The Holonomy Group

IV. The General L_n with Symmetric Connection*

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1. Introduction. This paper is a continuation of the papers [1], [2], [3] of the same title.† It uses the same symbolism and some results of these papers. In particular, we shall again denote by L_n an n -dimensional space endowed with a symmetric connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \Gamma_{\mu}{}^{\nu}{}_{\lambda}$ and by $R_{\omega\mu\lambda}{}^{\nu}$ its curvature tensor and put

$$R_{\omega\mu} \stackrel{\text{def}}{=} R_{\omega\mu\alpha}{}^{\alpha}.$$

The connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ (or the space L_n) is termed volume preserving if

$$R_{\omega\mu} = 0.$$

This condition characterized [III], where we dealt with a Riemann space V_n (*i.e.* an L_n with covariant constant quadratic tensor $g_{\lambda\mu} = g_{\mu\lambda}$ of rank n). In this paper we shall deal exclusively *with an L_n which is not volume preserving*:

$$(1.1) \quad R_{\omega\mu} \neq 0.$$

* * *

The paper contains three parts. The first part serves as an introduction to the second part. We define an L_n in a manner which is slightly different from the usual definition. This enables us to stress the condition (1.1) by means of a vector Q_{μ} and its alternating derivative $\partial_{[\omega} Q_{\mu]}$. The second part, comprising sections 5–12, deals with the holonomy group $G_l(R)$ of an L_n which enables us to state in a concise form the complete set of integrability conditions of certain systems of linear partial differential equations. The condition (1.1) is very important for the structure of the holonomy group. It turns out that there are two different types of holonomy groups. Both contain at least two subgroups, one of them being always normal (*i.e.* invariant with respect to the transformations of the adjoint group). In the second type of holonomy group the remaining

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†These papers will be denoted by [I], [II], [III].

subgroup is also normal and belongs to the center of the holonomy group. The Lie group induced by the tensor

$$W_{\omega\mu\lambda}{}^\nu \stackrel{\text{def}}{=} R_{\omega\mu\lambda}{}^\nu - \frac{1}{n} R_{\omega\mu}{}^\nu \delta_\lambda^\nu$$

has the same structural constants as $G_i(R)$. It is in one-to-one correspondence with the holonomy group of the first type. If the holonomy group is of the second type, then the Lie group induced by $W_{\omega\mu\lambda}{}^\nu$ is identical with one of the two above mentioned subgroups.

These results hold for any L_n with (1.1) and therefore also for a Weyl space which will be studied in the next paper.

The third part displays a method which yields for a large class of holonomy groups the values 0 and ± 1 for the structural constants. Therefore all structural problems of this large class of holonomy groups may be mastered by this method.*

I.

2. Elementary objects in an X_n . With a real coordinate system x^ν we may associate the vectors

$$(2.1) \quad e_\lambda^\nu \stackrel{*}{=} \delta_\lambda^\nu, \quad e_\lambda^\nu \stackrel{*}{=} \delta_\lambda^\nu \dagger$$

and the simple n -vectors

$$(2.2) \quad E^{\nu_1 \dots \nu_n} \stackrel{\text{def}}{=} n! e_{\underset{\text{I}}{\nu_1}}^{\nu_1} \dots e_{\underset{\text{N}}{\nu_n}}^{\nu_n}, \quad e_{\lambda_1 \dots \lambda_n} \stackrel{\text{def}}{=} n! e_{[\lambda_1}^{\text{I}} \dots e_{\lambda_n]}^{\text{N}}.$$

The n -vector $E^{\nu_1 \dots \nu_n}$ gives rise to a density $\mathfrak{E}$ of weight $p = -1$ and to a density $\mathfrak{E}$ of weight $p = -1$:

$$(2.3) \quad (a) \quad \mathfrak{E} \stackrel{\text{def}}{=} E^{\text{I} \dots \text{N}}, \quad (b) \quad \mathfrak{E} \stackrel{\text{def}}{=} |E^{\text{I} \dots \text{N}}|.$$

They are characterized by the transformation rules

$$(2.4) \quad (a) \quad \mathfrak{E}' = \Delta \mathfrak{E}, \quad (b) \quad \mathfrak{E}' = |\Delta| \mathfrak{E}.$$

In a similar way $e_{\lambda_1 \dots \lambda_n}$ gives rise to two densities $\tilde{\mathfrak{e}}$ and $\mathfrak{e}$ of weight $p = 1$:

$$(2.5) \quad (a) \quad \tilde{\mathfrak{e}} \stackrel{\text{def}}{=} n! e_{\text{I} \dots \text{N}}, \quad (b) \quad \mathfrak{e} \stackrel{\text{def}}{=} n! |e_{\text{I} \dots \text{N}}|$$

with the transformation rule

$$(2.6) \quad (a) \quad \tilde{\mathfrak{e}}' = \Delta^{-1} \tilde{\mathfrak{e}}, \quad (b) \quad \mathfrak{e}' = |\Delta|^{-1} \mathfrak{e}.$$

*The range of Greek indices consists of the first n Roman numerals I, II, ..., N, where N stands for the n^{th} Roman numeral. Coordinate systems will be denoted by $x^\nu, x'^\nu, x''^\nu, \dots$.

$\dagger \delta_\lambda^\nu$ is the Kronecker delta. The equality sign $\stackrel{*}{=}$ holds only in the coordinate system under consideration.

In general a density $\tilde{\alpha}$ of weight p has the transformation rule

$$(2.7a) \quad \tilde{\alpha}' = \Delta^{-p} \tilde{\alpha},$$

while a density α of weight p has the transformation rule

$$(2.7b) \quad \alpha' = |\Delta|^{-p} \alpha.$$

In a similar way we distinguish between tensorial densities $\tilde{\mathfrak{T}}:::$ and $\mathfrak{T}:::$.

Remark I. Both types of (scalar or tensorial) densities merge into one type for any even p .

Remark II. Put

$$\xi \stackrel{\text{def}}{=} \frac{\tilde{\mathfrak{G}}}{\mathfrak{G}}, \quad \epsilon_{\Delta} \stackrel{\text{def}}{=} \text{sgn } \Delta.$$

The transformation rule for ξ is

$$\xi' = \epsilon_{\Delta} \xi,$$

so that every density $\tilde{\alpha}$ of weight p gives rise to a density $\alpha \stackrel{\text{def}}{=} \tilde{\alpha} \xi^{-p}$ of the same weight.

Lemma (2.1). *There exists a density $\mathfrak{G} > 0$ with the following properties:*

- (a) *It is of weight $p = -2/n$.**
- (b) *It is known up to the gauge transformation*

$$(2.8) \quad *\mathfrak{G} = (e)'\mathfrak{G}$$

with arbitrary real f .

- (c) *It does not depend on the choice of the coordinate system.*

Proof. Associate with the coordinate systems x'' , x''' and x'''' the densities $\mathfrak{G}^{2/n}$, $\mathfrak{G}_1^{2/n}$, $\mathfrak{G}_2^{2/n}$ of weight $p = -2/n$ defined by

$$\mathfrak{G}^{2/n} \stackrel{*}{=} 1, \quad \mathfrak{G}_1^{2/n} \stackrel{*}{=} 1, \quad \mathfrak{G}_2^{2/n} \stackrel{*}{=} 1. \dagger$$

The components $\mathfrak{G}_1^{2/n}$ and $\mathfrak{G}_2^{2/n}$ (of $\mathfrak{G}_1^{2/n}$ and $\mathfrak{G}_2^{2/n}$) in the x'' -system are

$$\mathfrak{G}_1^{2/n} \stackrel{*}{=} |\Delta_1|^{-2/n} \mathfrak{G}^{2/n}, \quad \mathfrak{G}_2^{2/n} = |\Delta_2|^{-2/n} \mathfrak{G}^{2/n},$$

so that

$$(2.9) \quad \mathfrak{G}_2^{2/n} = \mathfrak{G}_1^{2/n} (e)^f, \quad (e)^f = \left| \frac{\Delta_1}{\Delta_2} \right|^{2/n}.$$

Because x'' , x'''' are arbitrary, f is arbitrary and real. Hence we have in the x'' -system a density of weight $p = -2/n$, namely $\mathfrak{G} \stackrel{\text{def}}{=} \mathfrak{G}_1^{2/n}$ known up to the

*Hence $\mathfrak{G}^n$ transforms by means of $\Delta^2 = |\Delta|^2$.

†Observe that these derivatives depend on the coordinate system.

gauge transformation (2.8) with $*\mathfrak{G} \stackrel{\text{def}}{=} \mathfrak{G}_2^{2/n}$ and independent of the coordinate system x^ν we started with.

Definition (2.1). *The gauge transformation (2.8) will be termed an f -transformation. An object (an equation) invariant with respect to all f -transformations will be termed f -invariant.*

In subsequent considerations we shall need also the indicators defined as follows:

$$(2.10) \quad \tilde{\mathfrak{E}}_{\nu_1 \dots \nu_n} \stackrel{\text{def}}{=} \frac{E^{\nu_1 \dots \nu_n}}{E^{1 \dots n}}, \quad \tilde{e}_{\lambda_1 \dots \lambda_n} = \frac{e_{\lambda_1 \dots \lambda_n}}{e_{1 \dots n}}.$$

Their rule of transformation is

$$(2.11) \quad \begin{aligned} (a) \quad \tilde{\mathfrak{E}}^{\nu'_1 \dots \nu'_n} &= \Delta^{-1} \frac{\partial x^{\nu'_1}}{\partial x^{\lambda_1}} \dots \frac{\partial x^{\nu'_n}}{\partial x^{\lambda_n}} \tilde{\mathfrak{E}}^{\lambda_1 \dots \lambda_n}, \\ (b) \quad \tilde{e}_{\lambda'_1 \dots \lambda'_n} &= \Delta \frac{\partial x^{\nu_1}}{\partial x^{\lambda'_1}} \dots \frac{\partial x^{\nu_n}}{\partial x^{\lambda'_n}} \tilde{e}_{\nu_1 \dots \nu_n}. \end{aligned}$$

$\tilde{\mathfrak{E}}^{\nu_1 \dots \nu_n} (\tilde{e}_{\lambda_1 \dots \lambda_n})$ is an n -vector density of weight $p = 1$ ($p = -1$). Its components are 1, -1 , 0 in every coordinate system.

3. Basic concept of an L_n . Let us prove first the following

Lemma (3.1). *Let $\mathfrak{h}_{\lambda\mu}$ be an f -invariant symmetric tensor density of weight $p = -2/n$ and of rank n . Its determinant $h \neq 0$ is a scalar and*

$$(3.1) \quad \mathfrak{g}_{\lambda\mu} \stackrel{\text{def}}{=} \mathfrak{h}_{\lambda\mu} |h|^{-1/n}$$

is an f -invariant symmetric tensor density of weight $p = -2/n$ and of rank n . Its determinant g is

$$(3.2) \quad g \stackrel{\text{def}}{=} \gamma, \quad \gamma \stackrel{\text{def}}{=} \text{sgn } h.$$

There is only one f -invariant symmetric tensor density $\mathfrak{g}^{\lambda\nu}$ of weight $p = 2/n$ and of rank n which satisfies the f -invariant equation

$$(3.3) \quad \mathfrak{g}^{\lambda\nu} \mathfrak{g}_{\nu\mu} = \delta_\mu^\lambda.$$

This density is given by

$$(3.4) \quad \mathfrak{g}^{\lambda\nu} = \frac{\partial \ln g}{\partial \mathfrak{g}_{\lambda\nu}}.$$

Proof. We have

$$(3.5) \quad h = \frac{1}{n!} \tilde{\mathfrak{E}}^{\mu_1 \dots \mu_n} \tilde{\mathfrak{E}}^{\lambda_1 \dots \lambda_n} \mathfrak{h}_{\mu_1 \lambda_1} \dots \mathfrak{h}_{\mu_n \lambda_n},$$

*In taking the derivative $\partial/\partial \mathfrak{g}_{\lambda\nu}$ of g we have to distinguish between $\mathfrak{g}_{\lambda\nu}$ and $\mathfrak{g}_{\nu\lambda}$.

and this equation shows that h is a scalar. The remaining parts of the statements are obvious.

Theorem (3.1a). *Let $g_{\lambda\mu}$ be a given tensor density with the properties mentioned in the previous lemma. Let $\mathfrak{G}_{\mu\lambda\nu} = \mathfrak{G}_{\mu\nu\lambda}$ be a given f -invariant tensor density of weight $p = -2/n$ satisfying the condition*

$$(3.6) \quad \mathfrak{G}_{\mu\lambda\nu} g^{\lambda\nu} = 0.$$

Then there is a unique connection $\Gamma_{\lambda}^{\nu}{}_{\mu} = \Gamma_{\mu}^{\nu}{}_{\lambda}$ satisfying the requirements:

(a) *We have*

$$(3.7) \quad \nabla_{\mu} g_{\lambda\nu} = -\mathfrak{G}_{\mu\lambda\nu} .^*$$

(b) *The given vector*

$$(3.8a) \quad Q_{\mu} = \mathfrak{G}^{-1} \nabla_{\mu} \mathfrak{G}$$

is not a gradient:

$$(3.8b) \quad \partial_{[\mu} Q_{\omega]} \neq 0.$$

The connection $\Gamma_{\lambda}^{\nu}{}_{\mu}$ is given by

$$(3.9a) \quad 2\Gamma_{\lambda}^{\nu}{}_{\mu} = g^{\alpha\nu}(\Delta_{\lambda} g_{\alpha\mu} + \Delta_{\mu} g_{\alpha\lambda} - \Delta_{\alpha} g_{\lambda\mu} + \mathfrak{G}_{\lambda\alpha\mu} + \mathfrak{G}_{\mu\alpha\lambda} - \mathfrak{G}_{\alpha\lambda\mu})$$

where

$$(3.9b) \quad \Delta_{\lambda} \stackrel{\text{def}}{=} \partial_{\lambda} - \partial_{\lambda} \ln \mathfrak{G} + Q_{\lambda} .$$

It is f -invariant and not volume preserving:

$$(3.10) \quad {}^* \Gamma_{\lambda}^{\nu}{}_{\mu} = \Gamma_{\lambda}^{\nu}{}_{\mu} ,$$

$$(3.11) \quad R_{\omega\mu} \neq 0.$$

Proof. The equations (3.7) and (3.8a) are equivalent to

$$(3.12) \quad (a) \quad \partial_{\mu} g_{\lambda\nu} = \Gamma_{\lambda}^{\alpha}{}_{\mu} g_{\alpha\nu} + \Gamma_{\nu}^{\alpha}{}_{\mu} g_{\lambda\alpha} - \frac{2}{n} \Gamma_{\alpha}^{\alpha}{}_{\mu} g_{\lambda\nu} - \mathfrak{G}_{\mu\lambda\nu} ,$$

$$(b) \quad \partial_{\mu} \ln \mathfrak{G} + \frac{2}{n} \Gamma_{\alpha}^{\alpha}{}_{\mu} = Q_{\mu} .$$

Eliminating $\Gamma_{\alpha}^{\alpha}{}_{\mu}$ from (3.12a, b) and applying to the resulting equation CHRISTOFFEL's method of elimination, we obtain (3.9a). The relations (3.10) and (3.11) will be proved simultaneously with the following two theorems.

Theorem (3.2). *The vector Q_{μ} transforms by an f -transformation according to*

$$(3.13) \quad {}^* Q_{\mu} = Q_{\mu} + \partial_{\mu} f.$$

* ∇_{μ} is the symbol of covariant derivative with respect to $\Gamma_{\lambda}^{\nu}{}_{\mu}$.

Theorem (3.3). *The first set of integrability conditions of (3.8a) is*

$$(3.14) \quad R_{\omega\mu} = n \partial_{[\mu} Q_{\omega]} .$$

Proof. The equation (3.13) follows at once from (3.8a) and (2.8). The equation (3.13) together with (2.8) yields

$$(3.15) \quad {}^*Q_\mu - \partial_\mu \ln {}^*\mathfrak{G} = Q_\mu - \partial_\mu \ln \mathfrak{G},$$

and this equation shows that (3.10) holds.

In order to prove Theorem (3.3) let us rewrite (3.8a) in the form

$$(3.16) \quad \mathfrak{G} Q_\mu = \nabla_\mu \mathfrak{G}.$$

Hence

$$(3.17a) \quad \nabla_{[\omega} \mathfrak{G} Q_{\mu]} = \mathfrak{G} Q_{[\omega} Q_{\mu]} + \mathfrak{G} \partial_{[\omega} Q_{\mu]} .$$

On the other hand,

$$(3.17b) \quad \nabla_{[\omega} \nabla_{\mu]} \mathfrak{G} = -\frac{1}{n} R_{\omega\mu} \mathfrak{G},$$

so that (3.17a, b) lead to (3.14). This equation together with the requirement (3.8b) yields (3.11).

