On Yilmaz' New Approach to General Relativity

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I. The Theorem of Yilmaz. YILMAZ¹ has presented a theory of gravitation in which he regards the metrical tensor $g_{\mu\nu}$ as a functional of a scalar function φ . His basic equations are

$$(1) \quad R_{\mu}^{\nu} - \frac{1}{2} \delta_{\mu}^{\nu} R = 8\pi T_{\mu}^{\nu}$$

and

$$(2) \quad g^{\mu\nu} \varphi_{,\mu\nu} \equiv (-g)^{-\frac{1}{2}} \{ g^{\mu\nu} (-g)^{\frac{1}{2}} \varphi_{/\mu} \}_{/\nu} = 0$$

where the comma denotes the covariant derivative with respect to the metric $g_{\mu\nu}$, the solidus denotes the partial derivative, and the tensor T_{μ}^{ν} is defined by

$$(3) \quad T_{\mu}^{\nu} = \frac{1}{8\pi} (2g^{\nu\lambda} \varphi_{/\mu} \varphi_{/\lambda} - \delta_{\mu}^{\nu} g^{\lambda\tau} \varphi_{/\lambda} \varphi_{/\tau}),$$

this being the usual energy-momentum tensor for the scalar field $\varphi/(8\pi)^{\frac{1}{2}}$.

He exhibits a static field based on the following remarkable theorem that he discovered.

Take the line element to have the special form

$$(4) \quad ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = e^{-2\varphi} dt^2 - e^{2\varphi} (dx^2 + dy^2 + dz^2)$$

with φ independent of time, and note that, for this line element, the field equation (2) reduces to the "flat" Newtonian equation

$$(5) \quad \nabla^2 \varphi \equiv \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0.$$

Now compute the Einstein tensor $R_{\mu}^{\nu} - \frac{1}{2} \delta_{\mu}^{\nu} R$ for the metric in (4), and also compute the tensor T_{μ}^{ν} defined in (3) using this metric. Then the two expressions will be identical—except for one discrepancy in the 4-4 components. We have, in fact, for this special line element,

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¹H. YILMAZ, *Phys. Rev.* **111**, 1417 (1958).

$$(6) \quad R_{\mu}^{\nu} - \frac{1}{2} \delta_{\mu}^{\nu} R \equiv 8\pi T_{\mu}^{\nu} + 2 \delta_{\mu}^{\nu} \delta_4^4 e^{-2\varphi} \nabla^2 \varphi.$$

However, in view of (5), the last term in (6) is zero. Accordingly, if φ satisfies (5), the static line element (4) will satisfy (1).

II. Remarks concerning Yilmaz' Theory. Since φ here satisfies (5) except at singular sources it may be thought of as the Newtonian potential due to these sources, and this fact leads YILMAZ to look on φ as representing the gravitational field, with the metrical tensor just a reflection of that field, so that (2) is the only field equation.² There are difficulties in this view. One that may be mentioned is the lack of invariance of (4). If $g_{\mu\nu}$ and φ are regarded as separate fields, and both (1) and (2) as field equations, then (4) and (5) can be regarded as giving a particular solution of these field equations. But if (4) is meant to define $g_{\mu\nu}$ as a functional of φ , one would expect (4) to be expressed in an invariant form, which it is not. Any transformation more complicated than a change of origin in the four-space plus a Euclidean rotation transformation of the coordinates x, y, z will upset the form of (4). The invariance can be somewhat improved by writing (4) in the more general form

$$(7) \quad ds^2 = e^{-2\varphi} dt^2 - e^{2\varphi} d\sigma^2$$

where $d\sigma^2$ is the line element in a flat three-space for any coordinate system in that three-space. Apparently this is what YILMAZ had in mind. His theorem still holds for this line element (7), provided we write (5) in covariant form for the sub-metric defined by $d\sigma^2$.

Of various problems connected with YILMAZ' theory, I consider below two concerning which I have been able to form conclusions having reasonably sharp focus.

One must question YILMAZ' derivation of the relation

$$(8) \quad E = M_0 c^2$$

quite apart from the lack of discussion of the energy-momentum pseudo-tensor of the gravitational field. There is, for example, a numerical discrepancy in the use of the factor $1/4\pi$ in his equation (21), for that factor is taken from his equation (14) which apparently should have had $1/8\pi$; one finds the $1/8\pi$ in both of his equations (17). With $1/8\pi$ his equation (21) would not coincide with (8) here but would be off by a factor of 2.

There is, however, a more serious difficulty. For in his equation (21) YILMAZ treats the energy of the field, to within the above factor of 2, as

$$(9) \quad E = \int T^{44}(g_{44})^{\frac{1}{2}} dv.$$

²Reference 1, page 1424. But note that on page 1419, above Eq. (18), YILMAZ refers to his equations (17) (which correspond to our (1)) as field equations—as I think one must.

