Psi, \nabla w + \nabla\Psi) \equiv f^*(x, t, w, \nabla w).$$

Now, since (6) holds, we have

$$(17) \quad |f^*(x, t, w, \nabla w)| \leq A + |L\Psi|_0 + B |\Psi|_0 + C |\nabla\Psi|_0 + B |w| + C |\nabla w|.$$

Hence, applying the result already proved to the function w , the existence of a solution u of the system (0.2), (0.3) follows. That u belongs to $C^{1+\delta}$ in D follows from Remark 1 in Part I, §7. That u belongs to $C^{2+\gamma}$ in D follows from [1]. The proof of Theorem 2 is thus completed.

Remark 1. If f satisfies the inequality

$$(18) \quad |f(x, t, u, \nabla u)| \leq A + B |u| + C |\nabla u| + D |u|^\lambda + E |\nabla u|^\mu$$

where $\lambda < 1$, $\mu < 1$, the constants A, B, D, E are arbitrary and C is sufficiently small, then f also satisfies (6) with new constants A, B, C and the new constant C can be taken to be small. Hence Theorem 2 holds also in case (6) is replaced by the apparently more general inequality (18).

Remark 2. If f does not depend on ∇u , and if one can establish an *a priori* bound on $|u|_0$ where u is a solution of (0.2), (0.3) in D , then the existence of a solution of the system (0.2), (0.3) follows.

Remark 3. Using (7.3) one can prove (by the fixed-point-theorem method of [2]) existence of a solution in D_σ for the system (0.2), (0.3) for *any* locally Hölder continuous function $f(x, t, u, w)$, provided σ is sufficiently small.

Remark 4. We recall [2] that uniqueness holds under one of the following conditions:

(i) $|f(x, t, u, w) - f(x, t, u', w)| \leq M |u - u'|$ for all $(x, t, u, w), (x, t, u', w)$ in the domain (0.4).

(ii) $f(x, t, u, w)$ is a monotone nonincreasing function in u , for all fixed values of x, t, w .

Using (i) it is easy to see that uniqueness holds also under the following much weaker condition:

(iii) $f(x, t, u, w)$ is locally Lipschitz continuous in u .

Indeed, if u_1, u_2 are two different solutions of (0.2), (0.3), then they are also solutions of (0.2), (0.3) with f replaced by $\tilde{f}$ where $\tilde{f}$ is obtained by modifying the definition of $f(x, t, u, w)$ for

$$|u| > \max_{i=1,2} (\text{l.u.b. } |u_i|), \quad |w| > \max_{i=1,2} (\text{l.u.b. } |\nabla u_i|).$$

We can easily construct $\tilde{f}$ in such a manner that it satisfies (i). Upon applying (i) we conclude that $u_1 \equiv u_2$.

In case of uniqueness, a step by step process (or an application of Theorem 2 to domains D_ρ with ρ increasing to infinity) yields an existence theorem for the system (0.2), (0.3) in case D is bounded by $t = 0, t = \infty$ (in particular, when D is the cylinder $\{(x, t); x \in B, 0 < t < \infty\}$). Thus, for instance, if (6) holds with $C = 0$ and if either (ii) or (iii) holds, then the existence and uniqueness of a solution of the system (0.2), (0.3) in a domain D bounded by $t = 0, t = \infty$ follows (D_ρ is supposed to be a bounded domain for every $\rho < \infty$).

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