Remark I. Using (3.7) and (3.3) we obtain easily

$$(3.18) \quad \begin{aligned} (a) \quad & \nabla_\mu \mathfrak{g}^{\lambda\nu} = \mathfrak{G}_\mu^{\lambda\nu}, \\ (b) \quad & \mathfrak{G}_\mu^{\lambda\nu} \stackrel{\text{def}}{=} \mathfrak{G}_{\mu\alpha\beta} \mathfrak{g}^{\alpha\lambda} \mathfrak{g}^{\beta\nu} = \mathfrak{G}_\mu^{\nu\lambda}. \end{aligned}$$

Hence $\mathfrak{G}_\mu^{\lambda\nu}$ is a tensor density of weight $p = 2/n$.

Remark II. Because g is a scalar, we have

$$\nabla_\mu g = \partial_\mu g,$$

so that according to (3.2) and (3.7)

$$-\mathfrak{g}^{\lambda\nu} \nabla_\mu \mathfrak{g}_{\lambda\nu} = -\partial_\mu \ln g = 0 = \mathfrak{G}_{\mu\lambda\nu} \mathfrak{g}^{\lambda\nu}.$$

Hence the requirement (3.6) follows from (3.2).

Theorem (3.1b). *Any a priori given symmetric connection $\Gamma_{\lambda}{}^{\nu}{}_\mu$ may be expressed in the form (3.9).*

Proof. Start with given quantities $\Gamma_{\lambda}{}^{\nu}{}_\mu$ and $\mathfrak{g}_{\lambda\mu}$. Define $\mathfrak{G}_{\mu\lambda\nu}$ and Q_μ by (3.7) and (3.8a). Then $\mathfrak{G}_{\mu\lambda\nu}$ satisfies (3.6). The equations (3.7) and (3.8a) are equivalent to (3.12), and these latter equations yield (3.9a).

Remark. If the connection is volume preserving, then Q_μ is a gradient and a convenient f -transformation leads to ${}^*Q_\mu = 0$. If the connection is not volume preserving we have always $Q_\mu \neq 0$.

4. Continuation.

Theorem (4.1). *The tensors*

$$(4.1) \quad (a) \quad g_{\lambda\mu} = g_{\lambda\mu} \mathfrak{G}^{-1}, \quad (b) \quad g^{\lambda\nu} = g^{\lambda\nu} \mathfrak{G}$$

have the following properties:

(a) *They are symmetric and satisfy the f-invariant relation*

$$(4.2) \quad g_{\lambda\mu} g^{\lambda\nu} = \delta_{\mu}^{\nu}.$$

(b) *They transform by an f-transformation according to*

$$(4.3) \quad (a) \quad *g_{\lambda\mu} = (e)^{-f} g_{\lambda\mu}, \quad (b) \quad *g^{\lambda\nu} = (e)^f g^{\lambda\nu}.$$

(c) *The determinant*

$$(4.4a) \quad g = \text{Det} ((g_{\lambda\mu})) = 1/\text{Det} ((g^{\lambda\nu}))$$

satisfies the condition

$$(4.4b) \quad g = \mathfrak{G}^{-n} \gamma,$$

so that

$$(4.5) \quad *g = (e)^{-nf} g$$

and

$$(4.6) \quad (a) \quad g_{\lambda\mu} = g_{\lambda\mu} |g|^{1/n}, \quad (b) \quad g^{\lambda\nu} = g^{\lambda\nu} |g|^{-1/n}.$$

(d) *If we put*

$$(4.7) \quad \begin{aligned} (a) \quad Q_{\mu\lambda\nu} &= \mathfrak{G}_{\mu\lambda\nu} \mathfrak{G}^{-1}, \\ (b) \quad Q_{\mu}^{\lambda\nu} &= Q_{\mu\alpha\beta} g^{\alpha\lambda} g^{\beta\nu}, \end{aligned}$$

then the following f-invariant equations hold:

$$(4.8) \quad \begin{aligned} (a) \quad \nabla_{\mu} g_{\lambda\nu} &= -Q_{\mu} g_{\lambda\nu} - Q_{\mu\lambda\nu}, \\ (b) \quad \nabla_{\mu} g^{\lambda\nu} &= Q_{\mu} g^{\lambda\nu} + Q_{\mu}^{\lambda\nu}, \\ (c) \quad \nabla_{\mu} g &= -n Q_{\mu} g. \end{aligned}$$

Proof. The statement (a) is obvious. The equations (4.3) follow from (4.1) and (2.8). The equation (4.4b) follows from (4.4a) and (3.2). The remaining part of the statement (c) is obvious.* The equations (4.8a, b) follow immediately from (4.1), (3.7) and (3.8a). The equation (4.8c) is an immediate consequence of (3.8a) and (4.4b). From (4.7a, b), (4.3a, b) and (2.8) we have

$$(4.9) \quad \begin{aligned} (a) \quad *Q_{\mu\lambda\nu} &= (e)^{-f} Q_{\mu\lambda\nu}, \\ (b) \quad *Q_{\mu}^{\lambda\nu} &= (e)^f Q_{\mu}^{\lambda\nu}. \end{aligned}$$

* $\mathfrak{G} > 0$, so that $\mathfrak{G}^{-n} > 0$. Hence we see from (4.4b) that $g_{\gamma} = |g|$, i.e. $\gamma = \text{sgn } g$.

These equations, together with (4.3a, b) and (3.13), show the f -invariance of (4.8).

Theorem (4.2). *The equation (3.9a) is equivalent to*

$$(4.10a) \quad \Gamma_{\lambda}^{\nu}{}_{\mu} = \{\lambda^{\nu}{}_{\mu}\} + \frac{1}{2}(Q_{\mu}\delta_{\lambda}^{\nu} + Q_{\lambda}\delta_{\mu}^{\nu} - g^{\alpha\nu}Q_{\alpha}g_{\lambda\mu}) \\ + \frac{1}{2}g^{\alpha\nu}(Q_{\mu\alpha\lambda} + Q_{\lambda\alpha\mu} - Q_{\alpha\lambda\mu}),$$

where $\{\lambda^{\nu}{}_{\mu}\}$ are Christoffel's symbols of $g_{\lambda\mu}$:

$$(4.10b) \quad \{\lambda^{\nu}{}_{\mu}\} \stackrel{\text{def}}{=} \frac{1}{2}g^{\nu\alpha}(\partial_{\lambda}g_{\alpha\mu} + \partial_{\mu}g_{\alpha\lambda} - \partial_{\alpha}g_{\lambda\mu}).$$

Proof. We have, by virtue of (4.1a),

$$\Delta_{\lambda}g_{\alpha\mu} = \partial_{\lambda}g_{\alpha\mu}\mathfrak{G} - g_{\alpha\mu}\mathfrak{G}(\partial_{\lambda}\ln\mathfrak{G} - Q_{\lambda}) = (\partial_{\lambda}g_{\alpha\mu} + Q_{\lambda}g_{\alpha\mu})\mathfrak{G}.$$

Substituting this value into (3.9a) and remembering (4.1b) and (4.7a), we obtain (4.10).

In subsequent considerations we shall need besides the curvature tensor $R_{\omega\mu\lambda}^{\nu}$ also the tensor

$$(4.11) \quad W_{\omega\mu\lambda}^{\nu} \stackrel{\text{def}}{=} R_{\omega\mu\lambda}^{\nu} - \frac{1}{n}R_{\omega\mu}\delta_{\lambda}^{\nu}.$$

Theorem (4.3). *We have for $n \neq 2$*

$$(4.12) \quad R_{\omega\mu} = \frac{3n}{2-n}W_{[\omega\mu\alpha]}^{\alpha}.$$

Proof. Applying the identity

$$R_{[\omega\mu\lambda]}^{\nu} = 0$$

to (4.11), we obtain

$$W_{[\omega\mu\lambda]}^{\nu} = -\frac{1}{n}R_{[\omega\mu}\delta_{\lambda]}^{\nu},$$

and this equation leads at once to (4.12).

II.

5. Holonomy group of an L_n^* . It is known that the ${}^{\nu}_{\lambda}$ -domain of the tensors

$$(5.1) \quad R_{\omega\mu\lambda}^{\nu}, \quad \nabla_{\xi_s} \cdots \nabla_{\xi_1} R_{\omega\mu\lambda}^{\nu}, \quad s = 1, 2, \dots,$$

is described by l linearly independent tensors $L_{a\lambda}^{\nu}$:

$$(5.2) \quad (a) \quad R_{\omega\mu\lambda}^{\nu} = M_{\omega\mu}^a L_{a\lambda}^{\nu}, \\ (b) \quad \nabla_{\xi_s} \cdots \nabla_{\xi_1} R_{\omega\mu\lambda}^{\nu} = M_{\omega\mu\xi_s \cdots \xi_1}^a L_{a\lambda}^{\nu},$$

where

$$(5.3) \quad M_{\omega\mu\xi_s \cdots \xi_1}^a = D_{\xi_s} \cdots D_{\xi_1} M_{\omega\mu}^a.$$

*This section displays without proofs some results obtained in Section [II] 4. Unless stated explicitly otherwise, German indices run from 1 to l . (In [II] and [III] we wrote r instead of l .)

There is at least one set of tensors $L^{a\lambda}{}_{\nu}$, such that

$$(5.4) \quad L_{a\lambda}{}^{\nu} L^{a\lambda}{}_{\nu} = \delta_a^a.$$

If y^{ν} are Cartesian coordinates of the local tangential E_n , the holonomy group $G_e(R)$ is given by l independent infinitesimal operators

$$(5.5) \quad X_a \stackrel{\text{def}}{=} L_{a\lambda}{}^{\nu} y^{\lambda} \frac{\partial}{\partial y^{\nu}}.$$

The structural constants* $C_{a\alpha}{}^{\beta}$ of $G_e(R)$ are given by

$$(5.6) \quad C_{a\alpha}{}^{\beta} = 2L_{[a|\lambda|}{}^{\alpha} L_{\beta] \alpha}{}^{\nu} L^{b\lambda}{}_{\nu}.$$

They satisfy the relation

$$(5.7) \quad 2L_{[a|\lambda|}{}^{\alpha} L_{\beta] \alpha}{}^{\nu} = C_{a\alpha}{}^{\beta} L_{b\lambda}{}^{\nu}.$$

The tensors $L_{a\lambda}{}^{\nu}$ are covariant constant with respect to the operator D_{ξ} :

$$(5.8) \quad D_{\xi} L_{a\lambda}{}^{\nu} \stackrel{\text{def}}{=} \nabla_{\xi} L_{a\lambda}{}^{\nu} - \Gamma_a{}^{\alpha}{}_{\xi} L_{\alpha\lambda}{}^{\nu} = 0.$$

If $T^{\lambda}{}_{\alpha\beta\gamma}$ is a tensor satisfying the equation

$$(5.9) \quad \nabla_{\mu} T^{\lambda}{}_{\alpha\beta\gamma} = T^{\lambda}{}_{\alpha\beta\gamma} \partial_{\mu} F,$$

the complete set of integrability conditions of (5.9) reduces to

$$(5.10) \quad L_{a\lambda}{}^{\alpha} T^{\nu}{}_{\alpha\beta\gamma} + \dots - L_{a\alpha}{}^{\nu} T^{\alpha}{}_{\lambda\beta\gamma} - \dots = 0.$$

The tensors $L_{a\lambda}{}^{\nu}$ are defined up to arbitrary linear transformations

$$(5.11a) \quad L_{a'\lambda}{}^{\nu} = A_a{}^{a'} L_{a\lambda}{}^{\nu}.$$

Here $A_a{}^{a'}$ are subject only to the restriction

$$(5.11b) \quad \text{Det} ((A_a{}^{a'})) \neq 0.$$

Hence a in $L_{a\lambda}{}^{\nu}$ may be looked upon as a covariant index in the group space G_i .

6. Integrability conditions. Let us put

$$(6.1) \quad \begin{aligned} (a) \quad L_a &\stackrel{\text{def}}{=} \frac{1}{n} L_{a\lambda}{}^{\lambda}, \\ (b) \quad W_{a\lambda}{}^{\nu} &\stackrel{\text{def}}{=} L_{a\lambda}{}^{\nu} - L_a \delta_{\lambda}^{\nu}. \end{aligned}$$

The following two statements will be proved simultaneously.

Lemma (6.1). *We have*

$$(6.2) \quad L_a \neq 0. \dagger$$

*I.e. constants with respect to the y^{ν} .

† L_a is a covariant vector in the group space G_e . Therefore (6.2) has to be understood to claim that at least one of the expressions $L_1, \dots, L_l$ is different from zero.

Theorem (6.1). *We have*

$$(6.3) \quad M^a{}_{\omega\mu} L_a = -\nabla_{[\omega} Q_{\mu]} = \frac{1}{n} R_{\omega\mu} \neq 0.$$

Proof. The equation, as well as the inequality (6.3), follows at once from (3.14), (3.11), (5.2a) and (6.1a). The inequality (6.3) yields at once (6.2).

Theorem (6.2). *The λ -domain of the tensor set*

$$(6.4) \quad W_{\omega\mu\lambda}{}^\nu, \quad \nabla_{\xi_1} \cdots \nabla_{\xi_i} W_{\omega\mu\lambda}{}^\nu$$

is described by the tensors $W_{a\lambda}{}^\nu$ defined by (6.1b):

$$(6.5) \quad \begin{aligned} (a) \quad & W_{\omega\mu\lambda}{}^\nu = M^a{}_{\omega\mu} W_{a\lambda}{}^\nu, \\ (b) \quad & \nabla_{\xi_1} \cdots \nabla_{\xi_i} W_{\omega\mu\lambda}{}^\nu = M^a{}_{\omega\mu\xi_1 \cdots \xi_i} W_{a\lambda}{}^\nu. \end{aligned}$$

Proof. We have, by virtue of (5.2a), (6.1b) and (6.3),

$$W_{\omega\mu\lambda}{}^\nu = R_{\omega\mu\lambda}{}^\nu - \frac{1}{n} R_{\omega\mu} \delta_\lambda^\nu = M^a{}_{\omega\mu} L_{a\lambda}{}^\nu - M^a{}_{\omega\mu} L_a \delta_\lambda^\nu = M^a{}_{\omega\mu} W_{a\lambda}{}^\nu.$$

The proof of (6.5b), based on (5.8), follows the same pattern.

Remark. Notice that we did not claim the linear independence of the tensors $W_{a\lambda}{}^\nu$. As a matter of fact, we shall see later that these tensors are not necessarily linearly independent.

Theorem (6.3). *Let $\mathfrak{T}_{\lambda_1 \cdots \lambda_v}^{\nu_1 \cdots \nu_u}$ be an f -invariant tensor density of weight $p = (u - v)/n$, integral of*

$$(6.6) \quad \begin{aligned} \nabla_\mu \mathfrak{T}_{\lambda_1 \cdots \lambda_v}^{\nu_1 \cdots \nu_u} &= -\mathfrak{T}_{\lambda_1 \cdots \lambda_v}^{\nu_1 \cdots \nu_u} F_\mu \\ (F_\mu &\stackrel{\text{def}}{=} \partial_\mu \ln F). \end{aligned}$$

The complete set of integrability conditions of (6.6) reduces to

$$(6.7) \quad \begin{aligned} \mathfrak{T}_{a\lambda_1 \cdots \lambda_v}^{\nu_1 \cdots \nu_u} &\stackrel{\text{def}}{=} W_{a\lambda_1}{}^\alpha \mathfrak{T}_{\alpha\lambda_2 \cdots \lambda_v}^{\nu_1 \cdots \nu_u} + \cdots + W_{a\lambda_v}{}^\alpha \mathfrak{T}_{\lambda_1 \cdots \lambda_{v-1}\alpha}^{\nu_1 \cdots \nu_u} \\ &\quad - W_{a\alpha}{}^{\nu_1} \mathfrak{T}_{\lambda_1 \cdots \lambda_v}^{\alpha\nu_2 \cdots \nu_u} - \cdots - W_{a\alpha}{}^{\nu_u} \mathfrak{T}_{\lambda_1 \cdots \lambda_v}^{\nu_1 \cdots \nu_{u-1}\alpha} = 0. \end{aligned}$$

Proof. We have, by virtue of (6.5a),

$$(6.8a) \quad \begin{aligned} 2\nabla_{[\omega} \nabla_{\mu]} \mathfrak{T}_{\lambda_1 \cdots \lambda_v}^{\nu_1 \cdots \nu_u} &= W_{\omega\mu\lambda_1}{}^\alpha \mathfrak{T}_{\alpha\lambda_2 \cdots \lambda_v}^{\nu_1 \cdots \nu_u} + \cdots + W_{\omega\mu\lambda_v}{}^\alpha \mathfrak{T}_{\lambda_1 \cdots \lambda_{v-1}\alpha}^{\nu_1 \cdots \nu_u} \\ &\quad - W_{\omega\mu\alpha}{}^{\nu_1} \mathfrak{T}_{\lambda_1 \cdots \lambda_v}^{\alpha\nu_2 \cdots \nu_u} - \cdots - W_{\omega\mu\alpha}{}^{\nu_u} \mathfrak{T}_{\lambda_1 \cdots \lambda_v}^{\nu_1 \cdots \nu_{u-1}\alpha} = M^a{}_{\omega\mu} \mathfrak{T}_{a\lambda_1 \cdots \lambda_v}^{\nu_1 \cdots \nu_u} \end{aligned}$$

and, by virtue of (6.6),

$$(6.8b) \quad 2\nabla_{[\omega} F_{\mu]} \mathfrak{T}_{\lambda_1 \cdots \lambda_v}^{\nu_1 \cdots \nu_u} = 0.$$

