On Hadamard's Variation Formula for Green's Function*

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Introduction. Although the Hadamard variation formula is not restricted to two dimensions, some of its important applications are to complex function theory and two dimensional fluid flow. It was first applied to conformal mapping by G. JULIA [7]; its wide significance for the solution of extremal problems in the complex domain and for various questions in conformal mapping was brought out primarily by M. SCHIFFER [11, 12, 13, 14], S. BERGMAN [2], P. GARABEDIAN & SCHIFFER [3, 4], SCHIFFER & D. C. SPENCER [15]. Other applications in connection with fluid dynamics were made by some of these and other authors.

The Hadamard variation formula [5, 13] may be stated as follows. Let Ω be a region bounded by $n + 1$ contours C_v , $v = 0, 1, \dots, n$, where $C_1, C_2, \dots, C_n$ are interior to C_0 and exterior to each other. Suppose $C = \bigcup_{v=0}^n C_v$ is represented in parametric form by $z = \zeta(t)$, $0 \leq t \leq 1$, where $\zeta(t)$ traverses the C_v consecutively as t varies from 0 to 1. For a given continuously differentiable function $h(t)$ along each C_v , let $\zeta^*(t) = \zeta(t) + \epsilon h(t)$ define a deformation $C^* = \bigcup_{v=0}^n C_v^*$ of C consisting of $n + 1$ contours C_v^* which bound a region Ω^* . Let $G(z, z_0)$ and $G^*(z, z_0)$ denote the Green's functions for Laplace's equation of Ω and Ω^* , respectively, with the singularity at z_0 . Then, if C is sufficiently smooth,

$$G(z, z_0) - G^*(z, z_0) = \frac{1}{2\pi} \int_C \operatorname{Im} \left\{ (\zeta^* - \zeta) \frac{|d\zeta|}{d\zeta} \right\} \frac{\partial G(\zeta, z_0)}{\partial n} \frac{\partial G(\zeta, z)}{\partial n} |d\zeta| + O(\epsilon^2)$$

as $\epsilon \rightarrow 0$, uniformly for z, z_0 in any compact subset B of $\Omega \cap \Omega^*$.

The proofs of this formula in the literature known to this writer (see *e.g.* [8], p. 42 and [13], p. 292) assume either that $\zeta(t)$ is analytic or, at least, that $\zeta(t)$ is twice differentiable with bounded second derivative (and that $\dot{\zeta}(t) \neq 0$). These rather stringent smoothness assumptions limit the applicability of the Hadamard formula. In fact, this circumstance led SCHIFFER (see, for example, [13]) to the development of his interior variation which does not involve the

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boundary. It seems, therefore, of interest to establish the Hadamard formula under minimal smoothness assumptions on C , and this is the aim of the present paper.

The usefulness of the formula is not impaired if the error term is only known to be $o(\epsilon)$ rather than $O(\epsilon^2)$, and that is the estimate we shall obtain. In order that the integral in the formula exist in the sense of Lebesgue, it is *necessary* (and also sufficient) that C be rectifiable and that, for some $a \in \Omega$,

$$(a) \quad \int_C \left(\frac{\partial G(\zeta, a)}{\partial n} \right)^2 |d\zeta| < \infty.$$

To this assumption we add a rather mild smoothness condition: There exists a constant $c > 1$ such that for any two points z_1, z_2 on any C_κ the length Δs of the (shorter) arc of C_κ between z_1 and z_2 satisfies the inequality

$$(b) \quad \Delta s \leq c |z_1 - z_2|.$$

These two conditions are essentially sufficient to establish the Hadamard formula for such deformations C^* for which the integral

$$\int_{C^*} \left(\frac{\partial G^*(\zeta^*, a)}{\partial n} \right)^2 |d\zeta^*|$$

is uniformly bounded in ϵ (Theorem 2). Since the condition (a) may be difficult to verify, we give two simple *geometrical* criteria for C which insure that the integral in (a) and the corresponding integral for G^* are uniformly bounded.

1. Preliminaries. 1.1. Let C_0 be a closed rectifiable Jordan curve and let $w = \phi(z)$ map its interior conformally onto the circle $|w| < 1$. We shall say that $C_0 \in \Lambda^2$ if

$$\int_{C_0} |\phi'(z)|^2 |dz| < \infty.$$

Let $C = \bigcup_{v=0}^n C_v$, where each C_v is a closed rectifiable Jordan curve and $C_1, C_2, \dots, C_n$ are interior to C_0 and exterior to each other. Suppose that $w = \phi_0(z)$ maps the interior of C_0 and $w = \phi_v(z)$ ($1 \leq v \leq n$) maps the exterior of C_v onto $|w| < 1$. We shall say that $C \in \Lambda^2$ if

$$\sum_{v=0}^n \int_{C_v} |\phi'_v(z)|^2 |dz| < \infty.$$

Let Ω be the region bounded by C . Suppose that all $\phi_v(z)$ are normalized by the condition that, for a point $a \in \Omega$, $\phi_v(a) = 0$, $v = 0, 1, \dots, n$. If M is then a constant such that

$$\sum_{v=0}^n \int_{C_v} |\phi'_v(z)|^2 |dz| \leq M^2,$$

we shall write $C \in \Lambda^2(a, M)$.¹

¹ Clearly $C \in \Lambda^2(a, M)$ implies that for any point $z_0 \in \Omega$, $C \in \Lambda^2(z_0, M_{z_0})$ with a corresponding M_{z_0} (cf. Lemma 3 below).

Since the definition of the class Λ^2 involves the mapping functions $\phi_*(z)$ which in general are not easily known, it is useful to have simple *geometrical* criteria which will insure that $C \in \Lambda^2$. The following are two such sufficiency conditions:

A. $C \in \Lambda^2$ if each C_* has sectionally continuous tangent angle and if, at the (finitely many) points of discontinuity of the tangent, C has only corners (no cusps).

