The Exterior Dirichlet Problem for Second Order Elliptic Partial Differential Equations

NORMAN MEYERS & JAMES SERRIN

In this paper we shall treat the exterior Dirichlet problem for linear elliptic equations of second order in $n \geq 2$ independent variables. Our main concern will be with the existence and uniqueness of solutions, and with the striking effect which the asymptotic behavior of the coefficients has on the results.

It will be convenient to preface our considerations with some remarks concerning the special case of the Laplace equation. In treating the exterior Dirichlet problem for that equation it is clear that some restriction on the behavior of the solution at distant points of space is necessary in order to insure unicity. A primary condition is that the solution be bounded. Indeed, for the case $n = 2$ this condition alone suffices for uniqueness, as is well known. On the other hand, in three or more dimensions there are an infinity of bounded solutions corresponding to any given boundary data, so that some additional restriction must be imposed. Since every bounded harmonic function defined in an exterior region tends to a limit at infinity, it is natural to assign this limit value as the specifying condition. The exterior problem then becomes uniquely solvable, and, moreover, as the limit value ranges over the real numbers, every bounded solution satisfying the given data is obtained.

Consider now the general elliptic equation of second order, which we write in the form

$$(1) \quad Lu = a_{ik}(x)u_{ik} + b_i(x)u_i + c(x)u = f(x),$$

where $x = (x_1, \dots, x_n)$ and

$$u_i = \frac{\partial u}{\partial x_i}, \quad a_{ik}u_{ik} = \sum_1^n a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k}, \quad \text{etc.}$$

By the exterior Dirichlet problem we mean the following: Let $\mathcal{E}$ be a given open region in the Euclidean number space E^n , containing a neighborhood of infinity and having a smooth internal boundary $F\mathcal{E}$. A continuous function $\phi(x)$ is prescribed on $F\mathcal{E}$. We wish then to find a function $u = u(x)$ which is of

class C^2 and satisfies $Lu = f$ in $\mathcal{E}$, and which continuously takes on the values $\phi(x)$ on the boundary $F\mathcal{E}$. The problem is further specified by certain conditions on the behavior of u at infinity. Corresponding to the preceding discussion of the Laplace equation, we shall consider two different cases:

- I. *The function u shall tend to an assigned limit l as x tends to infinity.*
- II. *The function u shall be bounded in $\mathcal{E}$.*

The problems arising from these two conditions will be referred to respectively as the exterior Dirichlet problem I and the exterior Dirichlet problem II. A problem will be said to be *well-set* for a given equation if for any assigned data there exists exactly one corresponding solution.

It will be observed that problem I is well-set for the Laplace equation in $n \geq 3$ independent variables, while problem II is well-set only for $n = 2$ variables. Such a simple distinction does not hold for the general equation (1), nor even for the special equation

$$(2) \quad \Delta u = a_{ik}u_{ik} = 0.$$

Indeed, there are simple equations of the form (2) in two independent variables for which problem I is well-set, while conversely there are equations in three variables for which problem II is well-set (*cf.* section 9). These examples are all the more remarkable because the operator Δ can in each case be uniformly elliptic. One of the main purposes of this paper is to clarify the rather anomalous behavior indicated by these examples, and to give criteria by which one can predict ahead of time whether problem I or problem II will be well-set for a particular equation. As a byproduct, this investigation has yielded new results even for the equation $\Delta u + cu = f$.

In connection with our work, we must mention certain previous investigations of GIRAUD and KRZYŻAŃSKI. The former author was the first to make a general study of the exterior problem for elliptic equations. His work, although a highly difficult exercise in the integral equation technique, is quite restricted both as to the nature of the operator L and the type of results obtained. KRZYŻAŃSKI's results are more similar to ours in method and outlook, but the specific problems which he treats are not immediately related to those discussed here. Altogether, the exterior Dirichlet problem (important though it may be) seems to be only superficially touched in the literature.

Description of results. Part I of the paper deals specifically with equations of the form (2), these being of considerable interest in their own right. In section 1 the concept of a ϕ -sequence is used to prove the existence of solutions of the exterior Dirichlet problem (certain technical continuity requirements on the coefficients in Δ may be taken for granted at this time). It is then shown that problem I is well-set if and only if there exists a barrier at infinity. The idea here is essentially the same as that used to describe a regular boundary point in potential theory, but its application to the exterior problem seems not to have been explicitly noticed before. In Theorems 2 and 3 we state several

sufficient conditions for the existence of a barrier. Theorem 3, in particular, is so simple that we repeat it here: *If $a_{ik} \rightarrow a_{ik}^0$ as x tends to infinity, and if the matrix (a_{ik}^0) has at least three positive eigenvalues, then there exists a barrier and problem I is well-set.* The Laplace equation with $n \geq 3$ occurs as a special case under this criterion. On the other hand, from this result it also follows that problem I can be well-set for certain non-uniformly elliptic equations.

The results for problem II have a comparable degree of completeness. In particular, this problem is well-set for an equation of the form (2) if and only if the equation enjoys an extended maximum principle. With the help of this result and the concept of an anti-barrier we then obtain certain conditions for problem II to be well-set (Theorems 5-7). An unexpected feature which arises here is the duality between problems I and II, which may be seen by comparing Theorems 1, 2, and 3, respectively, with Theorems 5, 6, and 7.

In the preceding two paragraphs we have discussed conditions under which one or the other of the two exterior problems will be well-set. The question naturally arises, do there exist equations for which neither problem is well-set? In section 3 a partial answer to this question is given: *For a uniformly elliptic equation of the form (2), whose coefficients are continuous at infinity, one or the other of the two problems is well-set. Moreover, among the solutions for varying data must occur every bounded solution. The requirement that the coefficients tend to limits at infinity can be weakened somewhat, and even entirely dropped when $n = 2$.*

Part II of the paper deals with the full elliptic equation (1). For the most part it will be supposed that $c \leq 0$, this requirement being essentially necessary in the problem (cf., however, section 8). The existence and uniqueness theorems obtained for equation (1) are of the same generality as those already discussed in connection with equation (2). The reader is referred particularly to Theorems 10 and 12, which are believed to be new even for the special equation $\Delta u + cu = f$. It should be observed that the added difficulties posed by equation (1), in comparison to equation (2), are due primarily to the strong effect which the function $f(x)$ exerts on the equation.

The simple and elegant Theorem 13 generalizes an interesting result of Yosida and Atkinson concerning the asymptotic behavior of solutions of the equation $\Delta u + cu = 0$, $c \leq -m^2$. Similarly, Theorem 15 has certain aspects in common with a recent theorem of Picone.

In Part III of the paper we present several examples illustrating the various possibilities which can arise in the solution of an exterior Dirichlet problem. These examples form an integral part of the paper, though they also have considerable independent interest. The result of Section 10 concerning the existence of bounded solutions of the Poisson equation may be cited in particular.

The final part of the paper deals with the exterior problem corresponding to Neumann and other non-Dirichlet type data on $F\mathcal{E}$. These problems can be treated in much the same way as the exterior Dirichlet problem, and this is our justification for including the material here. Theorem 16 is concerned with

the exterior problem when a fixed limit zero is assigned at infinity (problem I₀), and Theorem 17 when only a boundedness condition is imposed (problem II). With certain reservations on the kind of data assigned (for which it is best to refer directly to Part IV) the results may be roughly summarized by the statement that the exterior problems for non-Dirichlet data are well-set if and only if they are well-set for Dirichlet data.

Preliminary hypotheses and results. The following assumptions concerning the operators L and Δ will be made throughout the paper:

1°. The coefficients a_{ik} , b_i , and c , and the function f shall be Hölder continuous on bounded subsets of $\mathcal{E} + F\mathcal{E}$.

2°. At each point of $\mathcal{E} + F\mathcal{E}$, and for all real n -tuples $(\xi_1, \dots, \xi_n) \neq 0$,

$$a_{ik}(x)\xi_i\xi_k > 0.$$

By virtue of these hypotheses, equations (1) and (2) will be uniformly elliptic in any bounded part of $\mathcal{E}$. On the other hand, the coefficients *need not* be continuous at infinity and equations (1) and (2) are not necessarily uniformly elliptic over $\mathcal{E}$. One of the purposes of the paper is to avoid these specializing assumptions.