Hence the first set of integrability conditions is

$$(6.9) \quad M^a{}_{\omega\mu} \mathfrak{T}_{a\lambda_1 \cdots \lambda_v}^{\nu_1 \cdots \nu_u} = 0.$$

In order to obtain the second set we have to eliminate $\nabla_\mu \mathfrak{T}_{\lambda_1 \dots \lambda_s}^{v_1 \dots v_u}$ from

$$\nabla_{\xi_1} M^a_{\omega \mu \xi_1} \mathfrak{T}_{a \lambda_1 \dots \lambda_s}^{v_1 \dots v_u} = 0$$

and (6.6). In doing so we obtain, by virtue of (6.9),

$$M^a_{\omega \mu \xi_1} \mathfrak{T}_{a \lambda_1 \dots \lambda_s}^{v_1 \dots v_u} = 0.$$

Proceeding in the same way, we obtain the $(s+1)^{\text{th}}$ set of integrability conditions in the form

$$(6.10) \quad M^a_{\omega \mu \xi_1 \dots \xi_s} \mathfrak{T}_{a \lambda_1 \dots \lambda_s}^{v_1 \dots v_u} = 0.$$

Therefore (6.9) and (6.10) represent the whole set of integrability conditions. Because the $W_{a\lambda}^r$ describe the whole λ -domain of (6.4), the equations (6.9) can be satisfied if and only if (6.7) holds.*

Theorem (6.4). *Put*

$$(6.11) \quad T_{\lambda_1 \dots \lambda_s}^{v_1 \dots v_u} \stackrel{\text{def}}{=} \mathfrak{T}_{\lambda_1 \dots \lambda_s}^{v_1 \dots v_u} \mathfrak{G}^{\frac{1}{2}(u-v)} = \mathfrak{T}_{\lambda_1 \dots \lambda_s}^{v_1 \dots v_u} |\mathfrak{g}|^{(v-u)/2n}.$$

Then the tensor $T_{\lambda_1 \dots \lambda_s}^{v_1 \dots v_u}$ transforms by an f -transformation according to

$$(6.12a) \quad *T_{\lambda_1 \dots \lambda_s}^{v_1 \dots v_u} = (e)^{f(u-v)/2} T_{\lambda_1 \dots \lambda_s}^{v_1 \dots v_u},$$

and (6.6) is equivalent to the f -invariant equation

$$(6.12b) \quad \nabla_\mu T_{\lambda_1 \dots \lambda_s}^{v_1 \dots v_u} = T_{\lambda_1 \dots \lambda_s}^{v_1 \dots v_u} \left(\frac{u-v}{2} Q_\mu - F_\mu \right).$$

The complete set of integrability conditions of (6.12) reduces to

$$(6.13) \quad T_{a \lambda_1 \dots \lambda_s}^{v_1 \dots v_u} \stackrel{\text{def}}{=} W_{a \lambda_1}^\alpha T_{\alpha \lambda_2 \dots \lambda_s}^{v_1 \dots v_u} + \dots + W_{a \lambda_s}^\alpha T_{\lambda_1 \dots \lambda_{s-1} \alpha}^{v_1 \dots v_u} \\ - W_{a \alpha}^{v_1} T_{\lambda_1 \dots \lambda_s}^{\alpha v_2 \dots v_u} - \dots - W_{a \alpha}^{v_u} T_{\lambda_1 \dots \lambda_s}^{v_1 \dots \alpha} = 0.$$

Proof. We have, by virtue of (6.11), (4.8c) (or (3.8a)) and (6.6),

$$\begin{aligned} \nabla_\mu T_{\lambda_1 \dots \lambda_s}^{v_1 \dots v_u} &= (\nabla_\mu \mathfrak{T}_{\lambda_1 \dots \lambda_s}^{v_1 \dots v_u}) |\mathfrak{g}|^{(v-u)/2n} + \mathfrak{T}_{\lambda_1 \dots \lambda_s}^{v_1 \dots v_u} \nabla_\mu (\gamma \mathfrak{g})^{(v-u)/2n} \\ &= -F_\mu \mathfrak{T}_{\lambda_1 \dots \lambda_s}^{v_1 \dots v_u} |\mathfrak{g}|^{(v-u)/2n} + \frac{u-v}{2n} n Q_\mu \mathfrak{T}_{\lambda_1 \dots \lambda_s}^{v_1 \dots v_u} |\mathfrak{g}|^{(v-u)/2n} \\ &= T_{\lambda_1 \dots \lambda_s}^{v_1 \dots v_u} \left(-F_\mu + \frac{u-v}{2} Q_\mu \right). \end{aligned}$$

Hence (6.12b) and (6.6) are equivalent and so their integrability conditions must be. (6.13) is clearly equivalent to (6.7). The transformation (6.12a) follows from (6.11) and (2.8). The f -invariance of (6.12b) is then obvious.

*Let $l' \leq l$ be the number of linearly independent tensors $W_{a\lambda}^r$. Then we could choose the tensors $L_{a\lambda}^r$ in such a way that the l' linearly independent tensors W are $W_{a'\lambda}^{r'}$, $a' = \underline{1}, \dots, \underline{l'}$ while $W_{a'\lambda}^{r'} = 0$, $a' = \underline{l'+1}, \dots, \underline{l}$. This choice of the W 's leads immediately to the above statement.

Remark. Theorems (6.3) and (6.4) will be needed in the next paper.

7. Eigenvalues and eigenvectors. Let $T_\mu{}^\nu$ be an arbitrary tensor. A vector a^λ is termed an eigenvector of $T_\mu{}^\nu$ if it satisfies an equation of the form

$$\lambda a^\nu = T_\mu{}^\nu a^\mu.$$

The scalar λ is termed an eigenvalue of $T_\mu{}^\nu$ and a^ν is said to belong to λ . The eigenvalues are roots of the equation

$$\text{Det} ((\lambda \delta_\mu^\nu - T_\mu^\nu)) = 0.$$

In this section we shall deal with the eigenvalues and eigenvectors of $L_{a\lambda}{}^\nu$ and $W_{a\lambda}{}^\nu$. The eigenvalues of $L_{a\lambda}{}^\nu$ (of $W_{a\lambda}{}^\nu$) will be denoted by λ_a (by ω_a).*

Theorem (7.1). *The eigenvalues λ_a and ω_a are related by*

$$(7.1) \quad \lambda_a = \omega_a + L_a .$$

Moreover,

$$(7.2) \quad \begin{aligned} (a) \quad \sum_x \lambda_a &= L_{a\lambda}{}^\nu = n L_a , \\ (b) \quad \sum_x \omega_a &= W_{a\lambda}{}^\lambda = 0. \end{aligned}$$

Proof. The equations for λ_a and ω_a are

$$(7.3) \quad \begin{aligned} (a) \quad \text{Det} ((\lambda_a \delta_\lambda^\nu - L_{a\lambda}{}^\nu)) &= 0, \\ (b) \quad \text{Det} ((\omega_a \delta_\lambda^\nu - W_{a\lambda}{}^\nu)) &= 0. \end{aligned}$$

Substituting into (7.3a) from (6.1b) we obtain

$$(7.4) \quad \text{Det} ((\lambda_a - L_a) \delta_\lambda^\nu - W_{a\lambda}{}^\nu) = 0.$$

Comparing (7.3b) and (7.4) we obtain (7.1). The factor of $(\lambda_a)^{n-1}$ in (7.3a) is

$$\begin{aligned} -\frac{n}{n!} \tilde{\mathfrak{E}}^{\alpha_1 \dots \alpha_{n-1} \nu} \tilde{\mathfrak{e}}_{\beta_1 \dots \beta_{n-1} \lambda} \delta_{\alpha_1}^{\beta_1} \dots \delta_{\alpha_{n-1}}^{\beta_{n-1}} L_{a\nu}{}^\lambda \\ = -\frac{n}{n!} \tilde{\mathfrak{E}}^{\alpha_1 \dots \alpha_{n-1} \nu} \tilde{\mathfrak{e}}_{\alpha_1 \dots \alpha_{n-1} \lambda} L_{a\nu}{}^\lambda = -L_{a\lambda}{}^\lambda, \end{aligned}$$

while the factor of $(\lambda_a)^n$ is 1. Hence (7.2a) holds. The first equation (7.2b) may be proved in a similar way. The second equation (7.2b) follows from (7.1) and (7.2a).

*As in previous papers Latin indices run from 1 to n , unless stated explicitly otherwise. The indices x, y, z, t do not follow the usual summation convention.

In the next theorem we introduce the following abbreviation:

$$(7.5) \quad L_{\lambda}{}^{\nu} \equiv L_{\lambda}{}^{\nu}(v) \stackrel{\text{def}}{=} L_{a\lambda}{}^{\nu} v^a,$$

where v^a is an arbitrary vector of the group space. Moreover, we denote by $r(v)$ the rank of the matrix $((L_{\lambda}{}^{\nu}(v)))$.

Theorem (7.2a). *In order that either one of the equations*

$$(7.6) \quad (a) \quad \nabla_{\mu} a^{\nu} = 0, \quad (b) \quad \nabla_{\mu} b_{\lambda} = 0$$

admits exactly q_1 linearly independent solutions, $0 < q_1 < n$, it is necessary that

(a) *the μ -domain (the ν -domain)* of $L_{\mu}{}^{\nu}$ be contained in a covariant (contravariant) $(n - q_1)$ -vector, independent of v^a ,*

(b) *$r(v) \leq n - q_1$ and the value $n - q_1$ be actually reached for some v^a .†*

If these conditions are satisfied then

(a) *there is a number $q \geq q_1$ such that for every v^a*

$$(7.7) \ddagger \quad \begin{aligned} (a) \quad & \tilde{\mathfrak{E}}^{\lambda_1 \dots \lambda_n} \tilde{\mathfrak{e}}_{\mu_1 \dots \mu_n} L_{\lambda_1}{}^{\mu_1} \dots L_{\lambda_n}{}^{\mu_n} = 0, \\ (b) \quad & \tilde{\mathfrak{E}}^{\alpha_1 \dots \alpha_s \lambda_1 \dots \lambda_{n-s}} \tilde{\mathfrak{e}}_{\alpha_1 \dots \alpha_s \mu_1 \dots \mu_{n-s}} L_{\lambda_1}{}^{\mu_1} \dots L_{\lambda_{n-s}}{}^{\mu_{n-s}} = 0, \\ & s = 1, \dots, q-1, \\ (c) \quad & \tilde{\mathfrak{E}}^{\alpha_1 \dots \alpha_q \lambda_1 \dots \lambda_{n-q}} \tilde{\mathfrak{e}}_{\alpha_1 \dots \alpha_q \mu_1 \dots \mu_{n-q}} L_{\lambda_1}{}^{\mu_1} \dots L_{\lambda_{n-q}}{}^{\mu_{n-q}} \neq 0 \end{aligned}$$

and

(b) *any solution of (7.6a, b) may be written*

$$a^{\nu} = \alpha A^{\nu}, \quad b^{\nu} = \beta B^{\nu}$$

where A^{ν} , B^{ν} are solutions of

$$(7.8) \quad (a) \quad L_{a\mu}{}^{\nu} A^{\mu} = 0, \quad (b) \quad L_{a\mu}{}^{\nu} B_{\nu} = 0$$

such that

$$(7.9) \quad (a) \quad \nabla_{\mu} A^{\nu} = -A^{\nu} \partial_{\mu} \ln \alpha, \quad (b) \quad \nabla_{\mu} B_{\lambda} = -B_{\lambda} \partial_{\mu} \ln \beta.$$

Proof. The complete set of integrability conditions of (7.6a) is (7.8a), where A^{ν} is proportional to the unknown a^{ν} . The equation (7.8a) admits exactly q_1 linearly independent solutions if and only if

(I) the μ -domain of $L_{a\mu}{}^{\nu}$ is contained in a covariant $(n - q_1)$ -vector independent of a ,

*The first (the second) requirement refers to (7.6a) ((7.6b)).

†The first statement is a consequence of the second statement.

‡If $q = \begin{cases} 1 \\ n \end{cases}$ then only $\left(7.7 \begin{smallmatrix} a, c \\ a, b \end{smallmatrix} \right)$ hold.

(II) the μ -rank $r[\mu]^*$ of $L_{a\mu}{}^v$ is $r[\mu] = n - q_1$.† The conditions (I), (II) are equivalent to the conditions (a), (b). If they are satisfied then

$$(7.10) \quad L_{\mu}{}^v A^{\mu} = 0$$

admits exactly q_1 linearly independent solutions, which are the same for any v^a . Among them there are solutions such that (7.9a) is satisfied.** Hence $a^v = \alpha A^v$ is a solution of (7.6a). The statement about (7.6b) may be proved in a similar way.

If $r(v) \leq n - q_1$ then every one of the linearly independent solutions of (7.10) is an eigenvector of $L_{\lambda}{}^v$ belonging to the eigenvalue 0. Hence there must be a number $q \geq q_1$ such that $\text{Det}((\lambda \delta_{\mu}^v - L_{\mu}{}^v))$ is divisible by λ^q but not by λ^{q+1} . This happens if and only if (7.7a, b, c) are satisfied.

Remark I. Theorem (7.2) would be trivial for $q_1 = 0, n$. Hence we assumed $0 < q_1 < n$.

Remark II. There are cases where $q > q_1$, as the following example shows. Assume $n = 4, l = 1$ and define $L_{i\lambda}{}^v$ by

$$(7.11) \quad L_{iI}{}^{II} = L_{iI}{}^{IV} = 1, \quad \text{the remaining } L_{i\lambda}{}^v = 0.$$

The tensor $L_{\lambda}{}^v$ is in this simple case

$$L_{\lambda}{}^v = v L_{i\lambda}{}^v, \quad v \neq 0,$$

and $r(v) = n - 2$, so that $q_1 = 2$. In fact, all solutions of (7.10) are

$$A^v(0, 0, A^{III}, A^{IV}).$$

On the other hand,

$$\text{Det}((\lambda \delta_{\mu}^v - L_{\mu}{}^v)) = \lambda^4.$$

Hence $q = 4$ and $q_1 = 2$.

It is not difficult to find cases where $q = q_1$.

Theorem (7.2b). *Theorem (7.2a) holds also if we replace in turn*

$$a^v, \quad b_{\lambda}, \quad L_{\lambda}{}^v, \quad A^v, \quad B_{\lambda}$$

†The first statement is a consequence of the second statement.

*I.e. the rank of the matrix

$$((L_{aI}{}^v, L_{aII}{}^v, L_{aN}{}^v)).$$

**If A^v is an arbitrary system of q_1 linearly independent solutions of (7.10), then

$$\nabla_{\mu} A^v = \Lambda_{b\mu}^a A^v, \quad \partial_{[\omega} \Lambda_{|a|}^c{}_{|\mu]} + \Lambda_{a[\mu}^b \Lambda_{|b|}^c{}_{|\omega]} = 0, \quad a, b, c = 1, \dots, q_1,$$

so that the system

$$\partial_{\mu} \varphi^c + \Lambda_{a\mu}^c \varphi^a = -\varphi^c \partial_{\mu} \ln \alpha$$

is completely integrable and $A^v \stackrel{\text{def}}{=} \varphi^a A^v$ satisfies (7.9a).

by

$$a', \quad b_\lambda, \quad W_\lambda', \quad \mathfrak{A}', \quad \mathfrak{B}_\lambda$$

where $a', \mathfrak{A}'(b_\lambda, \mathfrak{B}_\lambda)$ are vector density of weight $p = 1/n$ (of weight $p = -1/n$) and

$$W_\lambda' \equiv W_\lambda'(v) \stackrel{\text{def}}{=} v^a W_{a\lambda}'.$$

Proof. Performing this substitution we obtain

$$(7.12) \quad (a) \quad \nabla_\mu a' = 0, \quad (b) \quad \nabla_\mu b_\lambda = 0$$

instead of (7.6). According to Theorem (6.3) the complete set of integrability conditions reduces to

$$(7.13) \quad (a) \quad W_{a\mu}' \mathfrak{A}^\mu = 0, \quad (b) \quad W_{a\mu}' \mathfrak{B}_\mu = 0,$$

and these equations replace (7.8a, b). The remaining part of the proof follows the pattern of the proof of Theorem (7.2a).

8. Continuation. We know that the tensors $L_{a\lambda}'$ are known up to linear transformations (5.11). This fact will be used in the following

Lemma (8.1). *The tensors $L_{a\lambda}'$ may be chosen in such a way that the following relations hold:*

$$(8.1) \quad L_a = \delta_a^{\dot{l}}$$

$$(8.2) \quad \begin{aligned} (a) \quad W_{\hat{a}\lambda}' &= L_{\hat{a}\lambda}', \quad \hat{a} = 1, \dots, \dot{l} - 1, \\ (b) \quad W_{\dot{l}\lambda}' &= L_{\dot{l}\lambda}' - \delta_\lambda^{\dot{l}}. \end{aligned}$$

Proof. We have from (5.11a) and (6.1a)

$$(8.3a) \quad L_{a'} = A_a^{\dot{a}} L_{\dot{a}}$$

We know from elementary algebra that if (6.2) holds, there is always a system of coefficients $A_a^{\dot{a}}$ such that

$$(8.3b) \quad \delta_{a'}^{\dot{l}'} = A_a^{\dot{a}} L_{\dot{a}}$$

Comparing (8.3a) and (8.3b) we obtain

$$L_{a'} = \delta_{a'}^{\dot{l}'},$$

and omitting primes in this equation we get (8.1). Then (8.2) follows from (8.1) and (6.1b).