B. A closed Jordan curve $C_0 \in \Lambda^2$ if it is of bounded rotation, *i.e.* $\tau(s)$, the tangent angle as function of the arc length, is of bounded variation, and if all the jumps of the tangent angle are $\geq -\pi\alpha$, where $0 < \alpha < 1$ (α is a constant), which means for all s

$$\tau(s+0) - \tau(s-0) \geq -\pi\alpha.$$

From Theorem B one obtains easily an analogous sufficient condition for a curve system C to be of class Λ^2 .

Theorems A and B will be proved elsewhere.²

1.2. Suppose that $C = \bigcup_{v=0}^n C_v$ is given in the form

$$(1) \quad z = \zeta(t), \quad t \in [0, 1],$$

where $\zeta(t)$ traverses the curves C_v consecutively as t varies from 0 to 1, *i.e.*, for $0 \leq t < b_0$, $\zeta(t)$ represents C_0 , for $b_0 \leq t < b_1$, $\zeta(t)$ represents C_1 , *etc.*, and $\zeta(1) = \zeta(1-0)$. We assume that, in each of these t -intervals, $\zeta(t)$ has bounded difference quotients and that $(\dot{\zeta}(t) = d\zeta/dt)$

$$(2) \quad 0 < k_1 < |\dot{\zeta}(t)| < k_2, \quad t \in [0, 1].$$

We shall assume in addition that C has the following property: There exists a constant $c > 1$ such that for any two points z_1 and z_2 on C_v ($v = 0, 1, \dots, n$) the length Δs of the (shorter) arc of C_v between z_1 and z_2 satisfies the inequality

$$(3) \quad \Delta s \leq c |z_2 - z_1|.$$

We shall use below also the following geometrical parameters connected with C . Let D denote the diameter of the region Ω bounded by C and let ρ be a constant such that the disk $|z - a| \leq \rho$ ($a \in \Omega$) is contained in Ω and, for some $a_v \in I(C_v)$ ($v = 1, 2, \dots, n$), the disk $|z - a_v| \leq \rho$ is contained in $I(C_v)$, where $I(C_v)$ is the interior of C_v .

Remark. If C_0 is a closed Jordan curve with the property (3), the condition $C_0 \in \Lambda^2$ is equivalent to the fact that the derivative $\phi'_0(z)$ of the mapping function is of class $H_2(C_0)$. For, if $f(w)$ is the inverse of $\phi(z)$, the condition $C_0 \in \Lambda^2$ means

² A is an extension of a theorem in [17] for contours with continuously turning tangent and is proved by a modification of the method used there. The proof of B is based on the representation of $\log |f'(w)|$, where $f(w)$ maps $|w| < 1$ onto the interior of C_0 , by means of a Stieltjes integral involving $d\tau$ (Stieltjes's integral, see [10]).

that

$$(4) \quad \int_0^{2\pi} \frac{d\theta}{|f'(e^{i\theta})|} \leq M,$$

while $\phi'(z) \in H_2(C_0)$ means that

$$(5) \quad \int_0^{2\pi} \frac{d\theta}{|f'(re^{i\theta})|} \leq M \quad \text{for } 0 \leq r < 1.$$

Of course, (5) implies (4), but under the condition (3) the converse is also true.³

1.3. We shall call the curve system $C^* = \bigcup_{v=0}^n C_v^*$ a deformation of C if it is composed of closed Jordan curves C_v^* given by

$$(6) \quad z = \zeta^*(t) = \zeta(t) + \epsilon h(t), \quad t \in [0, 1], \quad 0 < \epsilon \leq 1,$$

where $h(t)$ has bounded difference quotient in each interval $0 \leq t < b_0$, $b_{v-1} \leq t < b_v$ ($v = 1, 2, \dots, n$), $h(1) = h(1 - 0)$, and $|h(t)| \leq 1$, $|\dot{h}(t)| \leq 1$. We assume furthermore that ϵ is so small that $C_1^*, \dots, C_n^*$ are interior to C_0^* and exterior to each other. Let Ω^* denote the region bounded by C^* .

It is easily seen that for sufficiently small ϵ , $0 < \epsilon \leq \epsilon_0$, C^* satisfies condition (3) with c replaced by $2c$.

2. Hadamard's variation formula. 2.1. We now state a preliminary form of our main theorem.

Theorem 1. (i) Suppose $C = \bigcup_{v=0}^n C_v$ is a curve system given by (1) and (2) for which condition (3) is satisfied.

(ii) Let $C^* = \bigcup_{v=0}^n C_v^*$ be a deformation of C defined by (6) and suppose that $\Omega^* \subset \Omega$.

(iii) Suppose that, for some $a \subset \Omega^*$, $C \in \Lambda^2(a, M)$ and $C^* \in \Lambda^2(a, M)$.⁴

³ For the property (3) is a sufficient condition that C_0 be of Smirnov type (see [9], p. 183), i.e. that $\log |f'(w)|$ may be represented by the Poisson integral in $|w| \leq 1$:

$$\log |f'(re^{i\alpha})| = \frac{1}{2\pi} \int_0^{2\pi} \log |f'(e^{i\theta})| K(r, \theta - \alpha) d\theta,$$

$$K(r, \theta - \alpha) = \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \alpha)}.$$

Hence

$$\begin{aligned} \int_0^{2\pi} \frac{d\alpha}{|f'(re^{i\alpha})|} &= \int_0^{2\pi} \exp \left[-\frac{1}{2\pi} \int_0^{2\pi} \log |f'(e^{i\theta})| K(r, \theta - \alpha) d\theta \right] d\alpha \\ &\leq \int_0^{2\pi} \frac{d\alpha}{2\pi} \int_0^{2\pi} \exp [-\log |f'(e^{i\theta})|] K(r, \theta - \alpha) d\theta = \int_0^{2\pi} \frac{d\theta}{|f'(e^{i\theta})|}. \end{aligned}$$

⁴ It can be shown that if C satisfies the hypotheses of either Theorem A or B, and if in the first case $h(t)$ is continuous and in the second $h(t)$ is of bounded variation, then there exists, for given a , an M such that $C^* \in \Lambda^2(a, M)$ uniformly for all sufficiently small ϵ , $0 < \epsilon \leq \epsilon_0$.