In our existence theorems the required solution will be found by taking limits of solutions defined only on bounded subsets of the given exterior domain $\mathcal{E}$. This procedure is facilitated by the concept of a ϕ -sequence. Let $\{\mathcal{S}_n\}$, $n = 1, 2, 3, \dots$, be a sequence of open balls centered at the origin, with radii r_n tending monotonically to infinity and with $F\mathcal{E} \subset \mathcal{S}_1$. The function sequence $\{u_n\}$ will be called a ϕ -sequence for equation (1) if each function $u_n = u_n(x)$ satisfies (1) in the set $\mathcal{E} \cap \mathcal{S}_n$ and continuously takes on the values $\phi(x)$ on $F\mathcal{E}$.

Lemma 1. *Every locally bounded ϕ -sequence has a subsequence which converges to a solution in $\mathcal{E}$ taking on the boundary values $\phi(x)$.*

Proof. Hypotheses 1° and 2° above allow the application of SCHAUDER's interior estimates to the individual functions u_n . It follows that the sequence $\{u_n\}$ is locally equicontinuous as well as locally bounded, whence one can extract a subsequence which converges uniformly on compact subsets of $\mathcal{E}$ to a solution u . That this solution takes on the boundary values $\phi(x)$ may be proved with the aid of SCHAUDER's boundary estimates, or it may be demonstrated directly by the technique of barriers. We shall omit these rather standard arguments and only observe that, whichever method is employed, some degree of smoothness must be demanded of the boundary $F\mathcal{E}$. When the barrier method is adopted, in particular, it is enough that $F\mathcal{E}$ have bounded curvature. This latter condition will be assumed throughout the paper.

The smoothness assumption on $F\mathcal{E}$ also guarantees the *existence* of ϕ -sequences, according to standard theorems of elliptic partial differential equations. Indeed, one can even preassign the values of u_n on the sphere $r = r_n$, a convenience which we shall occasionally utilize.

For later use, we make the following definition: A function $\delta(t)$, $t_0 \leq t < \infty$, will be called a *Dini function* if and only if the (improper) integral

$$\int^{\infty} \delta(t) \frac{dt}{t}$$

exists and is finite. The functions $t^{-\alpha}$, $(\log t)^{-1-\alpha}$, and $\sin t$, where $\alpha > 0$, are examples of Dini functions.

PART I

1. The exterior problem I. In this and the following two sections we shall deal with the special equation

$$(2) \quad \Delta u = a_{ik} u_{ik} = 0.$$

This is done mainly for simplicity. As we shall see later, many of the results carry over to the general equation (1). We begin with the following simple observation:

There cannot be more than one solution of problem I.

This is an immediate consequence of the maximum principle, since the difference of two solutions would have zero data on $F\mathcal{E}$ and would tend to zero as $r \rightarrow \infty$. Here r denotes, as it will throughout the paper, the distance Ox , that is, $r^2 = \sum_{i=1}^n x_i^2$.

We now introduce the concept of a barrier. A function $v = v(x)$ will be called a *barrier at infinity for the equation $\Delta u = 0$* if (i) v is defined and positive in some neighborhood of infinity, (ii) v tends to zero as $r \rightarrow \infty$, and (iii) v is twice continuously differentiable and $\Delta v \leq 0$.

Property (iii) can be replaced by the following weaker condition: (iii*) v is continuous; if u satisfies $\Delta u = 0$ in some region $\mathcal{R}$ and if $u \leq v$ at every point of $F\mathcal{R}$, then $u \leq v$ throughout $\mathcal{R}$.

Theorem 1. *Problem I is well-set for the equation $\Delta u = 0$ if and only if this equation has a barrier at infinity.*

Proof. In view of the preceding observation on uniqueness, it is enough to show that Problem I is solvable for arbitrary assigned data if and only if there exists a barrier.

But if Problem I is always solvable, let v be that solution with boundary values equal to one and limit at infinity equal to zero. Then v is clearly a barrier.

On the other hand, suppose that v is a barrier. We first extend v so that it is defined over all of $\mathcal{E}$. To this end, let R be a number so large that v is defined for $r \geq R$. Let $v_0 = \min_{r=R} v$ and define

$$(3) \quad v^* = \begin{cases} \min(v_0, v) & \text{for } r \geq R \\ v_0 & \text{for } r < R. \end{cases}$$

It can easily be shown that v^* has the property (iii*).

Now we are in a position to construct a solution u such that $u = \varphi$ on $F\mathcal{E}$ and $\lim u = l$ as $r \rightarrow \infty$. First define

$$\bar{v} = c v^* + l \quad \text{and} \quad v = -c v^* + l,$$

where c is a constant chosen so large that $\bar{v} \geq \varphi$ and $v \leq \varphi$ on $F\mathcal{E}$. Let $\{u_n\}$ be a φ -sequence with $u_n = l$ on $r = r_n$. Because v^* satisfies (iii*), it follows that

$$(4) \quad \bar{v} \geq u_n \geq v.$$

By Lemma 1, the sequence $\{u_n\}$ determines a solution u in $\mathcal{E}$ with boundary values φ . Finally, since (4) holds for each function u_n , it must also hold for the limit function u , which implies that u tends to l at infinity. This completes the proof.

We turn now to several conditions which will guarantee existence of a barrier. It will be convenient to introduce the auxiliary function

$$\mathcal{Q} = \mathcal{Q}(x) = \frac{\sum a_{ik} x_k x_k}{\sum a_{ik} x_i x_k} = \frac{\sum a_{ii}}{Q},$$

where $Q \equiv (1/r^2) \sum a_{ik} x_i x_k$. Since $\sum a_{ii}$ is the sum of the eigenvalues of the matrix (a_{ik}) , while Q is a value of the quadratic form associated with (a_{ik}) , the function $\mathcal{Q}$ is evidently an orthogonal (but not an affine) invariant under coordinate changes in E^m . In general we have $1 < \mathcal{Q} < \infty$, while for uniformly elliptic equations $\mathcal{Q}$ is bounded away from both 1 and ∞ . In the special case of Laplace's equation, $\mathcal{Q} \equiv n$; in view of later theorems the reader may find it convenient to look upon $\mathcal{Q}$ as the "effective dimension" of the equation $\Delta u = 0$.

Theorem 2. *If $\mathcal{Q} \geq 2 + \epsilon$ in some neighborhood of infinity, where $\epsilon = \epsilon(r)$ is such that*

$$(5) \quad \Delta(t) \equiv \exp \left\{ - \int^t \epsilon(s) \frac{ds}{s} \right\}$$

is a Dini function, then problem I is well-set for the equation $\Delta u = 0$.

Proof. We shall find a barrier which is a function of r alone. To this end, consider a function $v = v(r)$ such that $v' < 0$. Then

$$(6) \quad \Delta v \equiv Q \left[v'' + \frac{v'}{r} (\mathcal{Q} - 1) \right] \leq Q \left[v'' + \frac{v'}{r} (1 + \epsilon) \right].$$

The particular function

$$v = \int_r^\infty \exp \left\{ - \int^t \epsilon(s) \frac{ds}{s} \right\} \frac{dt}{t}$$

makes the final expression in (6) equal to zero, and obviously satisfies $v' < 0$. It follows immediately that v is a barrier.

Remarks. Theorem 2 is sharp; for given any $\epsilon(r) > -1$ such that

$\exp \{-\int^t (\epsilon(s)/s)ds\}$ is *not* a Dini function, there exists an equation with $\alpha = 2 + \epsilon$ for which problem I is *not* well-set (section 9). However, the real importance of condition (5) does not lie in this result, but rather in the fact that it can be satisfied even in two dimensions by uniformly elliptic equations (section 9). Hence the scope of problem I is by no means restricted to higher dimensions. Furthermore, in all dimensions the condition allows certain types of non-uniform ellipticity which can become more pronounced as the dimension increases. The following geometric discussion, together with Theorem 3, justifies this statement.