We know that the tensors $L_{a\lambda}'$ are linearly independent. The question arises how many of the tensors $W_{a\lambda}'$ are linearly independent? This question is answered by the first of the following theorems, which will be proved simultaneously.

Theorem (8.1a). Let $\left\{ \begin{smallmatrix} l \\ w \end{smallmatrix} \right\}$ be the number of linearly independent tensors $\left\{ \begin{smallmatrix} L_{a\lambda} \\ W_{a\lambda} \end{smallmatrix} \right\}$.

Then

$$(8.4) \quad w = l - \theta$$

where

$$(8.5) \quad (a) \quad \theta = 0 \quad \text{or} \quad (b) \quad \theta = 1.$$

Theorem (8.1b). A necessary and sufficient condition that (8.5b) be satisfied is that there is a vector v^a in the group space G_e such that the tensor

$$(8.6) \quad L_{\lambda}{}^{\nu}(v) \stackrel{\text{def}}{=} v^a L_{a\lambda}{}^{\nu}$$

has the n -tuple eigenvalue $v^a L_a$. If this condition is satisfied, then

$$(8.7) \quad v^a L_a \neq 0.$$

The proof will be accomplished in three steps:

(1) The $L_{a\lambda}{}^{\nu}$ may always be chosen in such a way that (8.2) hold. Because they are linearly independent, we see from (8.2) that the number w of linearly independent tensors $W_{a\lambda}{}^{\nu}$ is at $\left\{ \begin{smallmatrix} \text{most} \\ \text{least} \end{smallmatrix} \right\}$ equal to $\frac{l}{l-1}$. Of course, it has to be proved that these limits can be reached.

(2) Consider the case $n > 2$, $l = 1$ and assume $\theta = 1$, so that

$$L_{1\lambda}{}^{\nu} = \delta_{\lambda}^{\nu}$$

and therefore, according to (5.2a),

$$R_{\omega\mu\lambda}{}^{\nu} = M^1{}_{\omega\mu} \delta_{\lambda}^{\nu}.$$

Then the identity $R_{[\omega\mu\lambda]}{}^{\nu} = 0$ yields for $n > 2$

$$R_{\omega\mu} = 0,$$

and the last equation contradicts (1.1). Hence we must have $\theta = 0$.

Assume now $l = 4$, $n = 2$; denote by u_1^{ν}, u_2^{ν} two linearly independent vectors and by $\overset{1}{u}_{\lambda}, \overset{2}{u}_{\lambda}$ their inverse. We may put

$$L_{1\lambda}{}^{\nu} = \overset{1}{u}_{\lambda} u_1^{\nu}, \quad L_{2\lambda}{}^{\nu} = \overset{2}{u}_{\lambda} u_2^{\nu}, \quad L_{3\lambda}{}^{\nu} = \overset{1}{u}_{\lambda} u_2^{\nu}, \quad L_{4\lambda}{}^{\nu} = \overset{2}{u}_{\lambda} u_1^{\nu},$$

so that, according to (6.1),

$$W_{1\lambda}{}^{\nu} = L_{1\lambda}{}^{\nu} - \frac{1}{n} \delta_{\lambda}^{\nu} = -L_{2\lambda}{}^{\nu} + \frac{1}{n} \delta_{\lambda}^{\nu} = -W_{2\lambda}{}^{\nu},$$

$$W_{3\lambda}{}^{\nu} = L_{3\lambda}{}^{\nu}, \quad W_{4\lambda}{}^{\nu} = L_{4\lambda}{}^{\nu}.$$

Hence in this case we have $\theta = 1$.

(3) Assume $\theta = 1$. Then there is a vector v^a such that

$$(8.8a) \quad v^a W_{a\lambda}{}' = 0,$$

or, according to (6.1b) and (8.6),

$$(8.8b) \quad L_{\lambda}{}'(v) = v^a L_a \delta_{\lambda}''.$$

Because the tensors $L_{a\lambda}{}'$ are linearly independent we must have

$$(8.8c) \quad L_{\lambda}{}'(v) \neq 0,$$

so that (8.7) holds. The equation (8.8b) implies that $L_{\lambda}{}'(v)$ has the n -tuple eigenvalue $v^a L_a$. Assume now that there is a vector v^a such that $L_{\lambda}{}'(v)$ has the n -tuple eigenvalue $v^a L_a \neq 0$. Then (8.8b, c) hold and (8.8a) is a consequence of (8.8b) and (6.1b). Hence we have $\theta = 1$. This proves Theorem (8.1b).

Theorem (8.2). *The $L_{a\lambda}{}'$ may always be chosen in such a way that*

$$(8.9a) \quad W_{\hat{a}\lambda}{}' = L_{\hat{a}\lambda}{}', \quad (1 - \theta)w_{\lambda}'' = L_{i\lambda}{}' - \delta_{\lambda}'', \quad \hat{a} = 1, \dots, l-1,$$

where

$$(8.9b) \quad (1 - \theta)w_{\lambda}'' = W_{i\lambda}{}', \quad w_{\lambda}'' \neq 0,$$

and for $\theta = \begin{cases} 0 \\ 1 \end{cases}$ the tensors $\begin{cases} W_{a\lambda}{}' \\ W_{\hat{a}\lambda}{}' \end{cases}$ are linearly independent.

The proof, based on (8.2), is almost trivial for $\theta = 0$. If $\theta = 1$ and $W_{i\lambda}{}' \neq 0$, there must be a vector v^a such that (8.8a) holds. Then a transformation brings the $W_{a\lambda}{}'$ to the form where $W_{\hat{a}\lambda}{}' = L_{\hat{a}\lambda}{}'$, $W_{i\lambda}{}' = 0$. Omitting the primes, we get (8.9). The remaining part of the theorem is obvious.

Definition (8.1). *The $L_{a\lambda}{}'$ and $W_{a\lambda}{}'$ satisfying the conditions of Theorem (8.2) will be termed canonical. The structural constants defined by (5.6), where the $L_{a\lambda}{}'$ are canonical, will be termed canonical. Canonical quantities will be denoted by script letters.*

Remark. According to this definition we shall write instead of (8.9)

$$(8.10a) \quad \begin{aligned} \mathcal{L}_{\hat{a}\lambda}{}' &= \mathfrak{W}_{\hat{a}\lambda}{}', \quad \text{for } \theta = 0 \\ \mathcal{L}_{i\lambda}{}' - \delta_{\lambda}'' &= \mathfrak{W}_{i\lambda}{}', \end{aligned}$$

and

$$(8.10b) \quad \begin{aligned} \mathcal{L}_{\hat{a}\lambda}{}' &= \mathfrak{W}_{\hat{a}\lambda}{}', \quad \text{for } \theta = 1. \\ \mathcal{L}_{i\lambda}{}' &= \delta_{\lambda}'', \end{aligned}$$

Theorem (8.3). *Let $\mathcal{L}_{\hat{a}\lambda}{}'$, $\mathfrak{W}_{\hat{a}\lambda}{}'$ be one system of canonical tensors. The most*

general system of canonical tensors is given by

$$\begin{aligned}
 (8.11) \quad & \begin{aligned}
 & \text{(a)} \quad \mathbb{W}_{\hat{a}'\lambda}{}^{\nu} = A_{\hat{a}}^{\hat{a}} W_{\hat{a}\lambda}{}^{\nu}, \\
 & \mathbb{W}_{\hat{l}'\lambda}{}^{\nu} = \mathbb{W}_{\hat{l}\lambda}{}^{\nu}, \quad \hat{a}, \hat{g} = 1, \dots, \hat{l} - 1, \\
 & \text{(b)} \quad \mathcal{L}_{\hat{a}'\lambda}{}^{\nu} = A_{\hat{a}}^{\hat{a}} \mathcal{L}_{\hat{a}\lambda}{}^{\nu}, \\
 & \mathcal{L}_{\hat{l}'\lambda}{}^{\nu} = \mathcal{L}_{\hat{l}\lambda}{}^{\nu},
 \end{aligned}
 \end{aligned}$$

where the arbitrary coefficients A 's are restricted only by the requirement

$$(8.11c) \quad \text{Det}((A_{\hat{a}}^{\hat{a}})) \neq 0.$$

The λ -domain described by $\mathcal{L}_{\hat{a}\lambda}{}^{\nu} = \mathbb{W}_{\hat{a}\lambda}{}^{\nu}$ is independent of the choice of the canonical system.

Proof. We have from (8.10) and (5.11)

$$\begin{aligned}
 (8.12) \quad & \mathcal{L}_{\hat{a}'\lambda}{}^{\nu} = \mathbb{W}_{\hat{a}'\lambda}{}^{\nu} = A_{\hat{a}}^{\hat{a}} \delta_{\lambda}^{\nu}, \\
 & \mathcal{L}_{\hat{l}'\lambda}{}^{\nu} = \mathbb{W}_{\hat{l}'\lambda}{}^{\nu} = A_{\hat{l}}^{\hat{l}} \delta_{\lambda}^{\nu}.
 \end{aligned}$$

Hence the new system is canonical if and only if

$$(8.13) \quad A_{\hat{a}}^{\hat{a}} = 0, \quad A_{\hat{l}}^{\hat{l}} = 1, \quad \text{Det}((A_{\hat{a}}^{\hat{a}})) \neq 0.$$

These equations prove the first statement. The second statement follows immediately from (8.11).

Definition (8.2). The λ -domain described by $\mathcal{L}_{\hat{a}\lambda}{}^{\nu} = \mathbb{W}_{\hat{a}\lambda}{}^{\nu}$ will be termed the principal λ -subdomain.

Remark. The principal λ -subdomain is intimately connected with the structure of the holonomy group (Sections 11 and 12). It will play also an important role in the next paper.

9. Consequences of previous results. In this section we shall derive some statements based on previous results. In the first place we shall be concerned with the maximal number $l_{\max}$ of parameters of the holonomy group.

Theorem (9.1). The maximal possible number of parameters of $G_l(R)$ is

$$(9.1) \quad l_{\max} = n^2 - 1 + \theta.$$

Proof. The $n^2 l$ components of $W_{\alpha\lambda}{}^{\nu}$ are restricted only by the conditions (7.2b). Hence the maximal possible number $w_{\max}$ of linearly independent tensors $W_{\alpha\lambda}{}^{\nu}$ is

$$w_{\max} = n^2 - 1.$$

This equation and the equation (8.4) (or 8.10) lead to (9.1).

Theorem (9.2). *In order that the principal λ -subdomain be the λ -domain of the set (6.4) it is necessary and sufficient that $\theta = 1$.*

The proof follows from the fact that the λ -domain of the set (6.4) is described for $\theta = \begin{cases} 0 \\ 1 \end{cases}$ by the linearly independent tensors $\left\{ \begin{matrix} \mathcal{W}_{a\lambda}^{\lambda} \\ \mathcal{W}_{\hat{a}\lambda}^{\lambda} \end{matrix} \right\}$, while the principal λ -subdomain is described by the linear independent tensors $\mathcal{W}_{\hat{a}\lambda}^{\lambda}$, $\hat{a} = 1, \dots, l-1$.

Using (5.2a) we see that to every canonical system $\mathcal{L}_{a\lambda}$ there corresponds one uniquely defined system $\mathfrak{N}_{\omega\mu}^{\hat{a}}$ which we call also canonical:

$$\mathfrak{N}_{\omega\mu}^{\hat{a}} = R_{\omega\mu\lambda}^{-1} \mathcal{L}_{a\lambda}^{\lambda}.$$

Theorem (9.3). *We have*

$$(9.2) \quad \mathfrak{N}_{\omega\mu}^{\hat{a}} = \frac{1}{n} R_{\omega\mu}.$$

Proof. We have from (6.1) and (8.10)

$$(9.3) \quad \mathcal{L}_a = \frac{1}{n} \mathcal{L}_{a\lambda}^{\lambda} = \delta_a^{\hat{a}}.$$

The equation (9.2) follows from (6.3) and (9.3).

Theorem (9.4). *Every eigenvector of $\begin{Bmatrix} W_{a\lambda}^{\lambda} \\ L_{a\lambda}^{\lambda} \end{Bmatrix}$ is at the same time an eigenvector of $\begin{Bmatrix} L_{a\lambda}^{\lambda} \\ W_{a\lambda}^{\lambda} \end{Bmatrix}$.*

The proof follows at once from (6.1).

Theorem (9.5). *A sufficient condition for the system*

$$(9.4) \quad \nabla_{\mu} T_{\lambda_1 \dots \lambda_v}^{\mu} = 0, \quad u \neq v,$$

not to admit any non-trivial solutions is $\theta = 1$.

Proof. The integrability conditions of (9.4) have the form (5.10), i.e.

$$(9.5) \quad L_{a\lambda_1}^{\alpha} T_{\alpha\lambda_2 \dots \lambda_v}^{\mu} + \dots + L_{a\lambda_v}^{\alpha} T_{\lambda_1 \dots \lambda_{v-1} \alpha}^{\mu} - L_{a\alpha}^{\nu} T_{\lambda_1 \dots \lambda_v}^{\nu\mu} - \dots - L_{a\alpha}^{\nu} T_{\lambda_1 \dots \lambda_v}^{\nu\mu-1\alpha} = 0.$$

Assume $\theta = 1$ and use the canonical tensors. Then the equation (9.5) reduces for $\alpha = \hat{a}$ to

$$(v-u) T_{\lambda_1 \dots \lambda_v}^{\mu} = 0,$$

and the theorem follows.

10. Canonical structural constants. Our starting point will be the equations (5.6), (5.7) and (6.1).

Theorem (10.1). *We have*

$$(10.1) \quad 2W_{[\alpha|\lambda]}{}^\alpha W_{\beta|\alpha}{}^\nu = C_{\alpha\beta}{}^\flat W_{\flat\lambda}{}^\nu$$

and

$$(10.2) \quad C_{\alpha\beta}{}^\flat L_\flat = 0.$$

Proof. Substituting into (5.7) from (6.1b) we obtain

$$(10.3a) \quad 2[W_{[\alpha|\lambda]}{}^\alpha + \delta_\lambda^\alpha L_{[\alpha]}][W_{\beta|\alpha}{}^\nu + L_{\beta]} \delta_\alpha^\nu] = C_{\alpha\beta}{}^\flat (W_{\flat\lambda}{}^\nu + L_{\flat\lambda} \delta_\lambda^\nu).$$

The left hand term in (10.3a) is equal to

$$2[W_{[\alpha|\lambda]}{}^\alpha + \delta_\lambda^\alpha L_{[\alpha]}][W_{\beta|\alpha}{}^\nu + L_{\beta]} \delta_\alpha^\nu] = 2W_{[\alpha|\lambda]}{}^\alpha W_{\beta|\alpha}{}^\nu.$$

Substituting this value into (10.3a) we obtain

$$(10.3b) \quad 2W_{[\alpha|\lambda]}{}^\alpha W_{\beta|\alpha}{}^\nu = C_{\alpha\beta}{}^\flat (W_{\flat\lambda}{}^\nu + L_{\flat\lambda} \delta_\lambda^\nu).$$

Moreover,

$$(10.4) \quad W_{[\alpha|\beta]}{}^\alpha W_{\gamma|\alpha}{}^\beta = 0.$$

If we put $\lambda = \nu$ in (10.3b) and substitute from (10.4) and (7.2b) into the equation so obtained, we get (10.2).

Theorem (10.2). *The canonical $\mathcal{C}_{\alpha\beta}{}^\flat$ satisfy the condition*

$$(10.5) \quad \mathcal{C}_{\alpha\beta}{}^{\flat\dot{l}} = 0,$$

and if $\theta = 1$, then

$$(10.6) \quad \mathcal{C}_{\dot{l}\beta}{}^\flat = 0.$$

Proof. Rewriting (10.2) in canonical form and using (9.3), we obtain (10.5). Rewriting (10.1) in canonical form, we obtain for $\theta = 1$ and $\alpha = \dot{l}$

$$\mathcal{C}_{\dot{l}\beta}{}^\flat \mathcal{W}_{\flat\lambda}{}^\nu = 0, \quad \hat{\mathfrak{h}} = \dot{1}, \dots, \dot{l} - \dot{1}.$$

Because the tensors $\mathcal{W}_{\flat\lambda}{}^\nu$ are linearly independent we must have, by virtue of the last equation,

$$\mathcal{C}_{\dot{l}\beta}{}^\flat = 0.$$

Combining this equation with (10.5), we obtain (10.6).