(iv) For $z_0 \in \Omega^*$ let $G(z, z_0)$ and $G^*(z, z_0)$ denote the Green's function of the logarithmic potential for Ω and Ω^* , respectively, with the singularity at $z = z_0$.

Then there exist two constants A and α which depend only on C —more precisely, only on the constants c , ρ , and D —such that for $z, z_0 \in \Omega^*$

$$G(z, z_0) - G^*(z, z_0) = \frac{1}{2\pi} \int_c \operatorname{Im} \left\{ (\zeta^* - \zeta) \frac{|d\zeta|}{d\zeta} \right\} \frac{\partial G(\zeta, z_0)}{\partial n} \frac{\partial G(\zeta, z)}{\partial n} |d\zeta| + \frac{A \cdot M}{d^\alpha} o(\epsilon).$$

Here $d = \min_{\zeta \in C} |z - \zeta|$ and $o(\epsilon)$ depends, aside from ϵ , on C and z_0 only.

2.2 Proof. A preliminary conformal transformation. We first map the interior $I(C_0)$ of C_0 onto $|w| < 1$ by means of $w = \phi_0(z)$ ($\phi_0(z_0) = 0$). Let

$$\phi_0(\Omega) = \Omega_w, \quad \phi_0(C) = \Gamma = \bigcup_{v=0}^n \Gamma_v, \quad \phi_0(C^*) = \Gamma^* = \bigcup_{v=1}^n \Gamma_v^*.$$

In particular, Γ_0 is the circle $|w| = 1$. Let

$$\omega(t) = \phi_0(\zeta(t)), \quad \omega^*(t) = \phi_0(\zeta^*(t)).$$

In Ω_w the Green's functions $G(z, z_0)$ and $G^*(z, z_0)$ will be denoted by $g(w)$ and $g^*(w)$, respectively.

For $\omega^* = \omega^*(t) \in \Gamma_0^*$, $\omega = \omega(t) \in \Gamma_0$ we have

$$g(\omega^*) = g(\omega^*) - g^*(\omega^*) = |\omega^* - \omega| \frac{\partial g(\omega)}{\partial \nu} \cos \beta + O(|\omega - \omega^*|^2)$$

where β is the angle which the vector $\omega^* - \omega$ makes with the normal to Γ_0 at $\omega(t)$ and $\partial/\partial \nu$ indicates differentiation with respect to the inner normal. We write this in the following form:

$$\begin{aligned} g(\omega^*) &= \operatorname{Im} \left\{ (\omega^* - \omega) \frac{|d\omega|}{d\omega} \right\} \frac{\partial g}{\partial \nu} + O(|\omega - \omega^*|^2) \\ &= \operatorname{Im} \left\{ \frac{\omega^* - \omega}{\zeta^* - \zeta} (\zeta^* - \zeta) \frac{|d\zeta|}{d\zeta} \left| \frac{d\omega}{d\zeta} \right| \frac{d\zeta}{d\omega} \right\} \frac{\partial g}{\partial \nu} + O(|\omega - \omega^*|^2). \end{aligned}$$

Here $d\omega/d\zeta$ and $d\zeta/d\omega$ exist except for a set of measure 0 in their respective intervals of definition. Since, for almost all t , $0 \leq t < b_0$,

$$(7) \quad \frac{\partial g(\omega)}{\partial \nu} \left| \frac{d\omega}{d\zeta} \right| = \frac{\partial G(\zeta, z_0)}{\partial n}, \quad \zeta(t) \in C_0,$$

where $\partial/\partial n$ indicates differentiation with respect to the inner normal along C_0 , we obtain

$$\begin{aligned} (8) \quad G(\zeta^*(t), z_0) &= \operatorname{Im} \left\{ (\zeta^* - \zeta) \frac{|d\zeta|}{d\zeta} \right\} \frac{\partial G(\zeta, z_0)}{\partial n} \\ &+ \operatorname{Im} \left\{ \left(\frac{\omega^* - \omega}{\zeta^* - \zeta} \frac{d\zeta}{d\omega} - 1 \right) (\zeta^* - \zeta) \frac{|d\zeta|}{d\zeta} \right\} \frac{\partial G(\zeta, z_0)}{\partial n} \\ &+ O \left(|\omega^* - \omega| \left| \frac{\omega^* - \omega}{\zeta^* - \zeta} \right| |\zeta^* - \zeta| \right). \end{aligned}$$

We write this in the form

$$(9) \quad G(\zeta^*(t), z_0) = \epsilon \psi(t) + \mu(t, \epsilon) + \kappa(t, \epsilon)$$

where we have set

$$(10) \quad \psi(t) = \operatorname{Im} \left\{ h(t) \frac{|d\zeta|}{d\zeta^*} \right\} \frac{\partial G(\zeta(t), z_0)}{\partial n}$$

and $\mu(t, \epsilon)$ and $\kappa(t, \epsilon)$ denote the two remaining terms on the right hand side of (8).

By mapping the exterior of any of the other contours C_v onto $|w| < 1$ so that z_0 corresponds to $w = 0$, we see that (8) and (9) are valid when $\zeta(t) \in C_v$, $\zeta^*(t) \in C_v^*$, i.e. for $t \in [0, 1]$. We note that $\psi(t) \in L^2[0, 1]$, as follows from the hypothesis that $C \in \Lambda^2$. In fact, if $\gamma(w)$ ($= \log 1/|w|$) denotes the Green's function of $|w| < 1$ with the logarithmic pole at $w = 0$, we have by the maximum principle, for $|\omega| = 1$,

$$(11) \quad \frac{\partial g(\omega)}{\partial \nu} \leq \frac{\partial \gamma(\omega)}{\partial \nu} = 1,$$

and therefore, by (7) for $\zeta(t) \in C_0$,

$$(12) \quad \frac{\partial G(\zeta(t), z_0)}{\partial n} \leq |\phi'_0(\zeta(t))|.$$

The analogous inequality holds for $\zeta(t) \in C_v$ ($1 \leq v \leq n$).