Let us replace the condition $\alpha \geq 2 + \epsilon$ by the stronger, though geometrically more illuminating, condition

$$(7) \quad \frac{\sum a_{ii}}{\lambda} \geq 2 + \epsilon,$$

where $\lambda = \lambda(x) =$ largest eigenvalue of $(a_{ik}(x))$. In the discussion we shall also assume that the coefficients are normalized so that $\sum a_{ii} \equiv 1$, whence (7) takes the simple form

$$(8) \quad \lambda \leq \frac{1}{2 + \epsilon}.$$

Now let H denote the $(n - 1)$ -dimensional set in R^n which is the convex hull of those points having one coordinate equal to one and all others equal to zero. Moreover, let $g(x)$ be the mapping from $\mathcal{E}$ to H which takes x into those points of H whose coordinates are the eigenvalues of $(a_{ik}(x))$, in any order. Since Δ is an elliptic operator, it is clear that $g(x)$ cannot lie on any face of H . Moreover, Δ is uniformly elliptic if and only if the closure of $g(\mathcal{E})$ has no points in common with a face of H .

Let H' denote the convex hull of those points in R^n which have two coordinates equal to $\frac{1}{2}$ and all others zero. If $g(r)$ denotes the image of points for which $\sum x_i^2 = r^2$, condition (8) will be fulfilled if $g(r)$ is contained in the interior of H' and does not approach too rapidly those $(n - 2)$ -dimensional faces where one coordinate equals $\frac{1}{2}$. The situation is easily visualized in the case $n = 3$ (see Figure 1). The case $n = 4$ can also be illustrated if we use barycentric coordinates in the 3-dimensional set H . Since the vertices of H are equidistant in R^4 , it follows that H can be represented as a tetrahedron in R^3 , and that H' is an octahedron inscribed in H (Figure 2). These remarks can be summarized by the statement that if $g(E)$ is interior to H then (2) is elliptic, while if $g(\infty)$ is in H' (and does not approach certain faces of H' too rapidly) then problem I is well-set.

Since the quantity α is not an affine invariant, it is clear that the decision as to whether Theorem 2 applies to a given equation is related to the particular coordinates used to represent the operator Δ . The following theorem, although somewhat weaker in its result, is free from this defect.

Figure 1. The sets H and H' as subsets of R^3 , (H' shaded).

Theorem 3 (*Three eigenvalue criterion*). If $(a_{ik}) \rightarrow (a_{ik}^0)$, $r \rightarrow \infty$, and if (a_{ik}^0) has at least three positive eigenvalues (equivalently, if $\text{Rank}(a_{ik}^0) \geq 3$), then problem I is well-set for the equation $\Delta u = 0$.

Proof. Let ν be the number of positive eigenvalues of (a_{ik}^0) , $3 \leq \nu \leq n$. By means of an affine coordinate transformation the matrix (a_{ik}^0) can be put in diagonal form with ones and zeros on the diagonal. In this new system, $\lim \sum a_{ii} = \nu$ and $\limsup Q \leq 1$ as $r \rightarrow \infty$. Thus $\liminf A \geq \nu$, and we can apply Theorem 2 with ϵ any constant satisfying $0 < \epsilon < \nu - 2$. The corresponding barrier has the form $v = \text{Const. } r^{-\epsilon}$, as is evident from formula (7).

Remark. An estimate for the asymptotic behavior of a solution of the exterior Dirichlet problem I can be obtained from the relation

$$u - l = O(v)$$

where v is an arbitrary barrier function (see the proof of Theorem 1). Thus, under the hypotheses of Theorem 3, a solution of the exterior problem I has the asymptotic behavior

$$u - l = O(r^{2-\nu+\delta}),$$

in which ν is the rank of (a_{ik}^0) and δ is an arbitrary positive constant. This result is a generalization of Theorem 3' of [2], insofar as the hypotheses on the coefficients a_{ik} are concerned.

Figure 2. The relation between the sets H and H' in the case $n = 4$, (H' shaded).

2. The exterior problem II. As in the preceding section we confine our attention to the special equation $\Delta u = 0$. To begin with, we have the simple result:

For an equation of the form $\Delta u = 0$, problem II is always solvable.

This fact, which is dual to the observation made at the beginning of section 1, may be proved as follows. Let $\{u_n\}$ be a φ -sequence such that $u_n = 0$ on $r = r_n$. Obviously $|u_n|$ is bounded by $\max |\varphi|$, and the result follows from Lemma 1.

Theorem 4. *Problem II is well-set for the equation $\Delta u = 0$ if and only if this equation enjoys the following extended maximum principle: For every bounded solution u defined on a set $r \geq R$,*

$$(9) \quad \limsup_{r \rightarrow \infty} |u| \leq \max_{r=R} |u|.$$

Proof. Suppose first that the equation $\Delta u = 0$ satisfies the extended maximum principle. Then there can be at most one solution of problem II, as one sees from a simple application of (9) and the ordinary maximum principle. Consequently problem II is well-set.

On the other hand, suppose there exists a bounded solution w which violates the extended maximum principle, i.e.,

$$\limsup_{r \rightarrow \infty} |w| > \max_{r=R} |w|.$$

From the ordinary maximum principle, it follows that R can be chosen so large that the region $r > R$ is contained in E . We will now show that problem II has more than one solution for zero data, from which the theorem immediately follows.

Define w^* by means of a φ -sequence based on the region $r > R$ and the boundary values $\varphi = w$ on $r = R$; we may further suppose that $w_n^* = 0$ on $r = r_n$. Evidently

$$\limsup_{r \rightarrow \infty} |w^*| \leq \max_{r=R} |w|,$$

whence $v = w - w^*$ is a bounded solution of our equation such that

$$(10) \quad v = 0 \quad \text{on} \quad r = R \quad \text{and} \quad \limsup_{r \rightarrow \infty} |v| > 0.$$

Now consider a φ -sequence in $\mathcal{E}$ with $\varphi = 0$ on $F\mathcal{E}$ and with $u_n = \sup |v|$ on $r = r_n$. (10) and the maximum principle imply that $u_n \geq |v|$. Therefore $u \geq |v|$. Thus $u = 0$ on $F\mathcal{E}$ and u is not the zero solution, thereby completing the proof.

The next problem is to formulate conditions under which the equation $\Delta u = 0$ will enjoy an extended maximum principle. To this end, let us say that a function $v = v(x)$ is an *anti-barrier*¹ at infinity for the equation $\Delta u = 0$ if (i) v tends to $+\infty$ as $r \rightarrow \infty$ and (ii) v is twice continuously differentiable and $\Delta v \leq 0$.

Theorem 5. *Problem II is well-set for the equation $\Delta u = 0$ if this equation has an anti-barrier at infinity.*

It suffices to show that $\Delta u = 0$ satisfies the extended maximum principle. The proof is exactly the same as that of Theorem 1 of reference [2] and will therefore be omitted. Just as Theorems 4 and 5 are dual to Theorem 1, so is the following result dual to Theorem 2.

Theorem 6. *If $\alpha \leq 2 + \epsilon$ in some neighborhood of infinity, where $\epsilon = \epsilon(r)$ is such that*

$$\Delta(t) \equiv \exp \left\{ - \int^t \epsilon(s) \frac{ds}{s} \right\}$$

is not a Dini function, then problem II is well-set for the equation $\Delta u = 0$.

Proof. We shall determine an anti-barrier which is a function of r alone. To this end, consider a function $v = v(r)$ satisfying $v' > 0$. Then

$$(11) \quad \Delta v = Q \left[v'' + \frac{v'}{r} (\alpha - 1) \right] \leq Q \left[v'' + \frac{v'}{r} (1 + \epsilon) \right].$$

The particular function

$$(12) \quad v = \int^r \exp \left\{ - \int^t \epsilon(s) \frac{ds}{s} \right\} \frac{dt}{t}$$

makes the final expression in (11) equal to zero, and obviously has the property $v' > 0$. Thus $\Delta v \leq 0$. Moreover, by hypothesis the integral (12) tends to infinity as $r \rightarrow \infty$. It follows, then, that v is an anti-barrier, and the proof is complete.

Theorem 7. *Consider the equation $\Delta u = 0$ in $n = 2$ variables. If $a_{ik} \rightarrow \delta_{ik}$ as $r \rightarrow \infty$ according to a Dini modulus, then problem II is well-set.*

¹ The concept of an anti-barrier appears also in references [4] and [7].