11. Holonomy Group $G_*(R)$.*

Lemma (11.1). *The highest possible $\mathfrak{h}$ -rank of $\mathcal{C}_{\alpha\beta}{}^\flat$ is $\dot{l} - 1$. For $\dot{l} = 2$ and $\theta = 1$ the holonomy group $G_2(R)$ is abelian.*

*In Sections 11 and 12 the German indices $\hat{\alpha}$, $\hat{\epsilon}$, $\hat{g}$, $\hat{h}$, $\hat{\mathfrak{f}}$ run from $\dot{1}$ to $\dot{l} - \dot{1}$. We shall assume also $C_{\alpha\beta}{}^\flat \neq 0$ unless stated explicitly otherwise.

Proof. Because of (10.5) the vector

$$V^i \stackrel{\text{def}}{=} \mathcal{C}_{\alpha\beta}^{\gamma} v^{\alpha\beta}$$

satisfies the condition $V^i = 0$ for any tensor $v^{\alpha\beta}$ whatsoever. Hence the highest possible $\mathfrak{h}$ -rank of $\mathcal{C}_{\alpha\beta}^{\gamma}$ is $l - 1$. The second statement follows directly from (10.6) or from (8.10b) for $l = 2$.

The following theorems will be proved simultaneously.

Theorem (11.1a). *The holonomy group $G_l(R)$ always has at least two subgroups: the subgroup $G'_{l-1}(R)$ defined by the infinitesimal operators*

$$(11.1) \quad X_{\hat{\alpha}} \stackrel{\text{def}}{=} \mathcal{L}_{\hat{\alpha}\lambda}^{\gamma} y^{\lambda} \frac{\partial}{\partial y^{\gamma}}$$

and the subgroup $G_l(R)$ defined by the infinitesimal operator

$$(11.2) \quad X_l = \mathcal{L}_{l\lambda}^{\gamma} y^{\lambda} \frac{\partial}{\partial y^{\gamma}}.$$

Theorem (11.1b). *The subgroup $G'_{l-1}(R)$ is normal, i.e. it is invariant.* If the $\mathfrak{h}$ -rank of $\mathcal{C}_{\alpha\beta}^{\gamma}$ is $l - 1$, then $G'_{l-1}(R)$ is the first derived subgroup of $G_l(R)$.*

Theorem (11.1c). *If $\theta = 1$ then $G_l(R)$ is invariant.† $G_l(R)$ has a center and $G_l(R)$ belongs to it.*

Proof. Let u^{α} be the coordinates in the group space G_l , so that a generic transformation U of $G_l(R)$ is defined by the infinitesimal operator.

$$(11.3) \quad u^{\alpha} X_{\alpha} = u^{\alpha} \mathcal{L}_{\alpha\lambda}^{\gamma} y^{\lambda} \frac{\partial}{\partial y^{\gamma}}.$$

Consider the independent operators (11.1). Every infinitesimal transformation $u^{\hat{\alpha}} X_{\hat{\alpha}}$ is mapped in the group space G_l on a point satisfying the condition

$$(11.4) \quad u^{\hat{l}} = 0.$$

According to (5.7) we have, by virtue of (10.5),

$$(11.5) \quad 2X_{[l\hat{\alpha}} X_{\hat{\beta}]} = \mathcal{C}_{\hat{\alpha}\hat{\beta}}^{\gamma} X_{\gamma}.$$

Hence the operators (11.1) define an $(l - 1)$ -parameter subgroup which we denote by $G'_{l-1}(R)$.

The subgroup $G_l(R)$ is mapped on the point

$$(11.6) \quad u^{\hat{\alpha}} = 0$$

of the group space G_l .

*I.e. invariant with respect to the adjoint group.

†See the previous footnote.

If T is a transformation of $G_l(R)$ defined by the infinitesimal operator $t^a X_a$, then the corresponding transformation of the adjoint group is given by the infinitesimal operator

$$(11.7) \quad t^a A_a \stackrel{\text{def}}{=} t^a u^a \mathcal{C}_{a\hat{a}}^{\hat{b}} \frac{\partial}{\partial u^{\hat{b}}}.$$

Hence

$$(11.8a) \quad t^a A_a(u^{\hat{c}}) = A_a^{\hat{c}} u^{\hat{a}},$$

where according to (10.5)

$$(11.8b)^* \quad A_a^{\hat{c}} \stackrel{\text{def}}{=} t^a \mathcal{C}_{a\hat{a}}^{\hat{c}} = t^a \mathcal{C}_{a\hat{a}}^{\hat{b}} \delta_{\hat{b}}^{\hat{c}}.$$

Hence the linear transformation (11.8) of the u 's may be split into two:

$$(11.9) \quad \begin{aligned} (a) \quad & t^a A_a u^{\hat{a}} = u^{\hat{a}} A_a^{\hat{a}}, \\ (b) \quad & t^a A_a u^{\hat{t}} = 0. \end{aligned}$$

For the subgroup $G'_{l-1}(R)$ the equation (11.9a) reduces, by virtue of (11.4), to

$$(11.9c) \quad t^a A_a u^{\hat{e}} = u^{\hat{e}} A_a^{\hat{e}}.$$

The equations (11.9b, c) show that $G'_{l-1}(R)$ is invariant (with respect to the adjoint group).

If $\theta = 1$, then (10.6) holds, so that

$$(11.10) \quad A_a^{\hat{c}} = t^{\hat{a}} \mathcal{C}_{\hat{a}\hat{b}}^{\hat{c}} \delta_{\hat{t}}^{\hat{b}} \delta_a^{\hat{t}} = A_{\hat{b}}^{\hat{t}} \delta_{\hat{t}}^{\hat{c}} \delta_a^{\hat{b}}.$$

Substituting this value into (11.9a) we obtain (11.9c) for any $u^{\hat{a}}$. Hence if we apply the transformation (11.7) of the adjoint group to (11.6), we obtain

$$(11.9d) \quad t^a A_a u^{\hat{e}} = 0.$$

The equations (11.9b, d) show that $G_1(R)$ is invariant (with respect to the adjoint group). Moreover, (10.6) shows that $G_1(R)$ commutes with every transformation of $G_l(R)$. Hence $G_l(R)$ has a center and $G_1(R)$ belongs to it.

In order to prove the remaining part of Theorem (11.1b) assume that the $\mathfrak{h}$ -rank of $\mathcal{C}_{a\hat{a}}^{\hat{b}}$ is s . Then the first derived subgroup of $G_e(R)$ is defined by the operators (11.3), where the u 's are solutions of

$$(11.11) \quad \mathcal{C}_{a_1 \hat{a}_1}^{(\hat{b}_1)} \dots \mathcal{C}_{a_s \hat{a}_s}^{(\hat{b}_s)} u^{\hat{a}_1} = 0.$$

Hence if $s = l - 1$, the only solution of (11.1) is (11.4), so that $G'_{l-1}(R)$ is the first derived subgroup of $G_e(R)$.

12. The Lie group $G_w(W)$. We shall now turn our attention to the group induced by the tensor defined by (4.11).† To this effect we shall prove first the

*The $A_a^{\hat{c}}$ of (11.8b) have no connection with the $A_a^{\hat{c}}$ in (5.11).

†See Section [II] 3 for the definition and theory of the Lie group induced by a tensor.

following

Lemma (12.1). *If $\theta = 1$, there is at least one system of tensors ${}^{-1}\mathbb{W}^{\hat{a}\lambda}$, which satisfy the condition*

$$(12.1) \quad {}^{-1}\mathbb{W}^{\hat{a}\lambda}, {}^{\alpha}\mathbb{W}_{\hat{a}\lambda}{}^{\nu} = \delta_{\hat{a}}^{\alpha}.$$

If $\theta = 0$, there is at least one system of tensors ${}^{-1}\mathbb{W}^{\hat{a}\lambda}$, which satisfy the condition

$$(12.2) \quad {}^{-1}\mathbb{W}^{\hat{a}\lambda}, {}^{\alpha}\mathbb{W}_{\hat{a}\lambda}{}^{\nu} = \delta_{\hat{a}}^{\alpha}.$$

Proof. If

$$(12.3) \quad \theta = \begin{cases} 0 \\ 1 \end{cases}$$

then the tensors $\begin{Bmatrix} {}^{\alpha}\mathbb{W}_{\hat{a}\lambda}{}^{\nu} \\ {}^{\alpha}\mathbb{W}_{\hat{a}\lambda} \end{Bmatrix}$ are linearly independent. Therefore there is at least one set of tensors $\begin{Bmatrix} {}^{-1}\mathbb{W}^{\hat{a}\lambda} \\ {}^{-1}\mathbb{W}^{\hat{a}\lambda} \end{Bmatrix}$ satisfying $\begin{cases} (12.2) \\ (12.1) \end{cases}$.

Remark. We see from (8.10b) that we may put for $\theta = 1$

$$(12.4) \quad {}^{-1}\mathbb{W}^{\hat{a}\lambda}, \stackrel{\text{def}}{=} \mathcal{L}^{\hat{a}\lambda}.$$

Theorem (12.1). *The tensor $W_{\omega\mu\lambda}$ induces a Lie group $G_{\omega}(W)$. Its infinitesimal operators are*

$$(12.5a) \quad Y_{\hat{a}} \stackrel{\text{def}}{=} {}^{\alpha}\mathbb{W}_{\hat{a}\lambda}{}^{\nu} y^{\lambda} \frac{\partial}{\partial y^{\nu}} = (\mathcal{L}_{\hat{a}\lambda}{}^{\nu} - \delta_{\hat{a}}^{\nu} \delta_{\lambda}^{\nu}) y^{\lambda} \frac{\partial}{\partial y^{\nu}}, \quad \theta = 0,$$

$$(12.5b) \quad Y_{\hat{a}} = {}^{\alpha}\mathbb{W}_{\hat{a}\lambda}{}^{\nu} y^{\lambda} \frac{\partial}{\partial y^{\nu}} \mathcal{L}_{\hat{a}\lambda}{}^{\nu} y^{\lambda} \frac{\partial}{\partial y^{\nu}}, \quad \theta = 1,$$

and its structural constants $\tilde{\mathcal{C}}_{\hat{a}\hat{b}}{}^{\hat{c}}$ are related to the structural constants $\mathcal{C}_{\hat{a}\hat{b}}{}^{\hat{c}}$ as follows:

$$(12.6a) \quad \tilde{\mathcal{C}}_{\hat{a}\hat{b}}{}^{\hat{c}} \stackrel{\text{def}}{=} \mathcal{C}_{\hat{a}\hat{b}}{}^{\hat{c}} \quad \text{for } \theta = 0,$$

$$(12.6b) \quad \tilde{\mathcal{C}}_{\hat{a}\hat{b}}{}^{\hat{c}} \stackrel{\text{def}}{=} \mathcal{C}_{\hat{a}\hat{b}}{}^{\hat{c}} \quad \text{for } \theta = 1,$$

where

$$(12.7a) \quad 2{}^{\alpha}\mathbb{W}_{[\hat{a}|\lambda]}{}^{\alpha}\mathbb{W}_{\hat{b}]\alpha}{}^{\nu} {}^{-1}\mathbb{W}^{\hat{b}\lambda}{}^{\nu} = \tilde{\mathcal{C}}_{\hat{a}\hat{b}}{}^{\hat{c}}, \quad \theta = 0,$$

$$(12.7b) \quad 2{}^{\alpha}\mathbb{W}_{[\hat{a}|\lambda]}{}^{\alpha}\mathbb{W}_{\hat{b}]\alpha}{}^{\nu} {}^{-1}\mathbb{W}^{\hat{b}\lambda}{}^{\nu} = \tilde{\mathcal{C}}_{\hat{a}\hat{b}}{}^{\hat{c}}, \quad \theta = 1.$$

Proof. According to Theorem (6.2) the tensors $W_{\alpha\lambda}$ describe the λ -domain

of the set (6.4). From (12.7) and (10.1) we obtain easily (12.6) and, according to (12.5a, b),

$$(12.8a) \quad 2Y_{[a}Y_{b]} = C_{ab}{}^c Y_c \quad \text{for } \theta = 0$$

and

$$(12.8b) \quad 2Y_{[a}Y_{b]} = C_{ab}{}^c Y_c \quad \text{for } \theta = 1.$$

Hence, according to the second fundamental Lie theorem, the independent operators (12.5) define a Lie group of w parameters. According to our definition of an induced Lie group, the group under consideration is induced by $W_{\omega\mu\lambda}$ and therefore denoted by $G_w(W)$.

Theorem (12.2a). *If $\theta = 0$, the groups $G_i(R)$ and $G_w(W)$ are in one-to-one correspondence. In particular, the invariant subgroup $G_{w-1}(W)$ defined by*

$$(12.9) \quad Y_a = W_{a\lambda}{}^\nu y^\lambda \frac{\partial}{\partial y^\nu}$$

is identical with the subgroup $G'_{i-1}(R)$ of $G_i(R)$.

The proof that the operators (12.9) define an invariant subgroup is the same as the proof of the first statement of Theorem (11.1b). The remaining part of our statement follows at once from (12.5), (11.1) and (11.2).

Theorem (12.2b). *If $\theta = 1$ then $G_w(W)$ is identical with the invariant subgroup $G'_{i-1}(R)$ of the holonomy group $G_i(R)$.*

The proof follows from (12.5b) and (11.1).

III.

In the previous sections 10–12 we studied the structure of the groups $G_i(R)$ and $G_w(W)$ by means of canonical structural constants. It is known that *all* problems concerning the structure of a Lie group are mastered by its structural constants. Hence if we know the structural constants of $G_i(R)$, we are able to solve any problem concerning the structure of the holonomy group. In the subsequent sections we shall display a method which reduces all $C_{ab}{}^c$ to 0, +1, -1 for a large class of holonomy groups. Here again the condition (1.1) plays a paramount role.

13. The ideal holonomy group. Let us consider n linearly independent

vectors a^ν_B and their inverse a^A_λ :

$$(13.1) \quad a^\lambda_B a^A_\lambda = \delta^A_B, \quad A, B = 1, \dots, n. *$$

*We are using capital Latin indices because small Latin indices will have another range.

The λ -domain of the tensor set (5.1) is described by a certain number l of n^2 linearly independent tensors

$$(13.2) \quad \underset{B}{L}_{\lambda}^A \stackrel{\text{def}}{=} \underset{B}{a}_{\lambda}^A a^B.$$

If and only if $l = n^2$, $G_l(R)$ is general.* In subsequent considerations we shall investigate a large class of holonomy groups, picking out some of the tensors (13.2).

Denote by S the set of numerals $1, \dots, n$, by $S_u \subset S$ ($u = 1, 2$) two of its subsets and by Q_0 the intersection

$$(13.3a) \quad Q_0 \stackrel{\text{def}}{=} S_1 \cap S_2.$$

We shall assume that Q_0 consists of q_0 elements. Denote by

$$(13.3b) \quad Q_u \text{ the complement of } Q_0 \text{ in } S_u,$$

and assume that it consists of q_u elements ($u = 1, 2$). Hence

$$(13.3c) \quad S_u = Q_0 \cup Q_u, \quad Q_1 \cap Q_2 = 0$$

and S_u consists of $q_0 + q_u$ elements. The elements of Q_u and S_u will be denoted by

$$(13.4) \quad \begin{aligned} (a) \quad & a_v, b_v, \dots, x_v, y_v, z_v, t_v \subset Q_v, \\ (b) \quad & a(u), b(u), \dots, x(u), y(u), z(u), t(u) \subset S_u, \\ & u = 1, 2, \quad v = 0, 1, 2, \end{aligned}$$

while the elements of $S_1 \cup S_2$ will be denoted by

$$(13.4c) \quad a, b, \dots, x, y, z, t \subset S_1 \cup S_2.$$

The last four symbols in (13.4) do not follow the usual summation convention.

Definition (13.1). *The holonomy group $G_l(R)$ is termed ideal if the λ -domain of the tensor set (5.1) is described by*

$$(13.5) \quad l = q_0^2 + q_0(q_1 + q_2) + q_1 q_2$$

tensors

$$(13.6a) \quad \underset{b(2)}{L}_{\lambda}^{a(1)} \stackrel{\text{def}}{=} \underset{b(2)}{a}_{\lambda}^{a(1)} a^b$$

which satisfy the condition

$$(13.6b) \quad \underset{b(2)}{L}_{\lambda}^{a(1)} = \delta_{b(2)}^{a(1)}.$$

Remark. Despite the very weak conditions of Definition (13.1), not every

*See definition II (6.3).

holonomy group is ideal. Thus, for instance, the tensors

$$\begin{matrix} 1 \\ L_{\lambda} \end{matrix} \begin{matrix} \nu \\ \end{matrix} = \begin{matrix} 1 \\ a_{\lambda} \end{matrix} \begin{matrix} \nu \\ \end{matrix}, \quad \begin{matrix} 2 \\ L_{\lambda} \end{matrix} \begin{matrix} \nu \\ \end{matrix} = \begin{matrix} 2 \\ a_{\lambda} \end{matrix} \begin{matrix} \nu \\ \end{matrix}, \quad \begin{matrix} 1 \\ L_{\lambda} \end{matrix} \begin{matrix} \nu \\ 2 \end{matrix} = \begin{matrix} 1 \\ a_{\lambda} \end{matrix} \begin{matrix} \nu \\ 2 \end{matrix}$$

lead to a group which is not ideal: $q_1 = q_2 = 0$, $q_0 = 2$, $l = 3$.