2.3. *Estimates of $\mu(t, \epsilon)$ and $\kappa(t, \epsilon)$.* We consider again first $t \in [0, b_0)$. Since C is rectifiable it follows that for almost all t in $[0, b_0)$

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mu(t, \epsilon) = 0.$$

We exhibit a function $M(t) \in L^2[0, 1]$ such that (see (7) and (11))

$$(13) \quad \left| \frac{\omega^* - \omega}{\zeta^* - \zeta} - \frac{d\omega}{d\zeta} \right| \left| \frac{d\zeta}{d\omega} \right| \frac{\partial G(\zeta(t), z_0)}{\partial n} \leq \left| \frac{\omega^* - \omega}{\zeta^* - \zeta} - \frac{d\omega}{d\zeta} \right| \leq M(t).$$

Since $|\zeta^* - \zeta| \leq \epsilon$ it will then follow by Lebesgue's convergence theorem that

$$(14) \quad \int_{C_0} \mu^2(t, \epsilon) dt \leq \epsilon^2 \int_{C_0} \left| \frac{\omega^* - \omega}{\zeta^* - \zeta} - \frac{d\omega}{d\zeta} \right|^2 dt = o(\epsilon^2) \quad \text{as } \epsilon \rightarrow 0.$$

By Lemma 1 (Section 3) there exists an ω_1 , $|\omega_1| = 1$, such that

$$\left| \frac{\omega_1 - \omega}{\omega^* - \omega} \right| \rightarrow 1$$

as $\omega^* \rightarrow \omega$, i.e. as $\epsilon \rightarrow 0$, and such that if $\phi_0(\zeta_1) = \omega_1$,

$$\left| \frac{\omega^* - \omega}{\zeta^* - \zeta} \right| \leq K \left| \frac{\omega_1 - \omega}{\zeta_1 - \zeta} \right| \quad (\omega = \omega(t), 0 \leq t < b_0).$$

Now, if $\omega_1 = \omega(t_1)$

$$|\omega_1 - \omega| \leq \left| \int_t^{t_1} |\phi'_0(\zeta(\tau))| |\zeta'(\tau)| d\tau \right|$$

and, by (3) and (2),

$$|\zeta_1 - \zeta| \geq \frac{\Delta s}{c} \geq \frac{k_1}{c} |t_1 - t|.$$

Hence

$$\left| \frac{\omega^* - \omega}{\zeta^* - \zeta} \right| \leq K \frac{c}{k_1} \sup_{0 \leq t_1 \leq b_0} \left[\frac{1}{t_1 - t} \int_t^{t_1} |\phi'_0(\zeta(\tau))| |\zeta'(\tau)| d\tau \right] = M_1(t).$$

By a theorem of HARDY & LITTLEWOOD [6] (see also [18], p. 244)

$$\int_{c_0} M_1^2(t) dt \leq 8 \int_0^{b_0} |\phi'_0(\zeta(t))|^2 |\zeta'(t)|^2 dt.$$

Since $d\omega/d\zeta = \phi'_0$ we can thus take $M(t) = M_1(t) + |\phi'_0(\zeta(t))|$.

Since any C_* may be used in place of C_0 , the definition of $M(t)$ may be extended analogously to all $t \in [0, 1]$, and

$$(15) \quad \int_C \mu^2(t, \epsilon) dt = o(\epsilon^2).$$

A similar argument shows that

$$(16) \quad \int_C \kappa^2(t, \epsilon) dt = o(\epsilon^2).$$

2.4. *Completion of the proof.* We define now

$$U(z) = \frac{\epsilon}{2\pi} \int_C \psi(t) \frac{\partial G(\zeta(t), z)}{\partial n} |d\zeta|$$

which is harmonic in Ω . Let $\eta > 0$ be given and let $q(t)$ be a continuous function on each C_* such that

$$(17) \quad \int_0^1 |\psi(t) - q(t)|^2 dt < \eta^2.$$

For $z \in \Omega$, let

$$(18) \quad U_a(z) = \frac{1}{2\pi} \int_C q(t) \frac{\partial G(\zeta(t), z)}{\partial n} |d\zeta|.$$

Because of the continuity of $q(t)$ on C there exists a $\delta = \delta(\eta, q) > 0$ such that

$$(19) \quad |U_a(z) - q(t)| < \eta \quad \text{for} \quad |z - \zeta(t)| < \delta.$$

Assume $\epsilon < \delta$ so that $|\zeta^*(t) - \zeta(t)| < \delta$. Then by (9)

$$\begin{aligned}
(20) \quad & \left\{ \int_0^1 |G(\zeta^*(t), z_0) - \epsilon U_a(\zeta^*(t))|^2 dt \right\}^{\frac{1}{2}} \leq \left\{ \int_0^1 |\epsilon \psi(t) - \epsilon q(t)|^2 dt \right\}^{\frac{1}{2}} \\
& + \left\{ \epsilon^2 \int_0^1 |q(t) - U_a(\zeta^*(t))|^2 dt \right\}^{\frac{1}{2}} + \left\{ \int_0^1 \mu^2(t, \epsilon) dt \right\}^{\frac{1}{2}} \\
& + \left\{ \int_0^1 \kappa^2(t, \epsilon) dt \right\}^{\frac{1}{2}} \leq 2\eta\epsilon + o(\epsilon) = o(\epsilon)
\end{aligned}$$

as $\epsilon \rightarrow 0$, or, noting that, by (2), $|\zeta^*(t)| \leq k_2 + \epsilon$,

$$(21) \quad \left\{ \int_{C^*} |G(\zeta^*, z_0) - G^*(\zeta^*, z_0) - \epsilon U_a(\zeta^*)|^2 |d\zeta^*| \right\}^{\frac{1}{2}} = o(\epsilon).$$