Proof. From the hypothesis it is clear that $\alpha \leq 2 + \epsilon$ where ϵ is a Dini function. But if ϵ is a Dini function then Δ is not. Thus Theorem 6 applies, and problem II is well-set. The restriction on the rate at which $a_{ik} \rightarrow \delta_{ik}$ is essential, as shown by an example in section 9.

Remark. Although not immediately related to the subject of the paper, it may be of interest to point out another property of operators Δ which possess an anti-barrier at infinity, namely the following Liouville theorem: *Let $u = u(x, y)$ be defined and non-negative in E^n , and suppose that $\Delta u \leq 0$, where Δ has an anti-barrier at infinity. Then $u \equiv \text{constant}$.*

The proof rests on the observation that u must satisfy an extended maximum principle,

$$(9') \quad \liminf_{r \rightarrow \infty} u \geq \min_{r=R} u.$$

This inequality is proved in the same way as Theorem 5. Now if (9') holds and u is defined throughout E^n , then certainly u takes on its minimum at some *finite* point. The ordinary maximum principle then implies that u constant, and the assertion is proved.

For equations $\Delta u = 0$ which do not possess an anti-barrier a Liouville theorem still may be valid, but the result lies much deeper (cf. reference [2]).

3. Uniformly elliptic equations. Under the assumption that the operator Δ is uniformly elliptic, certain further conclusions can be drawn. The main additional tool which is necessary is the following limit theorem for uniformly elliptic equations ([2], Theorem 3):

Let Δ be a uniformly elliptic operator, and, if $n \geq 3$, assume that the coefficients a_{ik} tend to limits at infinity. Let $u = u(x)$ be a bounded solution of $\Delta u = 0$ in $r \geq R$. Then u tends to a limit at infinity.

It is an immediate corollary of this theorem that if Δ satisfies the above hypotheses, then all possible bounded solutions of $\Delta u = 0$ will be obtained by varying the assigned data in either exterior problem I or exterior problem II. Moreover, under the above hypotheses one or the other of these problems must be well-set. For $n \geq 3$ this fact is nothing more than a special case of Theorem 3; the result for $n = 2$, however, is of separate interest:

Theorem 8. *Suppose that $n = 2$, and that Δ is uniformly elliptic. Then either problem I or problem II is well-set for the equation $\Delta u = 0$.*

Proof. If problem II is *not* well-set, then for some set of boundary values ϕ there are two distinct solutions u and v . Clearly u and v tend to distinct limits u_0 and v_0 at infinity. Letting u_0 be the larger limit, we see that the function $(u_0 - v_0) - (u - v)$ is a barrier at infinity. Problem I must therefore be well-set.

The results of Theorems 1, 4, 5, and 8 may be summarized in the following diagram, where the letter "U" placed over an arrow denotes that the proof makes use of the limit theorem for uniformly elliptic equations.

PART II

4. The general homogeneous equation. We consider briefly in this section the equation

$$(13) \quad Lu = 0 \quad (c \leq 0).$$

The results are not radically different from those already stated, except that, insofar as exterior problem I is concerned, *we require that the limit l be zero.* Non-zero limiting values are more naturally treated in connection with the general inhomogeneous equation.

With the proviso noted above, one sees easily that Theorems 1, 4, and 5 remain true for equation (13). The sufficient conditions for problems I and II, which are stated in Theorems 2 and 6, respectively, are likewise valid if the function α is replaced by α^* , where

$$(14) \quad \alpha^* = \frac{\sum (a_{ii} + b_i x_i)}{Q}.$$

Analogues of Theorems 3 and 7 can also be stated but these require additional restrictions on the growth of the coefficients at infinity.

5. The inhomogeneous equation. In this and the following sections we shall consider the full equation

$$(15) \quad Lu = f \quad (c \leq 0),$$

and we shall suppose that (15) is subject to the normalization

$$(16) \quad Q = \frac{1}{r^2} \sum a_{ik} x_i x_k = 1.$$

This normalization can be accomplished without loss of generality provided the origin is not in $\mathcal{E} + F\mathcal{E}$; the latter condition will be assumed henceforth. The restriction noted for c is more or less inherent in the problem, though in certain cases it can be lightened somewhat. This aspect of the problem will be put off to a later section. So far as problem I is concerned, it is sufficient to study the special case where the limit value l is zero, for the dependent variable $w = u - l$ satisfies an equation of the form (15) and tends to zero at infinity. Exterior problem I with the assigned limit $l = 0$ will be referred to as problem I_0 . We shall require the following simple Sturmian-type lemma.

Lemma 2. Consider the ordinary differential equation

$$\psi'' + B(t)\psi' + C(t)\psi = F(t), \quad a \leq t < b,$$

where B, C, F are continuous functions satisfying $C \leq 0, F \geq 0$. If $\psi(a) = \psi'(a) = 0$, then $\psi, \psi' \geq 0$ for $a \leq t < b$.

Proof. Consider the equation

$$(17) \quad \psi_n'' + B(t)\psi_n' + C(t)\psi_n = F(t) + \frac{1}{n}$$

with the initial conditions $\psi_n(a) = \psi_n'(a) = 0$. From (17) we have $\psi_n''(a) \geq 1/n$, implying that ψ_n is positive in a neighborhood of a . Now let $a < t \leq c$ be any interval in which $\psi_n > 0$. Then since

$$\psi_n'' + B\psi_n' = F + \frac{1}{n} - C\psi_n = G > 0,$$

we have

$$\psi_n' = \frac{1}{H(t)} \int_a^t G(s)H(s) ds, \quad a \leq t \leq c,$$

where $H(t) = \exp \{ \int B(t)dt \}$. Evidently $\psi_n' > 0$ in $a < t \leq c$, whence, by an obvious argument $\psi_n, \psi_n' > 0$ in the whole interval $a < t < b$. Letting $n \rightarrow \infty$ we get $\psi_n \rightarrow \psi, \psi_n' \rightarrow \psi'$, from which the conclusion of the lemma follows.

Lemma 3. Suppose that f vanishes in a neighborhood of infinity. Then if problem I_0 is well-set for $Lu = 0$ ($c \leq 0$), it is well-set for $Lu = f$.

Proof. Since problem I_0 is well-set for the homogeneous equation, it is sufficient to consider only the case where $\varphi \equiv 0$ on $F\mathcal{E}$.

Let R be so large that $f = 0$ for $r \geq R$. Also choose functions $B = B(r)$ and $C = C(r)$ such that

$$B \geq \frac{1}{r} (\alpha^* - 1), \quad C \leq c,$$

where α^* is defined by (14). Finally, let $F = F(r)$ be a continuous function satisfying $F \geq |f|$ and $F = 0$ for $r \geq R$. An easy computation shows that if $\psi = \psi(r) \geq 0, \psi' \leq 0$, and

$$(18) \quad \psi'' + B\psi' + C\psi = F,$$

then $L(\psi) \geq |f|$. From here on let ψ denote the particular solution of (18) such that $\psi(R) = \psi'(R) = 0$. Then $\psi \equiv 0$ for $r \geq R$, while

$$\psi \geq 0, \quad \psi' \leq 0 \quad \text{for } r \leq R,$$

the inequalities following from Lemma 2.

Now let $\{u_n\}$ be a φ -sequence for $Lu = f$ with $\varphi = 0$ on $F\mathcal{E}$. It may be assumed that $r_n \geq R$ for all n , and also that $u_n = 0$ on $r = r_n$. Since $L\psi \geq |Lu_n|$,

$$\psi \pm u_n \leq \max_{F\mathcal{E}} \psi \quad \text{in} \quad \mathcal{E} \cap (r \leq r_n).$$

Furthermore, because $\psi \geq 0$ it follows that $|u_n| \leq \max_{F\mathcal{E}} \psi$. In other words, the sequence $\{u_n\}$ is uniformly bounded. By Lemma 1, a subsequence $\{u_m\}$ converges to a solution u of $Lu = f$ with $u = \varphi = 0$ on $F\mathcal{E}$. It remains only to show that $u \rightarrow 0$ as $r \rightarrow \infty$.