This integral does not take account of the non-Cartesian character of the line element (4). The three dimensional dv needs to be multiplied by $(-\gamma)^{\frac{1}{2}}$, where γ is the determinant of the metrical tensor of the three-space $t = \text{const.}$ ³

For the line element (4) we have

$$(-\gamma)^{\frac{1}{2}} = e^{3\varphi}$$

and when this is inserted in the integrand of (9), φ being here proportional to $1/r$, it causes the integral to diverge.

Since

$$(10) \quad g_{44}(-\gamma) = -g,$$

the modified form of (9) is

$$(11) \quad E = \int T^{44}(-g)^{\frac{1}{2}} dv.$$

We can introduce a favorable factor by replacing the doubly contravariant component T^{44} by the more usual mixed component T^4_4 , writing

$$(12) \quad E = \int T^4_4(-g)^{\frac{1}{2}} dv;$$

but, despite the factor $e^{-2\varphi}$ thus introduced, the integral still diverges. Use of T_{44} in (12) would bring convergence but would not give the correct coefficient. Use of a relative tensor of *negative* weight⁴ would also bring convergence, and by a suitable choice of weight one could obtain the correct coefficient, but when such procedures are admitted there is too much arbitrariness for a derivation of (7) along these lines to carry conviction.

There are difficulties connected with YILMAZ' cosmological line element given by his equations (24), (25). YILMAZ obtains φ at a point with radial coordinate r by integrating over the matter contained within the sphere of radius r . But though a homogeneous spherical shell yields zero *force* inside, in Newtonian potential theory, this does not mean it will not contribute to the *potential* inside. The contribution is constant for a given shell, but the total contribution from beyond a radius r will depend on r . Thus, for the total potential at a point with radial coordinate r we have, not YILMAZ'

$$(13) \quad \varphi = \left(\frac{4}{3} \pi r^3 \sigma\right)/r = \frac{4}{3} \pi r^2 \sigma,$$

but

$$(14) \quad \varphi = \left(\frac{4}{3} \pi r^3 \sigma\right)/r + \sigma \int_r^\infty \frac{4\pi \rho^2 d\rho}{\rho} = \frac{4}{3} \pi r^2 \sigma + \infty - 2\pi r^2 \sigma = \infty - \frac{2}{3} \pi r^2 \sigma.$$

³Compare, in this connection the treatment in MØLLER, *The Theory of Relativity*, Oxford University Press, Oxford 1952, p. 238, p. 298, and Appendix 8.

⁴Compare J. N. GOLDBERG, Phys. Rev. 111, 315 (1958).

Ignoring the infinite constant, we find that the dependence of φ on r^2 has a different numerical coefficient from that of YILMAZ and, what is more important, an opposite sign. One easily verifies that the above φ satisfies the Poisson equation

$$(15) \quad \nabla^2 \varphi = -4\pi\sigma,$$

whereas the φ of YILMAZ gives

$$(16) \quad \nabla^2 \varphi = +8\pi\sigma.$$

While this discrepancy shows that YILMAZ' derivation of his cosmological line element is not valid, the argument just given is itself suspect. For, with a uniform distribution of matter throughout the universe, every point of the universe should be equivalent to every other point and therefore the potential ought to be a constant and not a function of r . The reason that the above calculation gives a function of r is apparently connected with the fact that the matter at infinity contributes an infinite amount to the potential.

III. Extension of Yilmaz' Theorem. Despite possible difficulties in the theory of YILMAZ, its basic mathematical theorem remains striking. How general is that theorem? The following calculations throw some light on this question.

Let Greek indices $\alpha, \beta, \gamma, \dots$ run from 0 to $n - 1$ and Latin indices $a, b, c, \dots$ from 1 to $n - 1$.

Consider a Riemannian space with line element in the special form

$$(17) \quad ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = g_{00} (dx^0)^2 + g_{ab} dx^a dx^b,$$

and do not allow transformations that mix x^0 with x^a . Furthermore, take $g_{\alpha\beta}$ to be independent of x^0 , so that

$$(18) \quad g_{\alpha\beta/0} = 0.$$

For convenience, write

$$(19) \quad g_{00} = e^{-2m\varphi}$$

where m is a constant and φ is a function of x^a but not of x^0 .

Denote the various curvature quantities belonging to $g_{\alpha\beta}$ by the generic symbol $\mathfrak{R}$ and those belonging to g_{ab} by R .