Now we have, for $z \in \Omega^*$,

$$\begin{aligned}
G(z, z_0) - G^*(z, z_0) - \epsilon U_a(z) \\
= \frac{1}{2\pi} \int_{C^*} [G(\zeta^*, z_0) - G^*(\zeta^*, z_0) - \epsilon U_a(\zeta^*)] \frac{\partial G^*(\zeta^*, z)}{\partial n} |d\zeta^*|,
\end{aligned}$$

and hence, by the inequality of Schwarz and by (21),

$$(22) \quad |G(z, z_0) - G^*(z, z_0) - \epsilon U_a(z)| \leq o(\epsilon) \left\{ \int_{C^*} \left(\frac{\partial G^*(\zeta^*, z)}{\partial n} \right)^2 |d\zeta^*| \right\}^{\frac{1}{2}}.$$

Similarly, since

$$\epsilon U_a(z) - U(z) = \frac{\epsilon}{2\pi} \int_C (q(t) - \psi(t)) \frac{\partial G(\zeta(t), z)}{\partial n} |d\zeta|,$$

we find for $z \in \Omega^*$ by (17)

$$(23) \quad |\epsilon U_a(z) - U(z)| \leq \epsilon \eta \left\{ \int_C \left(\frac{\partial G(\zeta, z)}{\partial n} \right)^2 |d\zeta| \right\}^{\frac{1}{2}}.$$

Thus, combining (22) and (23) and using Lemma 3 and the subsequent remark in Section 3, we obtain the conclusion of the theorem.

2.5. We now prove our principal result.

Theorem 2. *Suppose that the four hypotheses of Theorem 1 are satisfied. Let B be a compact subset of Ω^* . Then*

$$G(z, z_0) - G^*(z, z_0) = \frac{1}{2\pi} \int_C \operatorname{Im} \left\{ (\zeta^* - \zeta) \frac{|d\zeta|}{d\zeta} \right\} \frac{\partial G(\zeta, z_0)}{\partial n} \frac{\partial G(\zeta, z)}{\partial n} |d\zeta| + o(\epsilon)$$

as $\epsilon \rightarrow 0$, uniformly for all $z, z_0 \in B$.

Proof. We show that the estimates (15) and (16), as well as (17), (19), and (21), hold uniformly for all $z_0 \in B$. The conclusion follows then in the same manner as that of Theorem 1.

2.6. *Uniform estimate of $\kappa(t, \epsilon)$ and $\mu(t, \epsilon)$.* We refer to Section 2.3 and start once more with the interval $0 \leq t < b_0$. Let $\phi(z) = \phi_0(z, a)$ denote the function

which maps $I(C_0)$ conformally onto $|w| < 1$ such that $\phi(a) = 0$, where $a \in \Omega$. Then the function $\phi_0(z) = \phi(z, z_0)$ which we used in Section 2.3 is given by

$$\phi_0(z) = \frac{\phi(z) - \phi(z_0)}{1 - \overline{\phi(z_0)}\phi(z)} e^{i\gamma} \quad (\gamma \text{ a real constant}).$$

Let for $\zeta \in C_0$

$$\Delta\phi_0(\zeta) \equiv \frac{\phi_0(\zeta^*) - \phi_0(\zeta)}{\zeta^* - \zeta} - \phi'_0(\zeta)$$

and let $\Delta\phi(\zeta)$ be the analogous expression for ϕ . Then

$$\begin{aligned} |\Delta\phi_0(\zeta)| &\leq |\Delta\phi(\zeta)| \frac{1 - |\phi(z_0)|^2}{|1 - \overline{\phi(z_0)}\phi(\zeta)| |1 - \overline{\phi(z_0)}\phi(\zeta^*)|} \\ &\quad + \frac{(1 - |\phi(z_0)|^2)|\phi'(\zeta)| |\phi(z_0)|}{|1 - \overline{\phi(z_0)}\phi(\zeta)|^2 |1 - \overline{\phi(z_0)}\phi(\zeta^*)|} |\phi(\zeta^*) - \phi(\zeta)|. \end{aligned}$$

If the minimal distance of B from C is d , then

$$(24) \quad |\phi(z_0)| \leq r \leq 1 - m d^\alpha$$

by Lemma 2 below, where m and α depend only on ρ , D , and c . Thus r depends only on C_0 , B , and a . Using (24) we obtain

$$|\Delta\phi_0(\zeta)| \leq \frac{2}{1-r} |\Delta\phi(\zeta)| + \frac{2|\phi'(\zeta)|}{(1-r)^2} |\phi(\zeta) - \phi(\zeta^*)|.$$

Furthermore,

$$\sup_{t \in [0, b_0]} |\phi(\zeta) - \phi(\zeta^*)| \leq \eta(\epsilon),$$

where $\eta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ and depends, aside from ϵ , only on C_0 and a . Hence

$$(25) \quad \int_0^{b_0} |\Delta\phi_0(\zeta)|^2 dt \leq 2 \left[\frac{4}{(1-r)^2} \int_0^{b_0} |\Delta\phi(\zeta)|^2 dt + \frac{4\eta^2(\epsilon)}{(1-r)^4} \int_0^{b_0} |\phi'(\zeta)|^2 dt \right].$$

By (14) we have for fixed a (in place of z_0)

$$\epsilon^2 \int_0^{b_0} |\Delta\phi(\zeta(t))|^2 dt = o(\epsilon^2),$$

where $o(\epsilon^2)$ depends, aside from ϵ , only on C and a . Hence, from (25),

$$\int_0^{b_0} \mu^2(t, \epsilon) dt \leq \epsilon^2 \int_0^{b_0} |\Delta\phi_0(\zeta(t))|^2 dt \leq K_0(B) o(\epsilon^2),$$

where $K_0(B)$ depends only on B , C , and a .