To do this, choose a barrier v for the homogeneous equation such that

$$v \geq 1 + |u| \quad \text{on} \quad r = R.$$

This can certainly be done, since problem I_0 is well-set for $Lu = 0$. For all but a finite number of indices m , we infer that

$$|u_m| \leq v, \quad R \leq r \leq r_m.$$

Letting $m \rightarrow \infty$ yields $|u| \leq v$ for $r \geq R$, and hence u tends to zero as $r \rightarrow \infty$.

The concept of a barrier is again fundamental in treating problem I_0 , though it assumes a slightly different form. We shall say that a pair of functions $\bar{v}$ and $\underline{v}$ constitutes a *barrier at infinity for equation (15)* if and only if (i) $\bar{v} > \underline{v}$ in some neighborhood of infinity, (ii) $\bar{v}, \underline{v}$ tend to zero as $r \rightarrow \infty$, and (iii) $\bar{v}, \underline{v}$ are twice continuously differentiable and $L\bar{v} \leq f$, $L\underline{v} \geq f$.

Theorem 9. *Problem I_0 is well-set for equation (15) if and only if this equation has a barrier at infinity.*

Remark. In proving Theorem 9 the principle difficulty to be overcome is that, unlike the homogeneous case, there appears to be no simple method for extending the barrier to all of $\mathcal{E}$. But for this, the proof of Theorem 1 would carry over to Theorem 9 more or less intact. Lemma 3 is designed to circumvent this difficulty.

Proof of Theorem 9. If problem I_0 is well-set, then we can easily construct a barrier by the method of Theorem 1.

On the other hand, problem I_0 obviously cannot have more than one solution, so that it remains only to show that if equation (15) has a barrier, then problem I_0 is solvable. Suppose then that the given barrier functions are defined for $r \geq \frac{1}{2}R$. By the methods of Lemma 1 we can easily construct a function w such that

$$\underline{v} \leq w \leq \bar{v}, \quad Lw = f \quad (r > \tfrac{1}{2}R).$$

We leave the details to the reader.

Now let w^* be a C^2 function defined over $\mathcal{E} + F\mathcal{E}$ such that $w^* \equiv w$ for $r \geq R$; and introduce into the equation (15) the new dependent variable $u^* = u - w^*$. Evidently $Lu^* = g$, where $g \equiv 0$ for $r \geq R$. Problem I_0 is well-set for the equation $Lu^* = 0$, since $v = \bar{v} - \underline{v}$ constitutes a barrier (see section 4). Therefore, by Lemma 3, problem I_0 is solvable for $Lu^* = g$. Thus it must also be solvable for equation (15), completing the proof.

The following theorem is a direct analogue of Theorem 2.

Theorem 10. *If in some neighborhood of infinity*

$$\alpha^* \geq 2 + \epsilon \quad \text{and} \quad r^2|f| \leq \delta(r),$$

where ϵ is a positive constant and $\delta(r)$ is a Dini function, then problem I_0 is well-set for equation (15).

Proof. Suppose $v = v(r)$ and $v > 0$, $v' \leq 0$. Then by (14) and (16)

$$Lv \equiv v'' + \frac{v'}{r} (\alpha^* - 1) + cv \leq v'' + \frac{v'}{r} (1 + \epsilon),$$

since $c \leq 0$. Now consider the particular function

$$v = \int_r^\infty t^{-1-\epsilon} \int_1^t s^{-1+\epsilon} \delta(s) ds dt.$$

The fact that the integral converges follows by interchanging orders of integration; thus

$$v = \frac{1}{\epsilon} \left[r^{-\epsilon} \int_1^r s^\epsilon \delta(s) \frac{ds}{s} + \int_r^\infty \delta(s) \frac{ds}{s} \right].$$

In these formulas we have tacitly assumed δ to be defined over the interval $1 \leq r < \infty$, and we shall also suppose $\delta > 0$. Both these conditions can be imposed without loss of generality. It is now easily verified that

$$v > 0, \quad v' < 0, \quad v'' + \frac{v'}{r} (1 + \epsilon) = -\frac{\delta}{r^2},$$

while also $v \rightarrow 0$ as $r \rightarrow \infty$. The function pair $\bar{v} = v(r)$, $\underline{v} = -v(r)$ thus constitutes a barrier at infinity for equation (15).

Corollary. *If in some neighborhood of infinity,*

$$\alpha^* \geq 2 + \epsilon, \quad r^2(|c| + |f|) \leq \delta(r),$$

where ϵ is a positive constant and $\delta(r)$ is a Dini function, then problem I is well-set for equation (15).

That the condition imposed on f in the above theorem and corollary is sharp is shown by the example of section 10.

6. The inhomogeneous equation. Problem II. Uniqueness for the inhomogeneous equation is equivalent to uniqueness for the homogeneous equation; the relevant facts are found in section 4. The existence of solutions of Problem II is, however, less trivial.

Theorem 11. *Problem II is solvable for equation (15) if and only if there exist functions $\bar{w}$ and w which in some neighborhood of infinity are bounded, of class C^2 , and satisfy $Lw \geq f \geq L\bar{w}$.*

Proof. The argument is almost a duplication of the proof of Theorem 9. To begin with, by the method of Lemma 3 we find that problem II is solvable whenever f vanishes in some neighborhood of infinity. The general case is then reduced to this one by the method of Theorem 9, with the functions $\bar{w}$ and w playing the roles of $\bar{v}$ and v . The “barrier” functions $\bar{w}, w$ which enter explicitly into the solution of problem II for inhomogeneous equations are implicitly present as constants in the homogeneous case.

The next result is a direct analogue of Theorem 6.

Theorem 12. *If*

$$\alpha^* \leq 2 + \epsilon(r), \quad r^2|f| \leq \delta(r),$$

where $\epsilon(r)$ and $\delta(r) \log r$ are Dini functions, then problem II is well-set for equation (15).

Proof. In order to show that problem II has a solution we must construct functions $\bar{w}$ and w satisfying the conditions of the preceding theorem.

Consider the function

$$w = \int_1^r \Delta(t) \left(\int_t^\infty \frac{\delta(s)}{\Delta(s)} \frac{ds}{s} \right) \frac{dt}{t}, \quad r \geq 1,$$

where $\Delta(t)$ is defined by (5). From the hypothesis on ϵ , it is clear that there exist positive constants a and b such that $a \leq \Delta(t) \leq b, t \geq 1$. The inner integral is therefore convergent, and w is well defined. Moreover, w is a bounded function,

$$w \leq \frac{b}{a} \int_1^\infty \left(\int_t^\infty \delta(s) \frac{ds}{s} \right) \frac{dt}{t} = \frac{b}{a} \int_1^\infty \delta(s) \log s \frac{ds}{s}.$$

Finally, since $w \geq 0, w' \geq 0$ we have

$$Lw = w'' + \frac{w'}{r} (\alpha^* - 1) + cw \leq w'' + \frac{w'}{r} (1 + \epsilon) = -\delta(r)r^{-2} \leq -|f|.$$

Assuming, as we may, that $\delta > 0$, it follows that $\bar{w} = w, w = -w$ satisfy the conditions of Theorem 11. Thus problem II is solvable.

On the other hand, problem II has at most one solution since the homogeneous equation $Lu = 0$ admits only one solution whenever $\Delta(t)$ is not a Dini function (see section 4). But if $\epsilon(r)$ is a Dini function, then certainly $\Delta(t)$ is not.

Corollary. *Consider equation (15) in $n = 2$ variables. Suppose*

$$a_{ik} \rightarrow \delta_{ik}, \quad rb_i \rightarrow 0, \quad r^2f \rightarrow 0$$

as $r \rightarrow \infty$, the convergence in each case being governed by a Hölder modulus. Then the exterior Dirichlet problem II is well-set.

Even for the equation $\Delta u = f$ this corollary seems to be new. The conditions of Theorem 12 are best possible, as shown by the last example of section 10.

7. The special case $c \leq -m^2 < 0$. In the quite special case that $c(x)$ is bounded away from 0, certain more precise conclusions can be drawn (for previous work on this case, see references [1], [8]). We begin with the following uniqueness theorem, of interest even outside the context of the paper.