Then, after some routine computations, one finds

$$(20) \quad \mathfrak{R}_0^0 - \frac{1}{2} \delta_0^0 \mathfrak{R} = -\frac{1}{2} R - \frac{m(m-1)}{2} g^{cd} \varphi_{/c} \varphi_{/d},$$

$$(21) \quad \mathfrak{R}_a^0 = \mathfrak{R}_0^a = 0,$$

$$(22) \quad \begin{aligned} \mathfrak{R}_b^a - \frac{1}{2} \delta_b^a \mathfrak{R} &= R_b^a - \frac{1}{2} \delta_b^a R - m g^{ac} \varphi_{/b} \varphi_{/c} + m \delta_b^a g^{cd} \varphi_{/c} \varphi_{/d} \\ &\quad + m^2 g^{ac} \varphi_{/b} \varphi_{/c} - \frac{m(m+1)}{2} \delta_b^a g^{cd} \varphi_{/c} \varphi_{/d} \end{aligned}$$

where the commas denote covariant derivatives with respect to g_{ab}

Now make the following conformal transformation on the whole metric $g_{\alpha\beta}$:

$$(23) \quad g_{\alpha\beta} \rightarrow \bar{g}_{\alpha\beta} = e^{2\varphi} g_{\alpha\beta} .$$

This gives the new line element

$$(24) \quad d\bar{s}^2 = \bar{g}_{\alpha\beta} dx^\alpha dx^\beta = e^{-2(m-1)\varphi} (dx^0)^2 + e^{2\varphi} g_{ab} dx^a dx^b .$$

Denoting the curvature quantities belonging to $\bar{g}_{\alpha\beta}$ by $\bar{\mathcal{R}}$, we find,⁵ after expressing the covariant derivatives with respect to $g_{\alpha\beta}$ in terms of covariant derivatives with respect to g_{ab} , that

$$(25) \quad \begin{aligned} \bar{\mathcal{R}}_0^0 - \frac{1}{2} \delta_0^0 \bar{\mathcal{R}} \\ = e^{-2\varphi} \left[\mathcal{R}_0^0 - \frac{1}{2} \delta_0^0 \mathcal{R} - (n-2) g^{cd} \varphi_{,cd} - \frac{(n-2)(n-3)}{2} g^{cd} \varphi_{,c} \varphi_{,d} \right] , \end{aligned}$$

$$(26) \quad \mathcal{R}_a^0 = \bar{\mathcal{R}}_0^a = e^{-2\varphi} \mathcal{R}_a^0 ,$$

$$(27) \quad \begin{aligned} \bar{\mathcal{R}}_b^a - \frac{1}{2} \delta_b^a \bar{\mathcal{R}} = e^{-2\varphi} \left[\mathcal{R}_b^a - \frac{1}{2} \delta_b^a \mathcal{R} + (n-2) g^{ac} \varphi_{,bc} - (n-2) \delta_b^a g^{cd} \varphi_{,cd} \right. \\ \left. - (n-2) g^{ac} \varphi_{,b} \varphi_{,c} - \frac{(n-1)(n-2)}{2} \delta_b^a g^{cd} \varphi_{,c} \varphi_{,d} \right] . \end{aligned}$$

Comparing (20), (21), (22) with (25), (26), (27), we see that if we take $m = n - 2$ the second derivatives of φ will disappear from all components of $\bar{\mathcal{R}}_\beta^\alpha - \frac{1}{2} \delta_\beta^\alpha \bar{\mathcal{R}}$ except the 0-0 component. Moreover, for this value of m we have

$$(28) \quad d\bar{s}^2 = e^{-2(n-3)\varphi} (dx^0)^2 + e^{2\varphi} g_{ab} dx^a dx^b ,$$

$$(29) \quad \bar{\mathcal{R}}_0^0 - \frac{1}{2} \delta_0^0 \bar{\mathcal{R}} = e^{-2\varphi} \left[-\frac{1}{2} \mathcal{R} - (n-2)(n-3) g^{cd} \varphi_{,c} \varphi_{,d} - (n-2) g^{cd} \varphi_{,cd} \right] ,$$

$$(30) \quad \bar{\mathcal{R}}_a^0 = \bar{\mathcal{R}}_0^a = 0 ,$$

$$(31) \quad \begin{aligned} \bar{\mathcal{R}}_b^a - \frac{1}{2} \delta_b^a \bar{\mathcal{R}} \\ = e^{-2\varphi} \left[\mathcal{R}_b^a - \frac{1}{2} \delta_b^a \mathcal{R} + (n-2)(n-3) \{ g^{ac} \varphi_{,b} \varphi_{,c} - \delta_b^a g^{cd} \varphi_{,c} \varphi_{,d} \} \right] , \end{aligned}$$

and we note that the terms explicitly involving $\varphi_{,a}$ in (29), (30), (31) constitute the quantity