Theorem (13.1). *A general holonomy group $G_l(R)$ is ideal and*

$$(13.7) \quad \begin{aligned} (a) \quad & q_0 = n, \\ (b) \quad & q_1 = q_2 = 0, \\ (c) \quad & \theta = 1 \end{aligned}$$

Proof. For a general holonomy group we have $l = n^2$. This requirement implies (13.7a, b). The equation (13.7c) follows from (9.1) and from $l = n^2$.

Theorem (13.2). *If $G_l(R)$ is ideal, then the tensors (13.6) may be taken for $L_{a\lambda} \begin{smallmatrix} \nu \\ \end{smallmatrix}$.*

Proof. Both sets $L_{a\lambda} \begin{smallmatrix} \nu \\ \end{smallmatrix}$ and $L_{\lambda} \begin{smallmatrix} a(1) \\ b(2) \end{smallmatrix} \begin{smallmatrix} \nu \\ \end{smallmatrix}$ of l linearly independent tensors describe the same λ -domain, namely the λ -domain of the tensor set (5.1). Hence there exists a relation

$$(13.8a) \quad L_{\lambda} \begin{smallmatrix} \nu \\ \end{smallmatrix} = A_{b(2)}^{a(1)} L_{a\lambda} \begin{smallmatrix} \nu \\ \end{smallmatrix}$$

where the $l \times l$ matrix

$$(13.8b) \quad ((A)) \stackrel{\text{def}}{=} \begin{pmatrix} \left(\begin{matrix} \dots & A_{b(2)}^{1a(1)} & \dots \end{matrix} \right) \\ \vdots \\ \left(\begin{matrix} \dots & A_{b(2)}^{la(1)} & \dots \end{matrix} \right) \end{pmatrix}$$

is of rank l , so that

$$(13.8c) \quad \text{Det}((A)) \neq 0.$$

The relations (13.8a, c) are of the type (5.11a, b) (with a slightly different notation). This proves our statement.

Theorem (13.3). *We always have*

$$(13.9) \quad q_0 > 0.$$

Proof. Using (6.2) we see that according to (13.8a,c) and (13.6b) this condition implies

$$(13.10) \quad L_{\lambda} \begin{smallmatrix} x_0 \\ \end{smallmatrix} \neq 0,$$

at least for one x_0 . Hence (13.9) holds.

Remark I. The sets $a_{b(2)}^{\nu}$ and $a_{\lambda}^{a(1)}$ may be completed to sets $a_{\mathbf{B}}^{\nu}$, a_{λ}^A for which (13.1) holds. We shall need sometimes these completed sets, keeping in mind that *only the tensors* (13.6) *describe the* λ -*domain of the tensor set* (5.1).

Remark II. In subsequent considerations we shall deal exclusively with ideal holonomy groups, and therefore we shall drop the word "ideal".

14. Composed indices. Assume that the tensors $L_{a\lambda}^{\nu}$ in (13.8a) already have the form $L_{\lambda}^{a(1)\nu}$. Then $A_{b(2)}^{aa(1)}$ are equal to 0 or 1 and define a one-to-one correspondence between l German indices a and l pairs $b(2)$. This enables us to dispose of German indices and write

$$(14.1) \quad L_{a(1)\lambda}^{b(2)\nu} \stackrel{\text{def}}{=} L_{\lambda}^{a(1)\nu} = a_{\lambda}^{a(1)} a_{b(2)}^{\nu},$$

where the composed index $b(2)$ stands for some German index a .

The following agreement, although not necessary, is very advantageous for later purposes.

Agreement (14.1). In any composed index a we always assume $a \subset S_1$, $b \subset S_2$, i.e. $a = a(1)$, $b = b(2)$.

Using this agreement we shall not need to indicate the range of a , b in the composed index. Thus, for instance, we put

$$(14.2) \quad (a) \quad \delta_a^c \stackrel{\text{def}}{=} \delta_{b(2)}^{c(1)}, \quad (b) \quad \delta^b_c \stackrel{\text{def}}{=} \delta_{c(1)}^{b(2)}.$$

δ_a^c , δ^b_c are special cases of covariant and contravariant vectors in the group space. Another illustration of the use of composed indices is the Kronecker delta of the group space

$$(14.3) \quad \delta_a^f = \delta_{c(1)}^{f(1)} \delta_{b(2)}^{f(2)}.$$

In every case any composed index has to be thought of as a *single* index. Thus, for instance, T_a^b stands for the trace of the tensor T_c^b , while T_a^b is one single component of T_c^b . We shall always avoid repetition of single indices* in composed indices and shall write T_x^b instead of T_a^b , keeping in mind that x does not follow the usual summation convention.

In order to prove the immediate advantage of the use of composed indices

*I.e. of indices which follow the usual rule of summation.

we shall prove a theorem in which we shall use some of the vectors mentioned in Remark I of the previous section.

Theorem (14.1). *The set of inverse tensors $L^{f\lambda}$, to $L_{e\lambda}{}^{\nu}$, i.e. of tensors satisfying*

$$(14.4) \quad L^{f\lambda}{}_{\nu} L_{e\lambda}{}^{\nu} = \delta_e^f,$$

is given by

$$(14.5) \quad L^{f\lambda}{}_{\nu} \stackrel{\text{def}}{=} a_{\nu}^{(1)} a^{\lambda}_{(2)}.$$

Proof. Using (14.3), (14.1) and (14.5) we obtain

$$L^{f\lambda}{}_{\nu} L_{e\lambda}{}^{\nu} = a_{\nu}^{(1)} a^{\lambda}_{(2)} a_{\lambda}^{(1)} a^{\nu}_{(2)} = \delta_{e(1)}^{c(1)} \delta_{b(2)}^{f(2)} = \delta_e^f.$$

Remark. It ought to be stressed that, unlike $L_{e\lambda}{}^{\nu}$, the tensors $L^{f\lambda}{}_{\nu}$ do not describe the λ -domain of the set (5.1) if $S_1 \neq S_2$.

15. Eigenvectors and eigenvalues. In the following three theorems, which we shall prove simultaneously, use will be made of vectors introduced in Remark I of Section 13.

Theorem (15.1a). *Every tensor $L_{a\lambda}{}^{\nu}$, $a \neq b$, has one n -fold eigenvalue*

$$(15.1a) \quad \lambda_{1b}^a = \lambda_{2b}^a = \cdots = \lambda_{nb}^a = 0, \quad a \neq b,$$

and $n - 1$ corresponding eigenvectors a_B^{ν} , $B = 1, 2, \dots, a - 1, a + 1, \dots, n$.

Theorem (15.1b). *Every tensor $L_{x_0\lambda}{}^{\nu}$ has the eigenvalues*

$$(15.1b) \quad \lambda_{x_0 x_0}^{x_0} = 1, \quad \text{the remaining } \lambda_{x_0}^{x_0} = 0.$$

The non-zero eigenvalue is simple and the corresponding eigenvector is $a_{x_0}^{\nu}$. The zero eigenvalue is $(n - 1)$ -fold and the corresponding $n - 1$ eigenvectors are a_B^{ν} , $B = 1, 2, \dots, x_0 - 1, x_0 + 1, \dots, n$.*

Theorem (15.1c). *The vector $a_{b(2)}^{\nu}$ is a common eigenvector of all tensors $L_{e\lambda}{}^{\nu}$, $e \neq b(2)$. The vector $a_{x_0}^{\nu}$ is a common eigenvector of $L_{x_0\lambda}{}^{\nu}$ and $L_{e\lambda}{}^{\nu}$, $e \neq x_0$.*

Proof. The equation

*Of course, any linear combination of these $n - 1$ vectors is also an eigenvector.

$$\lambda a^r = L_{a^\mu}^r a^\mu = (\overset{a}{a_\mu} a^\mu) a^r, \quad a \neq b,$$

is satisfied only by $\lambda = 0$ and by $n - 1$ linearly independent vectors a^r , $B = 1, 2, \dots, a - 1, a + 1, \dots, n$.

The equation

$$\lambda a^r = L_{x_0^\mu}^r a^\mu = (\overset{x_0}{a_\mu} a^\mu) a^r$$

is satisfied by

$$\lambda = 1 \quad \text{and} \quad a^r = \underset{x_0}{a^r},$$

as well as by

$$\lambda = 0 \quad \text{and} \quad a^r = \underset{B}{a^r}, \quad B = 1, 2, \dots, x_0 - 1, x_0 + 1, \dots, n.$$

Theorem (15.1c) is then an immediate consequence of Theorems (15.1a, b).

Let us denote by $\tilde{S}_u$ the complement of S_u in S , $u = 1, 2$, and denote by $\tilde{b}_u \subset \tilde{S}_u$ its arbitrary element.* In the next theorem we shall use the vectors $a^\mu_{\tilde{b}_1}$.

Theorem (15.2a). *If $q_0 + q_1 < n$, then all tensors $L_{f^\lambda}^r$, have $n - q_0 - q_1$ common eigenvectors $a^\nu_{\tilde{b}_1}$.*

Proof. The equation

$$\lambda a^r = L_{f^\mu}^r a^\mu = (\overset{f(1)}{a_\mu} a^\mu) \underset{f(2)}{a^r}$$

is satisfied also by

$$\lambda = 0, \quad a^r = \underset{\tilde{b}_1}{a^r}.$$

Obviously there are only $n - q_0 - q_1$ linearly independent vectors $a^r_{\tilde{b}_1}$.†

Besides the contravariant eigenvectors a^r we may consider also the covariant eigenvectors a_λ , i.e. the vectors satisfying the condition

$$(15.2) \quad \lambda a_\mu = L_{f^\mu}^r a_r.$$

The following theorems will be proved simultaneously.

Theorem (15.3a). *The vectors a_μ , $A = 1, \dots, f - 1, f + 1, \dots, n$ are $n - 1$*

*Hence we have also $b_2 \subset \tilde{S}_1$, $b_1 \subset \tilde{S}_2$.

†Among them there are q_2 linearly independent vectors $a^r_{b_2}$ (of course, we have always $n - q_0 - q_1 \geq q_2$).

linearly independent eigenvectors of $L_{e\mu}{}^\nu$, $e \neq f$, belonging to the n -fold eigenvalue (15.1a).

Theorem (15.3b). The vector a_μ is the eigenvector of $L_{x_0\lambda}{}^\nu$ belonging to the simple non-zero eigenvalue λ_{x_0} . The vectors a_μ , $A = 1, \dots, x_0 - 1, x_0 + 1, \dots, n$ are $n - 1$ linearly independent eigenvectors belonging to the $(n - 1)$ -fold eigenvalue 0 of $L_{x_0\lambda}{}^\nu$.

Theorem (15.3c). The vector a_μ is a common eigenvector of all tensors $L_{e\lambda}{}^\nu$, $a(1) \neq f$. The vector a_λ is the common eigenvector of $L_{x_0\lambda}{}^\nu$ and $L_{e\lambda}{}^\nu$, $f \neq x_0$.

Theorem (15.2b). If $q_0 + q_2 < n$, then all tensors $L_{e\lambda}{}^\nu$ have $n - q_0 - q_2$ common eigenvectors a_λ .

The proof, based on (15.2), follows the pattern of the proof of Theorems (15.1) and (15.2a).

In the next corollary we shall use the equation (6.1a).

Corollary (15.1). We have

$$(15.3) \quad nL_a{}^\nu \stackrel{\text{def}}{=} L_{a\lambda}{}^\lambda = \delta_a{}^\nu = \lambda_{1a}{}^\nu + \dots + \lambda_{na}{}^\nu$$

where λ_a are the eigenvalue of $L_{a\lambda}{}^\nu$.

The proof follows from (7.2a) and (15.1a, b).

In the next theorem we shall use the equations (6.1a, b).

Theorem (15.4). The tensors $W_{a\lambda}{}^\nu$ defined by (6.1a, b) are

$$(15.4) \quad (a) \quad W_{ab}{}^\nu = L_{a\lambda}{}^\nu, \quad a \neq b,$$

$$(b) \quad W_{x_0\lambda}{}^\nu = L_{x_0\lambda}{}^\nu - \frac{1}{n} \delta_\lambda{}^\nu.$$

The proof follows at once from (6.1a, b) and (15.3).

In the next theorem we shall complete the statements of Theorem (13.1).

Theorem (15.5). If $q_0 < n$, then $\theta = 0$.

Proof. According to Theorem (8.1b), we have $\theta = 1$ if and only if there is a vector v^a_b in the group space such that

$$(15.5) \quad L_\lambda{}^\gamma(v) \stackrel{\text{def}}{=} v^a_b L_{a\lambda}{}^\nu$$

has the n -fold eigenvalue

$$(15.6a) \quad v^{\overset{a}{b}} L_a^{\overset{a}{b}} \neq 0$$

or, according to (15.3),

$$(15.6b) \quad \sum v^{\overset{x_0}{x_0}} L_{x_0}^{\overset{x_0}{x_0}} = \frac{1}{n} \sum v^{\overset{x_0}{x_0}}.$$

In other words, we have $\theta = 1$ if and only if

$$(15.7a) \quad L_{\lambda}^{\nu}(v) = \frac{1}{n} \delta_{\lambda}^{\nu} \sum v^{\overset{x_0}{x_0}}.$$

On the other hand,

$$(15.7b) \quad L_{\lambda}^{\nu}(v) = v^{\overset{a_0}{b_0}} L_{a_0 \lambda}^{\overset{a_0}{b_0} \nu} + v^{\overset{a_1}{b_0}} L_{a_1 \lambda}^{\overset{a_1}{b_0} \nu} + v^{\overset{a_0}{b_2}} L_{a_0 \lambda}^{\overset{a_0}{b_2} \nu} + v^{\overset{a_1}{b_2}} L_{a_1 \lambda}^{\overset{a_1}{b_2} \nu},$$

where the tensors $L_{a\lambda}^{\nu}$ are given by (14.1). Hence a necessary but not sufficient condition for (15.7a), *i.e.*, a necessary but not sufficient condition for $\theta = 1$, is $q_0 = n$. Hence if $q_0 < n$, we must have $\theta = 0$.

16. Covariant constant fields. In this section we shall investigate covariant constant fields in the case of an ideal holonomy group.*

Theorem (16.1). *The eigenvectors $a^{\nu}_{\bar{b}_1}$, $a_{\mu}^{\bar{b}_2}$ mentioned in Theorems (15.2a, b) may always be chosen in such a way that they are covariant constant:*

$$(16.1) \quad \begin{aligned} (a) \quad \nabla_{\mu} a^{\nu}_{\bar{b}_1} &= 0, \\ (b) \quad \nabla_{\mu} a_{\lambda}^{\bar{b}_2} &= 0. \end{aligned}$$

The proof will be accomplished in two steps:

(1) The complete set of integrability conditions of

$$(16.2) \quad \begin{aligned} (a) \quad \nabla_{\mu} a^{\nu} &= -a^{\nu} \partial_{\mu} \ln \varphi, \\ (b) \quad \nabla_{\mu} a_{\lambda} &= -a_{\lambda} \partial_{\mu} \ln \psi \end{aligned}$$

is

$$(16.3) \quad \begin{aligned} (a) \quad L_{a\lambda}^{\nu} a^{\lambda} &= 0, \\ (b) \quad L_{a\lambda}^{\nu} a_{\nu} &= 0. \end{aligned}$$

Hence a^{ν} (a_{λ}) must be a common eigenvector to all tensors $L_{a\lambda}^{\nu}$. According to

*See also Theorems (7.2a, b).