Theorem 13. *Let u be a solution of the homogeneous equation (15) in $\mathcal{E}$, with $u = 0$ on $F\mathcal{E}$. Suppose that in some neighborhood of infinity $M \leq \mathcal{Q}^* \leq N$, $c \leq -m^2$, where M , N , and m are constants ($m > 0$). Under these conditions, if*

$$u = o(r^{\frac{1}{2}(1-N)}e^{mr}) \quad \text{as } r \rightarrow \infty,$$

then $u \equiv 0$ in $\mathcal{E}$.

Proof. A simple computation establishes that

$$\begin{aligned} L(r^p e^{mr}) &= e^{mr}[(m^2 + c)r^p + m(2p + \mathcal{Q}^* - 1)r^{p-1} + p(p + \mathcal{Q}^* - 2)r^{p-2}] \\ &\leq e^{mr}[m(2p + N - 1)r^{p-1} + O(r^{p-2})]. \end{aligned}$$

Consequently

$$L[r^{\frac{1}{2}(1-N)}e^{mr}(1 + r^{-\frac{1}{2}})] \leq -mr^{-\frac{1}{2}(2+N)}e^{mr}[1 + O(r^{-\frac{1}{2}})],$$

and the latter expression is negative for r suitably large. The function

$$v = r^{\frac{1}{2}(1-N)}e^{mr}(1 + r^{-\frac{1}{2}})$$

is therefore an anti-barrier. Hence any function which is $o(v)$ satisfies the extended maximum principle (see reference [2], Theorem 1). Since $u = o(v)$ and has zero boundary data, it follows that $u \equiv 0$ in $\mathcal{E}$.

Theorem 13 allows us to establish conditions under which problem II is well-set. In particular, we have

Theorem 14. *If in some neighborhood of infinity $\mathcal{Q}^*$ and f are bounded and $c \leq -m^2 < 0$, then problem II is well-set for equation (15).*

Proof. From Theorem 13, there cannot be more than one solution. On the other hand, taking $\bar{w}$ and w to be constants, by Theorem 11 there exists a solution of problem II.

8. Relaxing the condition $c \leq 0$. As we have remarked earlier, the condition $c \leq 0$ is more or less essential in deriving the conclusions of this paper. It may happen, however, that by a transformation of the dependent variable our equation can be changed from a form in which c assumes positive values into a form in which $c \leq 0$. In such a case, the preceding work can be used to investigate the exterior Dirichlet problem. Theorem 15, which is an extension of Theorem 10, bears upon this problem.

Theorem 15. *Consider equation (15), except that the condition $c \leq 0$ is now dropped. Suppose that*

$$\mathcal{Q}^* \geq 2 + \epsilon, \quad r^2 c \leq \frac{\alpha(\epsilon - \alpha)}{1 + \beta r^\alpha}, \quad r^2 |f| \leq \delta(r)$$

where ϵ, α, β are positive constants, $\alpha \leq \epsilon$, and $\delta(r)$ is a Dini function. Then problem I_0 is well-set.

Proof. (By the same method it is possible to extend Theorem 12, but we have omitted this result because of its awkward statement.) Introduce the new dependent variable u' by means of the substitution

$$u = (1 + Ar^{-\alpha})u', \quad A = \frac{1}{\beta}.$$

Then $L'u' = f'$, where $a'_{ik} = a_{ik}$,

$$b'_i = b_i - 2\alpha r^{-2} a_{ik} x_k (1 + \beta r^\alpha)^{-1}, \quad f' = (1 + Ar^{-\alpha})^{-1} f,$$

and

$$r^2 c' = r^2 c - \alpha(\mathcal{Q}^* - 2 - \alpha)(1 + \beta r^\alpha)^{-1} \leq 0.$$

In addition, for r sufficiently large,

$$\mathcal{Q}^{*'} = \mathcal{Q}^* - 2\alpha(1 + \beta r^\alpha)^{-1} > 2 + \frac{1}{2}\epsilon,$$

while $|f'| \leq \text{const. } \delta(r)r^{-2}$. Theorem 10 therefore applies to $L'u' = f'$, and consequently problem I_0 is well-set for $Lu = f$.

Corollary. If the condition on c in Theorem 15 is replaced by the hypothesis that $\mathcal{E}$ is contained in the exterior of a sphere of radius R and if

$$r^2 c < \alpha(\epsilon - \alpha) \left(\frac{R}{r} \right)^\alpha \quad \text{in } \mathcal{E} + F\mathcal{E},$$

then the conclusion of the theorem remains valid.

Proof. In the set $\mathcal{E} \cap (R < r \leq 2R)$ there exists a constant k , $0 < k < 1$, such that

$$r^2 c < \alpha(\epsilon - \alpha) \left[\left(\frac{R}{r} \right)^\alpha - k \right].$$

Thus in $\mathcal{E}$ we have

$$r^2 c < \frac{\alpha(\epsilon - \alpha)}{1 + \beta r^\alpha}, \quad \beta = (2R)^{-\alpha} \text{Min} \left(\frac{k}{1 - k}, 2^\alpha - 1 \right),$$

and Theorem 15 applies.

PART III

9. Examples. In this and in the following section we present some equations which illustrate the essential sharpness of our results.

Consider the particular equation

$$(19) \quad \left(\delta_{ii} + \frac{g(r)}{r^2} x_i x_i \right) u_{ii} = 0, \quad g > -1,$$

which was first introduced in reference [2]. The eigenvalues of the coefficient matrix are 1 and $1 + g$, the former having multiplicity $n - 1$. Thus (19) is elliptic, and even uniformly elliptic if g is bounded away from -1 and $+\infty$. Furthermore,

$$\alpha = \frac{n + g}{1 + g} = 2 + \epsilon,$$

where

$$\epsilon = \epsilon(r) = \frac{n - 2 - g}{1 + g}, \quad -1 < \epsilon < \infty.$$

It is easily verified that (19) possesses a radially symmetric solution

$$(20) \quad u = u(r) = \int^r \exp \left\{ - \int^t \epsilon(s) \frac{ds}{s} \right\} \frac{dt}{t}.$$

In further considerations the asymptotic behavior of u will be essential. Since $u > 0$, $u' > 0$, it is clear that either

I. $u \rightarrow$ finite limit l , or

II. $u \rightarrow +\infty$ as $r \rightarrow \infty$.

In the first case the function $l - u$ is a barrier, hence the exterior Dirichlet problem I is well-set. In the second case, u itself is an anti-barrier and problem II is well-set. Since these cases are mutually exhaustive, it follows that *for equation (19), either problem I or problem II is well-set depending on whether the function*

$$\Delta(t) = \exp \left\{ - \int^t \epsilon(s) \frac{ds}{s} \right\}$$

is or is not a Dini function. This shows that Theorems 2 and 6 are best possible, as already stated.

In addition, since the exterior problem is shown to be governed only by the asymptotic behavior of $\epsilon(r)$, it is clear that neither the dimension of the underlying space, nor the hypothesis of uniform ellipticity, is of any aid (in itself) in determining which of the problems I or II is well-set.

Finally, we shall show that Theorem 7 is sharp, in that the conditions $a_{ik} \rightarrow \delta_{ik}$, $n = 2$, do not in themselves guarantee that problem II is well-set. Indeed, in equation (19) set $n = 2$ and

$$g = -\frac{2}{2 + \log r}.$$

Then as $r \rightarrow \infty$ we have $g \rightarrow 0$ and $a_{ik} \rightarrow \delta_{ik}$. However, (20) yields

$$u(r) = l + k (\log r)^{-1},$$

which tends to a finite limit. Hence problem I is well-set. It is only when the limit δ_{ik} is attained with sufficient rapidity (say according to a Dini modulus) that problem II will necessarily be well-set. This example may give pause to those who, on the sole basis of the fact $a_{ik} \rightarrow \delta_{ik}$ as $r \rightarrow \infty$, presume that their equation has all the usual qualitative properties of the Laplace equation.