An analogous reasoning applies to the other contours⁵ C_* and we have, therefore,

⁵ To obtain a bound for r corresponding to (24) in this case one applies the corollary to Lemma 2 (with a_* in place of a and a in place of b).

$$(15a) \quad \int_0^1 \mu^2(t, \epsilon) dt \leq K(B) o(\epsilon^2).$$

Similarly we find

$$(16a) \quad \int_0^1 \kappa^2(t, \epsilon) dt \leq K'(B) o(\epsilon^2),$$

where K, K' depend only on B, C , and a , and $o(\epsilon^2)$, apart from ϵ , depends only on C and a .

2.7. *Completion of the proof.* For $t \in [0, b_0]$ we can write the function $\psi(t)$ defined in (10) in the form

$$(26) \quad \psi(t) = \operatorname{Im} \left\{ h(t) \frac{|d\zeta|}{d\zeta} \right\} \frac{\partial g(\omega)}{\partial \nu} \frac{1 - |\phi(z_0)|^2}{|1 - \phi(\zeta)\phi(z_0)|^2} |\phi'(\zeta)|,$$

where $g(\omega) = G(\zeta, z_0)$ and $\omega = \phi(\zeta(t))$ (cf. Section 2.2). Let

$$\psi_a(t) = \operatorname{Im} \left\{ h(t) \frac{|d\zeta|}{d\zeta} \right\} |\phi'(\zeta(t))|.$$

Given any $\eta > 0$, there exists a continuous function $q_a(t)$ on C_0 such that

$$(27) \quad \int_0^{b_0} |\psi_a(t) - q_a(t)|^2 dt < \eta^2 \frac{(1-r)^2}{4(n+1)},$$

where r is the bound in (24) and $n+1$ is the number of contours C_* . The factor

$$\chi_a(t, z_0) = \frac{\partial g(\omega)}{\partial \nu} \frac{1 - |\phi(z_0)|^2}{|1 - \phi(z_0)\phi(\zeta)|^2} \quad (\zeta = \zeta(t))$$

of $\psi_a(t)$ in the expression (26) is *continuous in t and z_0 for $t \in [0, b_0 - 0]$, $z_0 \in B$ and hence also bounded.* However, one sees also directly from (11) and (24) that

$$(28) \quad |\chi_a(t, z_0)| \leq \frac{2}{1-r}.$$

Consequently, if we define the continuous function $q(t)$ by

$$q(t) = q_a(t) \chi_a(t, z_0),$$

we have by (27) and (28)

$$(29) \quad \int_{C_0} |\psi(t) - q(t)|^2 dt < \frac{\eta^2}{n+1}.$$

In an analogous manner one can approximate $\psi(t)$ by a continuous function $q(t) = q(t, z_0)$ on each of the other contours C_* such that

$$\int_{C_*} |\psi(t) - q(t)|^2 dt < \frac{\eta^2}{n+1} \quad (\text{for all } z_0 \in B),$$

and so, by addition, we have

$$(17a) \quad \int_C |\psi(t) - q(t)|^2 dt < \eta^2 \quad (\text{for all } z_0 \in B).$$

Here $q(t)$ has a modulus of continuity $\lambda(\tau)$ in t , which holds uniformly for all $z_0 \in B$, and $|q(t)| \leq Q$ for $t \in [0, 1]$, $z_0 \in B$.

Now we define $U_\epsilon(z)$ as in (18) for $z \in \Omega$. Because of the just stated continuity and boundedness properties of $q(t)$, there exists a $\delta = \delta(\eta, B, C, a)$ such that

$$(19a) \quad |U_\epsilon(z) - q(t)| < \eta \quad \text{for } |z - \zeta(t)| < \delta$$

uniformly for all $z_0 \in B$. Using (17a), (19a), (15a), and (16a) to estimate the four integrals on the right hand side in (20), we see that $o(\epsilon)$ in (20), (21), and (22) holds uniformly for all $z_0 \in B$. The proof is now completed by repeating the final step in the proof of Theorem 1.

3. Lemmas. We present now three lemmas which were used earlier. We shall consider a closed Jordan curve C which has the following property: There exists a constant $c > 0$ such that for any two points z_1 and z_2 of C the diameter Δ of one of the arcs of C between z_1 and z_2 satisfies the inequality

$$(3a) \quad \Delta \leq c |z_2 - z_1|.$$

Clearly, if C has property (3), then it also satisfies (3a). The converse is not necessarily true since C may not be rectifiable but still satisfy (3a).

3.1. Lemma 1. Suppose C is a closed Jordan curve which satisfies (3a). Suppose $z = f(w)$ maps the disk $|w| < 1$ conformally onto the interior of C . Then there exist two positive constants δ and H with the following property: If w_0 is any point on $|w| = 1$ and $\kappa_\rho = \kappa_\rho(w_0)$ is a circle of radius ρ which is orthogonal to $|w| = 1$, contains w_0 in its interior, and intersects the unit circle at points equidistant from w_0 , then for w_1 and w_2 on κ_ρ in $|w| \leq 1$, where $0 < \rho \leq \delta$,

$$\frac{1}{H} \leq \left| \frac{f(w_1) - f(w_0)}{f(w_2) - f(w_0)} \right| \leq H.$$