Theorems (15.2a, b) $\frac{a^\nu}{\bar{b}_1}, a_\lambda$ have this property. Hence if $n - q_0 - q_1 > 0$ ($n - q_0 - q_2 > 0$) we may always find* $n - q_0 - q_1$ ($n - q_0 - q_2$) linearly independent solutions $\frac{a^\nu}{\bar{b}_1}, a_\lambda$ of (16.3a) ((16.3b)) which satisfy the conditions:

$$(16.4) \quad \begin{aligned} \nabla_\mu \frac{a^\nu}{\bar{x}_1} &= -\frac{a^\nu}{\bar{x}_1} \partial_\mu \ln \varphi, \\ \nabla_\mu a_\lambda &= -a_\lambda \partial_\mu \ln \psi. \end{aligned}$$

(2) Denote by Q_{00} the intersection of the sets $\bar{S}_u$ used in Theorem (15.2a, b):

$$Q_{00} = \bar{S}_1 \cup \bar{S}_2,$$

and denote by $a_{00}, \dots, x_{00}, \dots$ its elements. The vectors $\frac{a^\nu}{\bar{b}_1}, a_\lambda$ satisfy the condition, derived from (13.1),

$$(16.5a) \quad \frac{a^\lambda}{\bar{b}_1} a_\lambda = \delta_{\bar{b}_1}^{\bar{a}_2}.$$

The right hand term is equal to one if and only if $\bar{a}_2 = \bar{b}_1$, i.e.

$$(16.5b) \quad \frac{a^\lambda}{x_{00}} a_\lambda = 1.$$

Therefore we have from (16.4)

$$-\nabla_\mu \frac{a^\lambda}{x_{00}} a_\lambda = \frac{a^\lambda}{x_{00}} a_\lambda (\partial_\lambda \ln \psi \varphi) = 0,$$

so that we may put, without loss of generality,

$$\varphi = \left(\frac{a^\lambda}{x_{00}} \right)^{-1}.$$

Hence if we put

$$\begin{aligned} \frac{a^\nu}{x_{00}} &= \varphi \frac{a^\nu}{x_{00} x_{00}}, \\ \frac{a^\lambda}{x_{00}} &= \left(\frac{a^\lambda}{x_{00}} \right)^{-1} a_\lambda, & \bar{x}_1, \bar{x}_2 \neq x_{00}, \\ \frac{a^\nu}{\bar{x}_1} &= \frac{\varphi a^\nu}{\bar{x}_1 \bar{x}_1}, \\ \frac{a_\lambda}{\bar{x}_2} &= \frac{\psi a_\lambda}{\bar{x}_2 \bar{x}_2}, \end{aligned}$$

we obtain

$$\nabla_\mu \frac{a^\nu}{\bar{b}_1} = 0, \quad \nabla_\mu \frac{a_\lambda}{\bar{b}_2} = 0$$

*If necessary we have to apply the method displayed in the footnote on p. 466.

and

$${}^{b(2)}a^{\lambda'} a_{\lambda} = \delta_{\bar{b}_1}^{\bar{a}_2},$$

Omitting the primes we have (16.1) and (16.5).*

Remark. One sees from (16.1b) that the vectors a_{λ} are gradients:

$$a_{\lambda} = \partial_{\lambda} a.$$

This fact is connected with the statement of the following

Theorem (16.2). The $q_0 + q_2$ vectors ${}^{b(2)}a^{\nu}$ are $X_{a_0+a_2}$ -forming. The scalars ${}^{\bar{a}_2}a$ are constant along each of these $X_{a_0+a_2}$. The $n - q_0 - q_1$ vectors ${}^{\bar{b}_1}a^{\nu}$ are $X_{n-a_0-a_1}$ -forming.

Proof. We have

$${}^{d(1)}a^{\lambda} a_{\lambda} = 0$$

and, by virtue of (16.1a),

$$2 {}^{d(1)}a^{\lambda} {}^{d(1)}a^{\mu} \nabla_{[\mu} a_{\lambda]} = - {}^{d(1)}a_{\lambda} [{}^{d(1)}a^{\mu} \nabla_{\mu} a^{\lambda} - {}^{d(1)}a^{\mu} \nabla_{\mu} a^{\lambda}] = 0.$$

Hence the $n - q_0 - q_1$ vectors ${}^{\bar{b}_1}a^{\nu}$ are $X_{n-a_0-a_1}$ -forming. Moreover, we have

$$(16.6a) \quad {}^{b(2)}a^{\lambda} a_{\lambda} = {}^{b(2)}a^{\lambda} \partial_{\lambda} a = 0$$

and, according to (16.1b),

$$(16.6b) \quad {}^{b(2)}a^{\lambda} {}^{c(2)}a^{\mu} \nabla_{[\mu} a_{\lambda]} = 0.$$

Hence the $q_0 + q_2$ vectors ${}^{b(2)}a^{\nu}$ are $X_{a_0+a_2}$ -forming and the scalars ${}^{\bar{a}_2}a$ are constant along each of these $X_{a_0+a_2}$.

Theorem (16.3). There is no vector density $\mathfrak{a}^{\nu}$ (a_{λ}) of weight $p = 1/n$ ($p = -1/n$) which is covariant constant.

Proof. For the existence of a covariant constant vector density $\mathfrak{a}^{\nu}$ (a_{λ}) of weight $p = 1/n$ ($p = -1/n$) it is necessary and sufficient that there be a density $\mathfrak{A}^{\nu}$ ($\mathfrak{A}_{\lambda}$) of the same weight which satisfies the condition

$$(16.7) \quad \nabla_{\mu} \mathfrak{A}^{\nu} = -\mathfrak{A}^{\nu} \partial_{\mu} F \quad (\nabla_{\mu} \mathfrak{A}_{\lambda} = -\mathfrak{A}_{\lambda} \partial_{\mu} F).$$

*Compare with Theorems (7.2a), (9.5) and (15.5).

The complete set of integrability conditions of (16.7) reduces, according to Theorem (6.3), to

$$W_{a\lambda}^{\nu} \mathfrak{U}^{\lambda} = 0 \quad (W_{a\lambda}^{\nu} \mathfrak{U}_{\nu} = 0)$$

or, according to (15.4),

$$(16.8) \quad \begin{aligned} (a) \quad & L_{a\lambda}^{\nu} \mathfrak{U}^{\lambda} = 0 \quad (L_{a\lambda}^{\nu} \mathfrak{U}_{\nu} = 0), \\ (b) \quad & L_{x_0\lambda}^{\nu} \mathfrak{U}^{\lambda} = \frac{1}{n} \mathfrak{U}^{\nu} \quad (L_{x_0\lambda}^{\nu} \mathfrak{U}_{\nu} = \frac{1}{n} \mathfrak{U}_{\lambda}). \end{aligned}$$

Because $n > 1$ the equation (16.8b) cannot be satisfied, even if $q_0 = 1$.*

17. Structural constants. The infinitesimal operators of $G_t(R)$ are given by (5.5), i.e.

$$(17.1a) \quad X_a \stackrel{\text{def}}{=} L_{a\lambda}^{\nu} y^{\lambda} \frac{\partial}{\partial y^{\nu}}.$$

An arbitrary infinitesimal operator, represented in the group space by the vector u^a_b , is given by

$$(17.1b) \quad X \stackrel{\text{def}}{=} u^a_b X_a.$$

The structural constants appear in (5.7), and this equation can be rewritten as follows:

$$(17.2) \quad L_{a\lambda}^{\alpha} L_{c\alpha}^{\nu} - L_{c\lambda}^{\alpha} L_{a\alpha}^{\nu} = C_{bd}^{\epsilon} L_{\epsilon}^{\nu}.$$

Theorem (17.1). *The structural constants are given by*

$$(17.3) \quad C_{bd}^{\epsilon} = -C_{ca}^{\epsilon} = \delta_c^{\epsilon} \delta_a^f - \delta_a^{\epsilon} \delta_c^f,$$

and this equation is equivalent to

$$(17.4) \quad \begin{aligned} (a) \quad & C_{x_0c}^f = -C_{c x_0}^f = -\delta_c^f, \quad c \neq b, \\ (b) \quad & C_{x_0y_0}^f = -C_{y_0x_0}^f = \delta_{x_0}^f - \delta_{y_0}^f, \quad x_0 \neq y_0, \\ (c) \quad & \text{the remaining } C_{bd}^{\epsilon} = -C_{ca}^{\epsilon} = 0. \end{aligned}$$

Proof. Substituting into the left hand term of (17.2) from (14.1), we obtain

*Cf. Theorem (15.1b).

$$a_{\lambda}^a a_b^c a_d^v - a_{\lambda}^c a_d^a a_b^v = \delta_b^c L_{a\lambda}^v - \delta_d^a L_{c\lambda}^v = (\delta_b^c \delta_a^f - \delta_d^a \delta_c^f) L_{ef\lambda}^v.$$

Comparing this result with the right hand term in (17.2), we get (17.3). This equation shows that C_{ac}^f is not identically equal to zero if at least one of the conditions

$$(17.5) \quad (a) \quad c = b, \quad (b) \quad a = d$$

is satisfied. The second condition can be satisfied only by

$$(17.6) \quad a = d = x_0 . *$$

Substituting this value into (17.3), we obtain (17.4a) for $c \neq b$ and (17.4b) for $c = b = y_0$.** The condition (17.5a) can be satisfied only by

$$c = b = x_0 .$$

Substituting this value into (17.3) and replacing a, d by c, b , we obtain again (17.4a, b). Finally both conditions (17.5) can be satisfied by

$$a = d = x_0 , \quad c = b = y_0 .$$

Substituting these values into (17.3), we obtain (17.4b). (Of course, the case $y_0 = x_0$ leads to the identity $0 = 0$.) These combinations of indices exhaust all possible cases in which the structural constants are not identically zero.

The following two theorems will be proved simultaneously.

Theorem (17.2). *The operators X_{x_0} define an abelian subgroup G_{x_0} of q_0 parameters.*

Theorem (17.3). *If $q_1 q_2 \neq 0$, the operators X_{a_1} define an abelian subgroup G_{a_1} of $q_1 q_2$ parameters.*

Proof. We have

$$(17.7) \quad 2X_{[b}^a X_{d]}^c = C_{bd}^{ac} X_e^f .$$

Hence if

$$(17.8a) \quad a \neq d, \quad b \neq c,$$

we have, according to Theorem (17.1),

$$(17.8b) \quad C_{bd}^{ac} = 0,$$

*We have $a = a(1), d = d(2)$.

**We have $c = c(1), b = b(2)$.

and the operators X_a , X_a define (for (17.8a)) an abelian subgroup. The conditions (17.8a) are satisfied for the operators mentioned in Theorem (17.3). The remaining parts of these theorems are obvious.

Remark. A generic operator of G_a is given by (17.1b) where

$$(17.9) \quad u^a_b = 0, \quad a \neq b.$$

This system of $l - q_0$ equations represents a q_0 -dimensional hyperplane in the group space.

A generic operator of $G_{a,q}$ is given by (17.1b) where

$$(17.10) \quad u^{a\circ}_{b\circ} = 0, \quad u^{a(1)}_{b\circ} = 0.$$

This system of $l - q_1 q_2$ equations represents a $q_1 q_2$ -dimensional hyperplane in the group space.

18. The center. The adjoint group. For subsequent considerations we shall need the following

Lemma (18.1). *We have*

$$(18.1) \quad u^a_b C^{e,f}_{ac} = \sum u^{e(1)}_{x\circ} \delta^{e(1)}_{x\circ} \delta^{f(2)}_{d(2)} - \sum u^{x\circ}_{f(2)} \delta^{x\circ}_{d(2)} \delta^{e(1)}_{c(1)}.$$

Proof. Substituting into the left hand term of (18.1) from (17.3), we obtain

$$(18.2) \quad u^a_b C^{e,f}_{ac} = \sum u^{y(1)}_{x(2)} C^{e,f}_{y(1)c} \\ = \sum u^{y(1)}_{x(2)} \delta^{e(1)}_{x(2)} \delta^{e(1)}_{y(1)} \delta^{f(2)}_{d(2)} - \sum u^{y(1)}_{x(2)} \delta^{y(1)}_{d(2)} \delta^{f(2)}_{x(2)} \delta^{e(1)}_{c(1)}.$$

$\delta^{e(1)}_{x(2)}$ is identically zero if $x(2) = x_2$. Hence only the term $\delta^{e(1)}_{x\circ}$ can survive. Moreover, $\delta^{y(1)}_{d(2)}$ is identically zero if $y(1) = y_1$. Hence only the term $\delta^{y\circ}_{d(2)}$ can survive. Therefore (18.2) reduces to

$$u^a_b C^{e,f}_{ac} = \sum u^{y(1)}_{x\circ} \delta^{e(1)}_{y(1)} \delta^{e(1)}_{x\circ} \delta^{f(2)}_{d(2)} - \sum u^{y\circ}_{x(2)} \delta^{f(2)}_{x(2)} \delta^{y\circ}_{d(2)} \delta^{e(1)}_{c(1)} \\ = \sum u^{e(1)}_{x\circ} \delta^{e(1)}_{x\circ} \delta^{f(2)}_{d(2)} - \sum u^{x\circ}_{f(2)} \delta^{x\circ}_{d(2)} \delta^{e(1)}_{c(1)}.$$

In the following three theorems, which will be proved simultaneously, we denote again by

$$(18.3) \quad X \stackrel{\text{def}}{=} u^a_b X_a$$

a generic infinitesimal operator of $G_l(R)$.

Theorem (18.1a). *If only one of the numbers q_1 , q_2 is equal to zero, then $G_l(R)$ has no center.*

Theorem (18.1b). *If $q_1 = q_2 = 0$, then the center of $G_i(R)$ is defined by the operator (18.3), where*

$$(18.4a) \quad u^a_b = \delta^a_b,$$

i.e. by

$$(18.4b) \quad X = \sum_{x_0} X_{x_0}.$$

Theorem (18.1c). *If $q_1 q_2 \neq 0$, then the center of $G_i(R)$ is the subgroup $G_{a_1 a_2}$, mentioned in Theorem (17.2). Its generic operator is (18.3) where*

$$(18.5) \quad u^{a_0}_{b^{(2)}} = 0, \quad u^{a^{(1)}}_{b^0} = 0.$$

Proof. A necessary and sufficient condition for the operator (18.3) to belong to the center of $G_i(R)$ is

$$(18.6a) \quad u^a_b C^{ef}_{ac} = 0$$

or, according to (18.1),

$$(18.6b) \quad \sum u^{e^{(1)}}_{x^0} \delta^{c^{(1)}}_{x^0} \delta^{f^{(2)}}_{d^{(2)}} = \sum u^{f^{(2)}}_{x^0} \delta^{x^0}_{d^{(2)}} \delta^{e^{(1)}}_{c^{(1)}}.$$

Assume first

$$(18.7a) \quad q_1 = q_2 = 0.$$

Then the indices $a_1 \subset Q_1$, $a_2 \subset Q_2$ do not exist and (18.6b) reduces to

$$\sum u^{e_0}_{x^0} \delta^{c_0}_{x^0} \delta^{f_0}_{d_0} = \sum u^{f_0}_{x^0} \delta^{x_0}_{d_0} \delta^{e_0}_{c_0},$$

equivalent to

$$(18.8a) \quad u^{e_0}_{c_0} \delta^{f_0}_{d_0} = u^{f_0}_{d_0} \delta^{e_0}_{c_0}.$$

This equation yields

$$(18.8) \quad \begin{aligned} (b) \quad u^{x_0}_{x^0} &= u^{y_0}_{y^0} \quad \text{for} \quad f_0 = d_0, \quad e_0 = c_0, \\ (c) \quad u^{x_0}_{y^0} &= 0 \quad \text{for} \quad \begin{cases} f_0 = d_0, & e_0 \neq c_0 \\ f_0 \neq d_0, & e_0 = c_0, \end{cases} \\ & \quad x_0 \neq y_0. \end{aligned}$$

Therefore we may assume, without loss of generality, $u^{x_0}_{x^0} = 1$ and obtain in this way Theorem (18.1b).

Assume now

$$(18.7b) \quad q_2 = 0, \quad q_1 \neq 0,$$

so that the indices $a_2 \subset Q_2$ do not exist, (18.6b) reduces to

$$(18.9a) \quad \sum u^{x^o} \delta_{x^o}^{e(1)} \delta_{a^o}^{f^o} = u^{f^o} \delta_c^{e(1)}$$

and this system splits into two, namely (18.8a) for $c(1) = c_0$, $e(1) = e_0$ and

$$(18.9b) \quad u^{e^1} \delta_{a^o}^{f^o} = 0 \quad \text{for } e(1) = e_1, \quad c(1) = c_0,$$

$$(18.9c) \quad u^{f^o} \delta_{c^1}^{e^1} = 0 \quad \text{for } e(1) = e_1, \quad c(1) = c_1$$

(while the case $e(1) = e_0$, $c(1) = c_1$ leads to the identity $0 = 0$). The equations (18.9b, c) lead to

$$(18.10a) \quad u^{a(1)} = 0. *$$

Assuming

$$(18.7c) \quad q_2 \neq 0, \quad q_1 = 0,$$

we obtain in the same way

$$(18.10b) \quad u^{b(2)} = 0.$$

Hence in the case (18.7b) or (18.7c) the center does not exist. Assume now

$$(18.7d) \quad q_1 q_2 \neq 0.$$

Then the condition (18.6b) leads again to the partial results (18.10), *i.e.* to (18.5). The equations (18.5) exhaust all components but u^{a_1} which do not appear in (18.6b). Hence u^{a_1} are arbitrary. This proves Theorem (18.1c).

If $X = t^a X_a$ is the infinitesimal operator defining the generic transformation T of $G_i(R)$ and if another transformation U of $G_i(R)$ is defined by (18.3), the infinitesimal operator (of the adjoint group) corresponding to T is given by (11.7), *i.e.*

$$(18.11) \quad t^d A_d^c = u^b t^d C_{ba}^c \frac{\partial}{\partial u^f}.$$

We shall use this result in the next theorem.