10. The Poisson equation. By means of the Poisson equation we shall now prove the sharpness of the asymptotic conditions imposed on the right hand side, $f(x)$, in Theorems 10 and 12. Consider, then, the equation

$$(21) \quad \Delta u = f(x) \quad (n \geq 3).$$

We know from Theorem 10 that if $r^2|f|$ is bounded by a Dini function, then problem I is well-set. Suppose, however, that

$$(22) \quad r^2 f \geq \psi(r) \geq 0,$$

where $\psi(r)$ is not a Dini function. Then (21) can not have a solution bounded from above, which is defined in a neighborhood of infinity.²

Proof. Let $u = u(x)$ be a solution of (21) defined for $r \geq R$, and suppose $u \leq K < \infty$. Also let

$$G(x, h) = \frac{1}{n-2} \left[\frac{1}{d^{n-2}} - \left(\frac{R}{rd'} \right)^{n-2} \right]$$

be the Green's function for the exterior region $r > R$, where $d = |x - y|$ and $d' = |x' - y|$. We observe that for a fixed x , $r > R$,

$$(23) \quad G(x, y) \geq (\text{pos. const.}) |y|^{2-n} \quad \text{as } |y| \rightarrow \infty.$$

Denote by $\{f_r\}$ a sequence of non-negative Hölder continuous functions converging upward to f and vanishing outside a compact set. Finally, set

$$u_r(x) = K - \frac{1}{\omega} \int_{|y| > R} G(x, y) f_r(y) dy$$

(ω = area of unit sphere). Now $\Delta u_r = f_r \leq f$, and $u_r = K$ when $r = R$, $r = \infty$. It follows from the maximum principle that $u_r \geq u$ everywhere.

From the Lebesgue monotone convergence theorem, however,

$$\lim_{r \rightarrow \infty} \int G(x, y) f_r(y) dy = \int G(x, y) f(y) dy.$$

² This can be phrased alternatively as a result about differential inequalities: If

$$\Delta u \geq r^{-2} \psi(r) \geq 0,$$

where $\psi(r)$ is not a Dini function, then $\limsup_{r \rightarrow \infty} u = +\infty$.

The latter integral is divergent by virtue of (22) and (23); hence, for each fixed x , $u_r(x) \rightarrow -\infty$. But this contradicts the condition $u_r \geq u$, thus completing the proof.

The corresponding situation when $n = 2$ is a little more delicate. Consider the equation

$$(24) \quad \Delta u = f \quad (n = 2).$$

From Theorem 12, if $r^2|f|$ is bounded by $\delta(r)$ and if $\delta \log r$ is a Dini function, then problem II is well-set. *However, if*

$$r^2 f \geq \psi(r) \geq 0,$$

where $\psi \log r$ is not a Dini function, then (24) can have no bounded solutions. If ψ itself is not a Dini function then (24) can have no solutions bounded from above.

Proof. As before, let u be a solution of (24) defined for $r \geq R$ and such that $u \leq K < \infty$. Let

$$G(x, y) = \log \frac{1}{d} - \log \frac{R}{r d'}$$

be the Green's function for $r > R$, and observe that for a fixed x

$$(25) \quad G(x, y) \rightarrow \log \frac{r}{R} \quad \text{as } |y| \rightarrow \infty.$$

With f_r defined as in the preceding proof, set

$$u_r(x) = K - \frac{1}{2\pi} \int_{|y| > R} G(x, y) f_r(y) dy.$$

Then $\Delta u_r = f_r \leq f$ and $u_r = K$ on $r = R$. We now assert that $u_r \geq u$ everywhere (this is no longer an immediate consequence of the maximum principle, as it was before). To prove the assertion set $w = u - u_r$, and observe that $\Delta w \geq 0$ and $w = 0$ on $r = R$. Also w is bounded above, since u_r has some finite lower bound (see equation (26)). It follows then from the maximum principle that

$$w \leq \epsilon \log \frac{r}{R} \quad \text{for any } \epsilon > 0,$$

and this in turn shows that $w \leq 0$, as asserted.

By the Lebesgue theorem,

$$\lim_{r \rightarrow \infty} \int G(x, y) f_r(y) dy = \int G(x, y) f(y) dy.$$

Since for fixed x and large y we have $G \geq (\text{pos. const.})$, the latter integral is divergent whenever ψ is not a Dini function. This proves the second part of the theorem.

To prove the first part, we have from (25) and the symmetry of the Green's function

$$(26) \quad \lim_{r \rightarrow \infty} u_r = K - \frac{1}{2\pi} \int \log \frac{|y|}{R} f_r(y) dy.$$

Since

$$\lim_{r \rightarrow \infty} \int \log \frac{|y|}{R} f_r(y) dy = \int \log \frac{|y|}{R} f(y) dy$$

and $u \leq u_r$, it follows from (26) that

$$(27) \quad \limsup_{r \rightarrow \infty} u \leq K - \frac{1}{2\pi} \int \log \frac{|y|}{R} f(y) dy = -\infty,$$

which implies that u is not bounded.

Notice that we have proved more than was asserted, for (27) implies that u has a limit at infinity. Therefore if $\psi \log r$ is *not* a Dini function and u is bounded from above, $\lim_{r \rightarrow \infty} u = -\infty$.

PART IV

11. Non-Dirichlet type data. Preliminary considerations. The final part of the paper is devoted to the exterior problem corresponding to non-Dirichlet type data assigned on the boundary $F\mathcal{E}$. In order to formulate the boundary conditions which we shall consider, let $\mathbf{v}$ be a field of unit vectors defined on $F\mathcal{E}$ and directed into $\mathcal{E}$. Also let

$$\frac{\partial u}{\partial \nu} = \mathbf{v} \cdot \nabla u \quad x \in F\mathcal{E},$$

be the inward directional derivative of u along $\mathbf{v}$, and let $k(x)$ be a fixed continuous function on $F\mathcal{E}$ with $k(x) \leq 0$. The applied boundary condition is then

$$(28) \quad \frac{\partial u}{\partial \nu} + ku = \varphi$$

where $\varphi = \varphi(x)$ is an arbitrary prescribed function on $F\mathcal{E}$. When $k \equiv 0$ and $\mathbf{v}$ is normal to $\mathcal{E}$, this is the Neumann problem. Other cases are sometimes referred to as the third boundary value problem; we shall not, however, insist on this terminology. As in the case of the exterior Dirichlet problem, it suffices to consider here the following two conditions on the behavior of a solution at infinity:

I₀. *The solution u shall tend to the limit zero as x tends to infinity.*

II. *The solution u shall be bounded in $\mathcal{E}$.*

These problems are conveniently designated as the exterior problem I₀ and the exterior problem II; no misunderstanding should result from the same nomenclature having been adopted already for the Dirichlet problem.

Now let $\{S_n\}$ be a sequence of open balls centered at the origin, with radii r_n tending monotonically to infinity and with $F\mathcal{E} \subset S_1$. The function sequence u_n will be called a φ -sequence if each function $u_n = u_n(x)$ satisfies (15) in $\mathcal{E} \cap S_n$ and (28) on $F\mathcal{E}$. Under suitable technical continuity conditions which will be given shortly, we have the following result, analogous to Lemma 1.

Lemma 4. *Every locally bounded φ -sequence has a subsequence which converges to a solution in $\mathcal{E}$ satisfying the boundary condition (28).*

Proof. The fact that a subsequence $\{u_n\}$ exists which converges uniformly on compact subsets of $\mathcal{E}$ to a solution u follows exactly as in Lemma 1. The only difficulty lies in showing that u satisfies (28), and this necessarily requires some degree of smoothness of $F\mathcal{E}$, as well as of the vector field $\mathbf{v}$, the function k , and the data φ . If, firstly, the local representations of $F\mathcal{E}$ are of class $C^{2+\lambda}$, and $\mathbf{v}$, k , and φ are in $C^{1+\lambda}$ on $F\mathcal{E}$, then standard Schauder-type boundary estimates imply that the second derivatives of u are equicontinuous in $(\mathcal{E} + F\mathcal{E}) \cap S_1$. From this it is immediate that u satisfies (28). The result can, however, be obtained from less strict assumptions, as we shall now show.