Proof. (i) Let $\zeta_0 = f(w_0)$ and $\zeta'_0 = f(-w_0)$. Because of the uniform continuity of $f^{-1}(\zeta)$ on C , the $\inf_{|w_0| \rightarrow 1} |f(w_0) - f(-w_0)| = 2\mu$ is positive. Let r_0 be so chosen that $cr_0 < \mu$. Denote the two arcs of C between ζ_0 and ζ'_0 by γ_+ and γ_- . Draw the circle $c_r : |z - \zeta_0| = r$ ($\leq r_0$) and traverse γ_+ and γ_- from ζ_0 to the last point of intersection of these arcs with c_r , thus obtaining the subarcs $\gamma_+(r)$ and $\gamma_-(r)$. Let $\Delta(r)$ be the radius of the least closed circle about ζ_0 which contains $\gamma_+(r)$ and $\gamma_-(r)$. Because of (3a) the diameter of $\gamma_+(r)$ or that of $\{C - \gamma_+(r)\}$ is at most equal to cr , but since $|\zeta_0 - \zeta'_0| \geq 2\mu > cr_0 \geq cr$, the diameter of $\gamma_+(r)$ is $\leq cr$. For a similar reason the diameter of $\gamma_-(r)$ is $\leq cr$. Hence

$$\Delta(r) \leq cr.$$

From this inequality we infer the following: Let τ_r denote a circular cross-cut of the interior of C on $|z - \zeta_0| = r$ which connects a point of γ_+ with a point of

γ_- . Let σ be an arbitrary cross-cut of $I(C)$ contained in $|z - \zeta_0| \leq r_0$ which also joins γ_+ and γ_- . Suppose $r_1 = \sup r$ and $r_2 = \inf r$ of all r for which σ meets a τ_r . Then σ is contained in the ring

$$(30) \quad \frac{r_1}{c} \leq |z - \zeta_0| \leq cr_2.$$

Finally we note that we can choose $\delta > 0$ such that for $w \in \kappa_\rho(w_0)$, $|w| \leq 1$, with $\rho \leq \delta$, $|f(w) - f(w_0)| < r_0$ uniformly for all w_0 , $|w_0| = 1$.

(ii) Now we apply the transformation

$$Z = \log \frac{1}{z - \zeta_0}, \quad Z = X + iY,$$

which maps $I(C)$ onto a semi-infinite strip S such that $z = \zeta_0$ corresponds to $X = +\infty$. We denote by Γ_+ and Γ_- the images in the Z -plane of γ_+ and γ_- , respectively.

We also map $|w| < 1$ by means of

$$W = W(w) = \log \frac{w_0 + w}{w_0 - w}, \quad W(0) = 0,$$

onto the strip Σ : $|V| < \frac{1}{2}\pi$ ($W = U + iV$). Under this mapping the image of the arc of $\kappa_\rho(w_0)$ which is contained in $|w| \leq 1$ is a vertical segment

$$s\{U_\rho\} = \{W \mid U = U_\rho, |V| \leq \tfrac{1}{2}\pi\},$$

and $U_\rho > U_\delta$ if $\rho < \delta$.

The function

$$Z = Z(W) = \log \frac{1}{f(w) - \zeta_0}, \quad w = w_0 \frac{e^W - 1}{e^W + 1}$$

maps Σ onto S in such a way that $X = +\infty$ corresponds to $U = +\infty$ and Γ_+ is carried into one, and Γ_- into the other, of the two boundary lines $V = \pm \frac{1}{2}\pi$ of Σ . Furthermore, the image of any point of Σ in $U \geq U_\delta$ lies in the halfplane $X \geq \log 1/r_0$.

(iii) We estimate now the oscillation of $\Re\{Z(W)\} = X(W)$ on a vertical segment $s = s\{U\}$ for $U > U_\delta$, i.e.

$$\lambda(U) = \max_{W \in s} X(W) - \min_{W \in s} X(W).$$

For this purpose we apply AHLFORS' (first) inequality [1] concerning the distortion in conformal mapping of infinite strips. We consider a Jordan arc $\beta \subset S$ (except for its endpoints) which connects $Z'_0 = \log 1/(\zeta'_0 - \zeta_0)$ (in $\Re Z < \log 1/r_0$) with $X = +\infty$, and define, for $X \geq \log 1/r_0$, θ_x as the cross-cut on $\Re Z = X$ which joins a point on Γ_+ to one on Γ_- and which is met *first* when β is traversed from Z'_0 to $+\infty$. We denote the length of θ_x by $\theta(X)$.

Let $\sigma = Z(s\{U\})$. If $X_1 = \inf X$ and $X_2 = \sup X$ of all values of X for which

σ meets a θ_x , then, by (30), σ is contained in the domain $X_1 - \log c \leq X \leq X_2 + \log c$, or

$$(31) \quad \lambda(U) \leq X_2 - X_1 + 2 \log c.$$

To estimate $X_2 - X_1$ we note that AHLFORS' inequality, [1], p. 12, is applicable (see [1], p. 11, Bemerkung) and obtain

$$\int_{x_1}^{x_2} \frac{dX}{\theta(X)} \leq 4.$$

Since $\theta(X) \leq 2\pi$, we have

$$X_2 - X_1 \leq 8\pi,$$

and therefore by (31)

$$\lambda(U) \leq 8\pi + 2 \log c.$$

Now, if w_1 and w_2 are two points in $|w| \leq 1$ on $\kappa_\delta(w_0)$, we have

$$\left| \log \left| \frac{f(w_1) - \zeta_0}{f(w_2) - \zeta_0} \right| \right| \leq 8\pi + 2 \log c,$$

and this is equivalent to the conclusion of the lemma.

3.2. Lemma 2. *Suppose C is a closed Jordan curve for which (3a) is satisfied and which is contained in the annulus $0 < \rho < |z - a| < D$. Let $f(w)$ map the circle $|w| < 1$ conformally onto the interior of C so that $f(0) = a$. Then there exist two positive constants β and μ , which depend only on ρ , D , and c , such that*

$$|f(re^{i\theta}) - f(e^{i\theta})| \leq \mu(1 - r)^\beta \quad (0 \leq r < 1).$$

This statement is a consequence of a more general result Theorem II in [16] applied, in the notation of that paper, with $\kappa = 1 + c$, $\eta_0 = 0$, $\delta_0 = \rho/c$. (It may also be proved directly by a much simplified version of the method employed in the proof of Theorem II.)