Theorem (18.2). *We have*

$$(18.12) \quad t^d A_d^c u^f = \sum (t^f u^{x^o} - t^{x^o} u^f).$$

Proof. Using (18.1) and (18.11) we get

* (18.8c) is a special case of (18.10a).

$$(18.13) \quad t^c_d A_c u^e_f = t^c u^a_b C_{ac}^e = \sum_{x_o, z^{(1)}, t^{(2)}} u^{x_o} \delta_{x_o}^{z^{(1)}} \delta_{t^{(2)}}^{f^{(2)}} t^{x^{(1)}} - \sum_{x_o, z^{(1)}, t^{(2)}} u^{x_o} \delta_{t^{(2)}}^{f^{(2)}} \delta_{z^{(1)}}^{e^{(1)}} t^{t^{(2)}}.$$

On the other hand,

$$(18.14a) \quad \sum_{x_o, z^{(1)}, t^{(2)}} u^{x_o} \delta_{x_o}^{z^{(1)}} \delta_{t^{(2)}}^{f^{(2)}} t^{t^{(2)}} = \sum_{x_o, z_o, t^{(2)}} u^{x_o} \delta_{x_o}^{z_o} \delta_{t^{(2)}}^{f^{(2)}} t^{t^{(2)}} = \sum_{x_o, t^{(2)}} u^{x_o} t^{t^{(2)}} \delta_{t^{(2)}}^{f^{(2)}} = \sum u^{x_o} t^f.$$

In a similar manner

$$(18.14b) \quad \sum_{x_o, z^{(1)}, t^{(2)}} u^{f^{(2)}} \delta_{t^{(2)}}^{x_o} \delta_{z^{(1)}}^{t^{(2)}} t^{z^{(1)}} = \sum u^f t^{x_o}.$$

Substituting from (18.14) into (18.13), we obtain (18.12).

Remark. We know that the center is normal, *i.e.* invariant with respect to the adjoint group. This could be checked directly. In fact, if we substitute from (18.4a) into (18.12) and remember that the indices $a_1 \subset Q_1$, $a_2 \subset Q_2$ do not exist in the case (18.7a), we obtain

$$(18.15) \quad t^{d_o}_c A_c u^{e_o}_f = \sum (t^{f_o} \delta^{e_o}_{x_o} - t^{e_o} \delta^{f_o}_{x_o}) = t^{f_o} - t^{e_o} = 0.$$

On the other hand, if (18.5) hold then

$$(18.16) \quad \begin{aligned} t^c_d A_c u^{e_o}_{f^{(2)}} &= \sum (t^{f^{(2)}} u^{e_o}_{x_o} - t^{e_o} u^{f^{(2)}}_{x_o}) = 0, \\ t^c_d A_c u^{e_1}_{f_o} &= \sum (t^{f_o} u^{e_1}_{x_o} - t^{e_1} u^{f_o}_{x_o}) = 0, \\ t^c_d A_c u^{e_1}_{f_2} &= \sum (t^{f_2} u^{e_1}_{x_o} - t^{e_1} u^{f_2}_{x_o}) = 0. \end{aligned}$$

The equations (18.15), (18.16) show that the subgroup (18.4) ((18.5)) is normal.

The questions arises whether the subgroup G_{a_o} , mentioned in Theorem (17.2), is normal.

Theorem (18.3). *The subgroup G_{a_o} is not normal.*

Proof. Substituting from (17.9) we obtain

$$(18.17) \quad \begin{aligned} (a) \quad t^c_d A_c u^e_f &= \sum (t^f u^e_{x_o} - t^e u^f_{x_o}) = -t^f (u^f - u^e), \quad e \neq f, \quad \text{no sum,} \\ (b) \quad t^c_d A_c u^{y_o}_{x_o} &= \sum (t^{y_o} u^{x_o}_{x_o} - t^{x_o} u^{y_o}_{x_o}) = 0. \end{aligned}$$

The equation (18.17a) shows that G'_{a_0} is not invariant with respect to the adjoint group.

19. The derived group. Let us define a set of vectors V^f in the group space by means of structural constants. Let $\overset{a}{b}, \overset{c}{d}$ be two arbitrary but fixed composed indices and put

$$(19.1) \quad V^f_{\overset{ac}{bd}} \stackrel{\text{def}}{=} C_{ac}^f \overset{f}{bd}.$$

These vectors are not independent of the transformations (5.11) in the group space. On the other hand, if the indices $\overset{a}{b}, \overset{c}{d}$ run through the whole range, the rank of the matrix of all vectors (19.1) is independent of the transformations of the group space. In subsequent consideration we shall investigate this rank.

Theorem (19.1). *The set (19.1) can be described by means of $l' = l - 1$ linearly independent vectors. These vectors are*

(1) $l - q_0$ linearly independent vectors

$$(19.2a) \quad V^f_{\overset{c}{b}} \stackrel{\text{def}}{=} -\delta_c^f, \quad c \neq b,$$

and

(2) $q_0 - 1$ linearly independent vectors

$$(19.2b) \quad V^f_{x'_0} \stackrel{\text{def}}{=} \delta_{x'_0}^f - \delta_{u_0}^f, \quad x'_0 \neq u_0.*$$

Proof. Using (19.1) and (17.4), we obtain

$$(19.3) \quad \begin{aligned} (a) \quad & V^f_{\overset{x_0 c}{b \ x_0}} = -V^f_{\overset{c \ x_0}{x_0 b}} = -\delta_c^f = V^f_{\overset{c}{b}}, \quad c \neq b, \\ (b) \quad & V^f_{\overset{x_0 y_0}{y_0 x_0}} = -V^f_{\overset{y_0 x_0}{x_0 y_0}} = \delta_{x_0}^f - \delta_{y_0}^f, \\ (c) \quad & \text{the remaining } V^f_{\overset{ac}{bd}} = 0. \end{aligned}$$

The sets (19.3) are obviously linearly independent, and the set (19.3a) is described by means of linearly independent vectors (19.2a). The set (19.2a) splits into four sets

$$(19.4) \quad V^f_{\overset{c_0}{b_0}}, \quad V^f_{\overset{c_0}{b_2}}, \quad V^f_{\overset{c_1}{b_2}}, \quad V^f_{\overset{c_1}{b_0}}, \quad c(1) \neq b(2),$$

* $u_0 \subset Q_0$ is an arbitrary fixed index.

and these sets have in turn

$$q_0(q_0 - 1), \quad q_0q_2, \quad q_1q_2, \quad q_1q_0$$

vectors. Hence the set (19.2a) contains

$$q_0^2 + q_0(q_1 + q_2) + q_1q_2 - q_0 = l - q_0$$

linearly independent vectors. The set (19.3b) splits into two sets

$$(19.5) \quad \begin{aligned} (a) \quad & V_{\substack{x_0' y_0' \\ y_0' x_0'}}^f = - V_{\substack{y_0' x_0' \\ x_0' y_0'}}^f = V_{x_0'}^f - V_{y_0'}^f, \quad x_0', y_0' \neq u_0, \\ (b) \quad & V_{\substack{x_0' u_0 \\ u_0 x_0'}}^f = - V_{\substack{u_0 x_0' \\ x_0' u_0}}^f = V_{x_0'}^f, \quad x_0' \neq u_0. \end{aligned}$$

Hence the set (19.3b) is described by means of $q_0 - 1$ linearly independent vectors (19.2b).

The following two theorems will be proved simultaneously.

Theorem (19.2). *The f -rank of C_{ac}^f is $l' = l - 1$ and the f -domain of C_{ac}^f is described by l' linearly independent vectors (19.2).*

Theorem (19.3).* *The first derived subgroup $G'_l(R)$ of $G_l(R)$ is defined by l' independent operators*

$$(19.6) \quad \begin{aligned} (a) \quad & X_a' \stackrel{\text{def}}{=} X_a, \quad a \neq b, \\ (b) \quad & X_{x_0'}' = X_{x_0'} - X_{u_0}. \end{aligned}$$

Proof. Theorem (19.2) follows at once from Theorem (19.1). We know that the operator (18.3) belongs to the first derived subgroup if and only if u^b is a linear combination of the vectors (19.2) which describe the f -domain of C_{ac}^f .** In this way we obtain (19.6).

Theorem (19.4). *$G'_l(R)$ is represented in the group space by the l' -dimensional hyperplane:*

$$(19.7) \quad \sum u^{y_0} = 0.$$

Proof. Starting with the operator (18.3) and using (19.6) we obtain

$$\begin{aligned} X &= \sum_{x \neq y} u^y X_x' + \sum_{\substack{y_0' \\ y_0' x_0'}} u^{y_0'} (X_{y_0'}' + X_{u_0}) + u^{u_0} X_{u_0} \\ &= \sum_{x \neq y} u^y X_x' + \sum_{\substack{y_0' \\ y_0' x_0'}} u^{y_0'} X_{y_0'}' + X_{u_0} \sum u^{y_0}. \end{aligned}$$

*See Theorem (11.1b).

**See the equation (11.11).

Hence this operator belongs to $G'_{l'}(R)$ if and only if (19.7) holds.

Remark I. Cf. Theorem (11.1b).

Remark II. One obtains from (18.12)

$$t^d A_c \sum_d u^{y_0} = \sum_{x_0, y_0} (t^{y_0} u^{x_0} - t^{x_0} u^{y_0}) = 0,$$

and this equation confirms the known fact that $G'_{l'}(R)$ is normal.

Theorem (19.5). *The structural constants of $G'_{l'}(R)$ are*

$$(19.8a) \quad C'_{ac} \binom{f}{bd} = -C'_{ca} \binom{f}{db} \stackrel{\text{def}}{=} C'_{ac} \binom{f}{bd}, * \quad a \neq b, \quad c \neq d,$$

$$(19.8) \quad (b) \quad C'_{ax_0'} \binom{f}{bx_0'} = -C'_{x_0'a} \binom{f}{x_0'b} \stackrel{\text{def}}{=} C'_{ax_0'} \binom{f}{bx_0'} - C'_{ax_0'} \binom{f}{bx_0'}, \quad a \neq b,$$

$$(c) \quad \text{the remaining } C'_{ac} \binom{f}{bd} = 0.$$

The equations (19.8) are equivalent to

$$(19.9) \quad \begin{aligned} (a) \quad C'_{ax_0'} \binom{f}{bx_0'} &= -C'_{x_0'a} \binom{f}{x_0'b} = -\delta_c \binom{f}{b}, & c, b, x_0 \neq 0, \\ (b) \quad C'_{x_0y_0} \binom{f}{y_0x_0} &= -C'_{y_0x_0} \binom{f}{x_0y_0} = \delta_{x_0} \binom{f}{x_0} - \delta_{y_0} \binom{f}{y_0}, & x_0 \neq y_0, \\ (c) \quad C'_{ax_0'} \binom{f}{bx_0'} &= -C'_{x_0'a} \binom{f}{x_0'b} \\ &= \delta_{x_0'} \delta_a \binom{f}{x_0'} - \delta_a \delta_{x_0'} \binom{f}{x_0'b} - \delta_{x_0'} \delta_a \binom{f}{u_0} + \delta_a \delta_{x_0'} \binom{f}{u_0}, & a \neq b, \\ (d) \quad \text{the remaining } C'_{ac} \binom{f}{bd} &= 0. \end{aligned}$$

Proof. We have for $a \neq b$, by virtue of (19.6),

$$\begin{aligned} 2X'_{[a} X_{x_0']} &= C'_{ax_0'} \binom{f}{bx_0'} X'_{f} = C'_{ax_0'} \binom{f}{bx_0'} X'_{(f)} + X_{u_0} \sum_{y_0} C'_{ax_0'} \binom{y_0}{bx_0'}, \\ 2X'_{[a} X_{x_0']} &= 2X'_{[a} X'_{x_0']} + 2X'_{[a} X_{u_0]}. \end{aligned}$$

Hence

$$(19.10) \quad \begin{aligned} 2X'_{[a} X'_{x_0']} &= C'_{ax_0'} \binom{f}{bx_0'} X'_{(f)} = C'_{ax_0'} \binom{f}{bx_0'} X'_{(f)} - C'_{ax_0'} \binom{f}{bx_0'} X'_{(f)} \\ &\quad + X_{u_0} \sum_{y_0} (C'_{ax_0'} \binom{y_0}{bx_0'} - C'_{ax_0'} \binom{y_0}{bx_0'}). \end{aligned}$$

*Here and in subsequent equations the index $\binom{f}{j}$ indicates that $j \neq u_0$.

On the other hand, we have, by virtue of (17.3),

$$\sum_{y_0} C_{ax_0}^{y_0} = \sum_{y_0} (\delta_{x_0}^y \delta_a^{y_0} - \delta_a^{y_0} \delta_{x_0}^y) = \sum_{y_0} (\delta_{b(2)}^{x_0} \delta_{y_0}^{x_0} \delta_{a(1)}^{y_0} - \delta_{a(1)}^{x_0} \delta_{y_0}^{x_0} \delta_{b(2)}^{y_0}) \\ = \delta_{b(2)}^{x_0} \delta_{a(1)}^{x_0} - \delta_{a(1)}^{x_0} \delta_{b(2)}^{x_0} = 0.$$

Therefore (19.10) simplifies to

$$(19.11) \quad 2X'_{[a} X'_{x_0']} = C'_{ax_0}^{(f)} X'_{(f)} = (C_{ax_0'}^{(f)} - C_{ax_0'}^{au_0}) X'_{(f)},$$

and this equation yields (19.8b). The remaining equations (19.8) could be obtained in a similar manner. The equations (19.9) follow from (19.8), (17.3) and (17.4).

The structural constants $C'_{ac}^{(f)}$ will be needed in the next section.

20. Continuation. In this section we shall consider higher derived groups. To this effect we need first the following

Theorem (20.1). *The f -rank of $C'_{ac}^{(f)}$ is $l' = l - 1$. The f -domain of $C'_{ac}^{(f)}$ is described by l' linearly independent vectors in the group space. These vectors are*

(a) $l - q_0$ vectors

$$(20.1a) \quad V'_{\frac{c}{b}}^{(f)} \stackrel{\text{def}}{=} V_{\frac{c}{b}}^{(f)} = -\delta_{\frac{b}{c}}^{(f)}, \quad c \neq b,$$

(b) $q_0 - 1$ vectors

$$(20.1b) \quad V'_{x_0'}^{(f)} \stackrel{\text{def}}{=} \delta_{x_0'}^{(f)}, \quad x_0' \neq u_0.$$

Proof. Let us define the following set of vectors:

$$(20.2) \quad V'_{\frac{ac}{bd}}^{(f)} \stackrel{\text{def}}{=} C'_{ac}^{(f)}.$$

Then we have, according to (19.9),

$$(20.3) \quad \begin{aligned} (a) \quad V'_{\frac{x_0 c}{b x_0}}^{(f)} &= -V'_{\frac{c x_0}{x_0 b}}^{(f)} = -\delta_{\frac{b}{c}}^{(f)}, & c, b, x_0 \neq 0, \\ (b) \quad V'_{\frac{x_0 y_0}{y_0 x_0}}^{(f)} &= -V'_{\frac{y_0 x_0}{x_0 y_0}}^{(f)} = \delta_{x_0}^{(f)} - \delta_{y_0}^{(f)}, & x_0 \neq y_0, \\ (c) \quad V'_{\frac{ax_0'}{bx_0'}}^{(f)} &= -V'_{\frac{x_0' a}{x_0' b}}^{(f)} \\ &= \delta_{x_0'}^{(f)} \delta_a^{(f)} - \delta_a^{(f)} \delta_{x_0'}^{(f)} - \delta_{\frac{b}{x_0'}} \delta_{u_0}^{(f)} + \delta_{u_0}^{(f)} \delta_{\frac{b}{x_0'}}^{(f)}, & a \neq b, \end{aligned}$$

$$(d) \quad \text{the remaining } V_{\substack{ac \\ bd}}^{(r)} = 0.$$

The vectors (20.3a, c) are linear combinations of the vectors (20.1a). The set (20.3b) splits into two, namely

$$(20.4) \quad \begin{aligned} V_{\substack{x_0' y_0' \\ y_0' x_0'}}^{(r)} &= - V_{\substack{y_0' x_0' \\ x_0' y_0'}}^{(r)} = \delta_{x_0'}^{(r)} - \delta_{y_0'}^{(r)}, & x_0' \neq y_0', \\ V_{\substack{x_0' u_0 \\ u_0 x_0'}}^{(r)} &= - V_{\substack{u_0 x_0' \\ x_0' u_0}}^{(r)} = \delta_{x_0'}^{(r)}. \end{aligned}$$

Hence the set (20.3b) consists of linear combinations of the vectors (20.1b). These facts lead at once to the statements of the theorem.

Theorem (20.2). *The second, third, ... derived group is identical with the first derived group $G'_1(R)$.*

Proof. This statement follows at once from the fact that the (r) -rank of the structural constants $C'_{ac}^{(r)}$ of $G'_1(R)$ is equal to the number l' of parameters of $G'_1(R)$. In fact, the operator $X = u^{(r)} X_{(r)}$ belongs to the second derived group if and only if $u^{(r)}$ are linear combinations of the vectors (20.1) which span the r -domain of $C'_{ac}^{(r)}$. Hence the second derived group is defined by the infinitesimal operators

$$X_a'' \stackrel{\text{def}}{=} X_a', \quad X_{x_0'}'' \stackrel{\text{def}}{=} X_{x_0'}', \quad a \neq b, \quad x_0' \neq u_0.$$

Remark. According to the previous theorem the holonomy group $G_*(R)$ is not integrable.

This concludes the investigations of ideal holonomy groups.

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