$$(32) \quad T_\beta^\alpha = (n-2)(n-3) [2\bar{g}^{\alpha\gamma} \varphi_{,\beta} \varphi_{,\gamma} - \delta_\beta^\alpha \bar{g}^{\gamma\delta} \varphi_{,\gamma} \varphi_{,\delta}]$$

when account is taken of the fact that φ is independent of x^0 . Note that the factor $e^{-2\varphi}$ is included in (32). The quantity T_β^α is $(n-2)(n-3)$ times the usual energy-momentum tensor of the scalar field φ . If we introduce the new scalar $\bar{\varphi}$ by

$$(33) \quad \bar{\varphi} = \left[\frac{(n-2)(n-3)}{2\omega_{n-1}} \right]^{\frac{1}{2}} \varphi ,$$

⁵Compare EISENHART, *Riemannian Geometry*, Princeton University Press (1926), pp. 89-90, noting the definition of his σ_{ij} in his equation (28,4).

where

$$\omega \equiv \frac{2\pi^{\frac{1}{2}(n-1)}}{\Gamma\left(\frac{n-1}{2}\right)}$$

is the solid angle in $(n-1)$ -dimensional space, we may define a $\bar{T}_\beta^\alpha$ by

$$(34) \quad \bar{T}_\beta^\alpha = \frac{1}{2\omega_{n-1}} (2\bar{g}^{\alpha\gamma}\bar{\varphi}_{/\beta}\bar{\varphi}_{/\gamma} - \delta_\beta^\alpha \bar{g}^{\gamma\delta}\bar{\varphi}_{/\gamma}\bar{\varphi}_{/\delta}),$$

and so write (29), (30), (31) as

$$(35) \quad \bar{\mathcal{R}}_0^0 - \frac{1}{2} \delta_0^0 \bar{\mathcal{R}} = \frac{1}{2\omega_{n-1}} \bar{T}_0^0 - \frac{1}{2} e^{-2\varphi} R - (n-2) \bar{g}^{cd} \varphi_{,cd},$$

$$(36) \quad \bar{\mathcal{R}}_a^0 = \mathcal{R}_a^a = \bar{T}_a^0 = \bar{T}_0^a = 0,$$

$$(37) \quad \bar{\mathcal{R}}_b^a - \frac{1}{2} \delta_b^a \bar{\mathcal{R}} = \frac{1}{2\omega_{n-1}} \bar{T}_b^a + e^{-2\varphi} [R_b^a - \frac{1}{2} \delta_b^a R],$$

always remembering that this is for the case in which all field quantities are independent of x^0 .

From (35), (36), (37) we see that YILMAZ' theorem is not valid unless

$$(38) \quad R_b^a - \frac{1}{2} \delta_b^a R = 0$$

which is equivalent to

$$(39) \quad R_b^a = 0.$$

However, when (39) is satisfied, then a generalized form of YILMAZ' theorem is valid for a line element of the special form (28) and not only holds for spaces of dimensionality other than 4, but also holds irrespective of signature. A consequence of the last fact is that in the space-time case discussed by YILMAZ one may specialize a coordinate other than the time coordinate and still obtain a solution of the basic equations. For example, by specializing x rather than t , one obtains the line element

$$(40) \quad ds^2 = -e^{-2\varphi} dx^2 + e^{2\varphi} (dt^2 - dy^2 - dz^2)$$

where φ is independent of x and satisfies the cylindrical wave equation

$$(41) \quad \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \varphi}{\partial z^2} = 0.$$

Solutions of this type were already known to YILMAZ. He informs me, too, that he has generalized his theorem to the case of n dimensions along lines different from the above. His generalization is concerned with the case in which the sub-metrics are flat, and his results for this case complement those given above.

In the space-time case, the sub-metric is three-dimensional. Since in three dimensions the Ricci tensor and the curvature tensor each have just six independent components, the condition (39) is equivalent to the vanishing of the curvature tensor. Thus for YILMAZ' space-time theory the line element $d\sigma^2$ that enters (7) must be flat, and this seems to emphasize the key gravitational significance of his φ , for in the solutions given by his theorem φ indeed enters as the sole progenitor of the geometrical effects of gravitation. However, when the overall number of dimensions exceeds four the condition (39) for the sub-metric does not imply flatness, and in that case one could have curvature entering the analogous solutions independently of φ and the question could arise as to the extent to which φ represented the gravitational field. Since space-time is four-dimensional this question may be academic. Yet extensions to higher dimensionalities, as in certain unified field theories, are not impossible. Indeed, YILMAZ' theorem may give interesting results in such cases, and then the above question could force a different interpretation of the role of the φ field from that favored by YILMAZ.

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