Suppose in fact that the frontier $F\mathcal{E}$ has bounded curvature,³ so that E. HOPF's boundary point maximum principle will hold. In addition, denoting by S an arbitrary open ball containing $F\mathcal{E}$, let us assume that the mixed problem

$$(29) \quad \begin{cases} Lu = f & \text{in } \mathcal{E} \cap S \\ \frac{\partial u}{\partial \nu} + ku = \varphi & \text{on } F\mathcal{E}, \quad u = \psi & \text{on } FS \end{cases}$$

is solvable for arbitrary functions ψ of class C^2 on FS . Sufficient conditions for this last assumption to hold are that $\mathbf{v}$, k , and φ should be in C^λ on $F\mathcal{E}$ (see [5], §23). To show now that u satisfies (28), let w be the solution of (29) such that $\psi \equiv u$ on FS . Setting $v = u_n - w$ we obtain $Lv = 0$ in $\mathcal{E} \cap S$, and

$$\frac{\partial v}{\partial \nu} + kv = 0 \quad \text{on } F\mathcal{E}, \quad v = u_n - u \quad \text{on } FS.$$

It follows then from the boundary point principle that $|v| \leq \text{Max}_{FS} |u_n - u|$, whence $\lim |u_n - w| = 0$. Thus $u \equiv w$, and condition (28) is certainly fulfilled.

The smoothness assumptions on $F\mathcal{E}$ and on $\mathbf{v}$, k , and φ mentioned in the foregoing proof clearly guarantee the *existence* of φ -sequences, and, as in the case of the Dirichlet problem, one can even preassign the values of u_n on the sphere $r = r_n$. It will be tacitly assumed in the remainder of the paper that the above conditions hold.

12. The exterior problem I_0 . The theorem of this section gives a necessary and sufficient condition for the exterior problem I_0 to be well-set for the in-

³ This is stronger than necessary (as it was also in the case of the Dirichlet problem), but it is in the spirit of our approach not to place stress on the technical details of best possible boundary hypotheses.

homogenous equation (15). Theorems 2, 3, and 10 can then be applied if one wishes specific criteria for a particular equation.

Theorem 16. *The exterior problem I_0 is well-set for the equation $Lu = f$ ($c \leq 0$) and the boundary condition*

$$Mu = \frac{\partial u}{\partial \nu} + ku = \varphi$$

if and only if the exterior Dirichlet problem I_0 is well-set.

Proof. The uniqueness of the solution is an immediate consequence of the boundary point principle and the ordinary maximum principle. It suffices then to show that problem I_0 is solvable if and only if there exists a barrier.

To begin with, suppose problem I_0 is solvable, and let $\bar{v}$ and $\underline{v}$ be solutions corresponding to $\varphi = 1$ and $\varphi = -1$, respectively. From the boundary point lemma it follows that $\bar{v} > \underline{v}$, and therefore the functions $\bar{v}$ and $\underline{v}$ constitute a barrier (see the definition stated prior to Theorem 9).

On the other hand, suppose that a barrier at infinity exists. Then by Theorem 9 the exterior Dirichlet problem I_0 is solvable. Let w_1 be the solution such that

$$Lw_1 = f \quad \text{in } \mathcal{E}, \quad w_1 = 0 \quad \text{on } F\mathcal{E},$$

and let w_2 be the solution defined by

$$Lw_2 = 0 \quad \text{in } \mathcal{E}, \quad w_2 = 1 \quad \text{on } F\mathcal{E}.$$

Evidently $0 < w_2 < 1$ in $\mathcal{E}$. Moreover $Mw_2 < 0$, so that there exists a positive constant C with the property

$$C \cdot Mw_2 \leq -|\varphi - Mw_1|.$$

Now let $\{u_n\}$ be a φ -sequence for the equation $Lu = f$, with $u_n = w_1$ on $r = r_n$. Setting $w = Cw_2 \pm (u_n - w_1)$ we obtain easily $Lw = 0$ in $\mathcal{E} \cap \mathcal{S}_n$, and

$$Mw \leq 0 \quad \text{on } F\mathcal{E}, \quad w \geq 0 \quad \text{on } F\mathcal{S}_n.$$

It follows then from the boundary point lemma that $w \geq 0$; thus we have proved

$$(30) \quad |u_n| \leq Cw_2 + |w_1|.$$

This shows that the sequence $\{u_n\}$ is uniformly bounded. Therefore by Lemma 4 there exists a solution u of $Lu = f$ which satisfies $Mu = \varphi$ on $F\mathcal{E}$. It remains only to show that u tends to zero as x tends to infinity. This, however, follows directly from (30), and the proof is completed.

13. The exterior problem II. Our object is to determine conditions under which this problem will be well-set. To begin with, the uniqueness of solutions of problem II under the boundary condition (28) requires that two conditions be satisfied: *First, the equation $Lu = 0$ must enjoy an extended maximum principle, and, second, at least one of the functions c and k must take on non-zero values.* Indeed, if these conditions are satisfied, then the difference of two solutions

must (in the first place) be constant, and then by the second condition the constant must be zero. On the other hand, the failure of either condition allows one to construct more than one solution of a given boundary problem. To see this, observe that if $c \equiv k \equiv 0$, then two solutions can differ by a constant; while if the extended maximum principle fails, then (as in the proof of Theorem 4) we can construct a non-trivial bounded solution of $Lu = 0$ satisfying $Mu = 0$ on $F\mathcal{E}$. This discussion shows, by the way, that the exterior Neumann problem II is not well-set if $c \equiv 0$.

Having disposed of the uniqueness question, we turn now to the more difficult problem of existence. The following result is an analogue of Theorem 11.

Theorem 17. *Consider the exterior problem II for the equation $Lu = f$ ($c \leq 0$) and the boundary condition*

$$Mu = \frac{\partial u}{\partial \nu} + ku = \varphi,$$

and suppose that at least one of the functions c and k takes on non-zero values. Then this problem is solvable if and only if the exterior Dirichlet problem II is solvable.

Remarks. The condition on c and k is essential. Indeed, if $c \equiv k \equiv 0$ there may be no bounded solutions; e.g. Laplace's equation in two dimensions has no bounded solution in $\mathcal{E}$ unless $\int_{F\mathcal{E}} (\partial u / \partial n) dS = 0$.

In order to apply this theorem to a specific equation one can use the criteria for the existence of a solution of the Dirichlet problem contained in Theorems 11, 12, and 14. Moreover, if the equation is homogeneous then the exterior Dirichlet problem is always solvable.

Proof of Theorem 17. Suppose first that problem II is solvable. If w is *any* solution, we can set $\bar{w} = w = w$ and thus verify the condition of Theorem 11.

On the other hand, let us assume that the exterior Dirichlet problem II is solvable. Then without loss of generality we can suppose that our problem is reduced to the homogeneous case $f \equiv 0$ (let w be *any* solution of the exterior Dirichlet problem II, and consider the new unknown $u - w$). With this understanding, let a boundary function φ^* be defined as follows:

- (1) If $k \equiv 0$, set $\varphi^* \equiv 1$.
- (2) If $k \equiv 0$, choose on $F\mathcal{E}$ two non-empty, disjoint, closed sets Γ_1 and Γ_2 with the property that $k < 0$ on $F\mathcal{E} - \Gamma_1$.

We then take for φ^* a smooth function satisfying

$$\varphi^* = \begin{cases} 1 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2. \end{cases}$$

Now let w be a solution of the exterior Dirichlet problem obtained from a φ^* -sequence with $w_n^* = 0$ on $r = r_n$. From the maximum principle we have

$$0 < w^* < 1 \quad \text{in } \mathcal{E};$$

for case (2) this is obvious, while for case (1) it is a consequence of the fact the $c \neq 0$.

In case (1), from the boundary point principle we have $Mw^* < 0$. In case (2), for the same reason, we have $Mw^* < 0$ on Γ_1 ; therefore, since $k < 0$ on the complement of Γ_1 ,

$$M(w^* + C_1) < 0 \quad \text{on } F\mathcal{E}$$

provided C_1 is a suitably large constant. Finally, in either case, there exists a positive constant C_2 such that

$$C_2 \cdot M(w^* + C_1) \leq -|\varphi| \quad \text{on } F\mathcal{E}.$$

Now let $\{u_n\}$ be a φ -sequence with $u_n = 0$ on $r = r_n$. Exactly as in the proof of Theorem 16 we derive

$$|u_n| \leq C_2(w^* + C_1) \leq C_2(1 + C_1).$$

Thus $\{u_n\}$ is bounded, and Lemma 4 implies that there exists a bounded solution of the problem in question. This completes the proof of Theorem 17.

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University of Minnesota
Minneapolis, Minnesota