Corollary. *Suppose C is a closed Jordan curve as described in Lemma 2. Suppose, moreover, that there exists a point b in the exterior $E(C)$ of C which is contained in $|z - a| < D$ and has at least the distance ρ from C . If $F(w)$ maps $|w| < 1$ conformally onto $E(C)$ so that $F(0) = b$, then there exist two constants μ' and β' which depend only on ρ , D , and c , such that for all r for which $|F(re^{i\theta}) - a| \leq D^6$*

$$|F(re^{i\theta}) - F(e^{i\theta})| \leq D^2 \mu' (1 - r)^{\beta'}.$$

To show this we map $E(C)$ conformally onto the interior of a closed Jordan curve C' by means of $z' = 1/(z - a)$. We apply Lemma 2 to $f(w) = 1/(F(w) - a)$, and thus $z = (b - a)^{-1}$ takes the place of $z = a$. The parameters ρ' , D' , c' per-

* In our application the condition $|F(re^{i\theta}) - a| \leq D$ is satisfied. In general one may use the fact that $|F(w) - a| \leq 2D |w - w_0|^{-1}$ where w_0 is the pole of $F(w)$.

taining to C' which correspond to ρ , D , c may be expressed as functions of ρ , D , and c only (e.g. $\rho' = \rho/D^2$). Hence, by Lemma 2,

$$\left| \frac{1}{F(re^{i\theta}) - a} - \frac{1}{F(e^{i\theta}) - a} \right| \leq \mu'(1 - r)^{\beta'},$$

and since $|F(e^{i\theta}) - a| \leq D$, $|F(re^{i\theta}) - a| \leq D$, we obtain the conclusion.

3.3. Lemma 3. Let $C = \bigcup_{v=1}^n C_v$ be the boundary of a bounded multiply connected region Ω . Suppose that C satisfies condition (3) and that, for some $a \in \Omega$, $C \in \Lambda^2(a, M)$. Let $G(z, z_0)$ be the Green's function of Ω with singularity at $z = z_0$. Then there exist two positive constants α and A which depend only on ρ , D , and c , such that for $z \in \Omega$

$$(32) \quad \left[\int_c \left\{ \frac{\partial G(\zeta, z)}{\partial n} \right\}^2 |d\zeta| \right]^{\frac{1}{2}} \leq \frac{A}{d^\alpha} M \quad d = \min_{\zeta \in C} |z - \zeta|.$$

Proof. Let $\phi(z)$ map the outer boundary C_0 of Ω onto $|w| < 1$ so that $\phi(a) = 0$, and let $g(\omega, w) = G(\zeta, z)$ where $\omega = \phi(\zeta)$, $w = \phi(z)$. Then almost everywhere on C_0

$$\frac{\partial G(\zeta, z)}{\partial n} = \frac{\partial g(\omega, w)}{\partial \nu} |\phi'(\zeta)|,$$

where $\partial/\partial n$ and $\partial/\partial \nu$ indicate differentiation with respect to the inner normal along C_0 and $|\omega| = 1$, respectively. If $w = \phi(z) = re^{i\beta}$ we obtain by the maximum principle

$$\frac{\partial g(\omega, w)}{\partial \nu} \leq \frac{1 - r^2}{|\omega - re^{i\alpha}|^2} \leq \frac{2}{1 - r}.$$

By Lemma 2, $1 - r \geq m_0^{-1} d^{\alpha_0}$ where α_0 and m_0 depend on ρ , D , c only. Hence

$$(33) \quad \int_{c_0} \left(\frac{\partial G(\zeta, z)}{\partial n} \right)^2 |d\zeta| \leq \left(\frac{2m_0}{d^{\alpha_0}} \right)^2 \int_{c_0} |\phi'(\zeta)|^2 |d\zeta|.$$

If C_κ is any one of the inner contours we map $E(C_\kappa)$ onto $|w| < 1$ by $w = \phi_\kappa(z)$, $\phi_\kappa(a) = 0$. From the corollary to Lemma 2 we infer the existence of α_κ and m_κ such that

$$(34) \quad \int_{c_\kappa} \left(\frac{\partial G(\zeta, z)}{\partial n} \right)^2 |d\zeta| \leq \left(\frac{2m_\kappa}{d^{\alpha_\kappa}} \right)^2 \int_{c_\kappa} |\phi'_\kappa(\zeta)|^2 |d\zeta| \quad (1 \leq \kappa \leq n).$$

Let

$$A = \max_{0 \leq \kappa \leq n} (2m_\kappa), \quad \alpha = \max_{0 \leq \kappa \leq n} \alpha_\kappa.$$

If $d \leq 1$, the conclusion (32) follows from (33) and (34). If $d > 1$, we have $d^\alpha \geq 1$ and the left hand side of (32) is $\leq A \cdot M \leq A(D/d)^\alpha \cdot M$.

Remark. Suppose C is the curve system defined in Section 1.2 which satisfies (2) and (3). Then the deformation C^* as given by (6) satisfies (3) with c replaced by $2c$, for all sufficiently small ϵ , $0 < \epsilon \leq \epsilon_0$. Furthermore, C^* is contained in

$|z - a| < D$, $I(C^*)$ contains the circle $|z - a_*| < \rho$, and for all sufficiently small ϵ the circle $|z - a| < \rho$ is contained in Ω^* . Consequently, if $C^* \in \Lambda^2(a, M)$, Lemma 3 is applicable to C^* and we find, for $z \in \Omega^*$,

$$\left[\int_{C^*} \left(\frac{\partial G^*(\zeta, z)}{\partial n} \right)^2 |d\zeta| \right]^{\frac{1}{2}} \leq \frac{A}{d^\alpha} M,$$

where A and α depend only on ρ , D , and c .

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