The Problem of Evolution in General Relativity

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Communicated by V. HLAVATÝ

Summary. It is shown that the problem of evolution in general relativity need have no unique solution determined by continuity alone if matter is present as a perfect fluid. This is demonstrated by two examples. One of these is a model universe, in agreement with the cosmological observations, which arises by a smooth transition from flat space-time.

1. Introduction. Existence theorems for the problem of evolution in general relativity have been given by FOURÈS-BRUHAT [1]. Her results are, roughly, the following. Corresponding to given Cauchy data on a space-like hypersurface the field equations

$$(1.1) \quad R_{ik} = 0$$

have one and only one solution. The theorem is subject to the Cauchy data and the metric tensor satisfying certain differentiability conditions. A similar theorem applies for the field equations

$$(1.2) \quad R_{ik} - \frac{1}{2}g_{ik}R = -8\pi\rho u_i u_k ,$$

that is to say, if “pure matter” is present.

A different way of stating these results is as follows. Given for $t \leq 0$ a metric tensor g_{ik} satisfying (1.1) (or (1.2)), then g_{ik} has a unique continuation satisfying (1.1) (or (1.2)) for $t > 0$, provided appropriate continuity conditions are satisfied on $t = 0$.

It will be proved in this paper that these results cannot in general be extended to the case where matter is present in the form of perfect fluid. The method will be to show by an example (in §2) that if the energy tensor corresponds to a perfect fluid, one can obtain for $t > 0$ a metric (with non-zero pressure and density) continuous at $t = 0$ with flat space-time. Flat space-time is a solution corresponding to a perfect fluid with zero pressure and density, and one continuation of it is, of course, flat space-time itself. It therefore follows that the problem of evolution is not uniquely determined by continuity alone in this case.

The metric so obtained is a cosmological model satisfying the known observational data, and representing a universe created out of flat space-time in the neighbourhood of $t = 0$. For a short period in its history, the model has a negative pressure, and is therefore open to some objection on physical grounds. For this reason I give in §3 another model, free of negative pressures, exhibiting a similar indeterminacy in its evolution.

2. A model universe created from flat space-time. Consider the metric

$$(2.1) \quad ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 \quad (t \leq 0),$$

$$(2.2) \quad ds^2 = -e^{g(t)}(dx^2 + dy^2 + dz^2) + dt^2 \quad (t > 0),$$

where

$$(2.3) \quad g(t) = \frac{4}{3} \log [1 + at^m e^{-k/t}],$$

m , a and k being real constants (a , $k > 0$). Then, as t tends to zero from above, we have

$$e^g \rightarrow 1, \quad \frac{d^n g}{dt^n} \rightarrow 0 \quad (\text{all } n).$$

It follows that the metric tensor, given by (2.1) and (2.2), and all its derivatives are continuous at $t = 0$. We may therefore assume that this tensor satisfies any requirements of continuity which one might wish to impose when discussing the problem of evolution.

We now allow the above metric to determine an energy tensor T_{ik} according to the equations

$$R_{ik} - \frac{1}{2}g_{ik}R = -8\pi T_{ik}.$$

For $t \leq 0$, T_{ik} is of course zero, and for $t > 0$ it has the form corresponding to a perfect fluid, as may be verified by direct calculation. (TOLMAN, [5], §98, formulae (98.6)). The proper pressure p and the proper density ρ are given by

$$(2.4) \quad 8\pi p = -\ddot{g} - \frac{3}{4}\dot{g}^2,$$

$$(2.5) \quad 8\pi\rho = \frac{3}{4}\dot{g}^2,$$

where $\dot{}$ indicates differentiation with respect to t , and as p , ρ and all their time derivatives are evidently continuous at $t = 0$, these quantities too satisfy any requirements of continuity. Since, as remarked in §1, one possible continuation for $t > 0$ of (2.1) is simply flat space-time itself, this example shows that *if the energy tensor is allowed to assume the form for a perfect fluid, the problem of evolution need have no unique solution determined by continuity alone.*

Turning now to the physical meaning of the metric, we first calculate p and ρ for $t > 0$, using (2.2), (2.3), (2.4) and (2.5), and find

$$(2.6) \quad 8\pi p = -\frac{4}{3}e^{-\frac{3}{2}g}ae^{-k/t}t^{m-4}[m(m-1)t^2 + 2k(m-1)t + k^2],$$

$$(2.7) \quad 8\pi\rho = \frac{4}{3}e^{-\frac{3}{2}g}a^2e^{-2k/t}t^{2m-4}(mt + k)^2.$$

The density is always positive, but the pressure is negative for small t . This is not surprising, for it is known from the work of MCCREA [2] and McVITTIE [3] that a negative pressure in relativistic cosmological models results in the creation of matter. What is perhaps unexpected is that the period during which the pressure is negative can be made arbitrarily small by appropriate choice of k . This can be seen from (2.6) which shows that p changes sign at time

$$t_1 = \frac{k}{m} [(1 - m)^{-\frac{1}{2}} - 1];$$

now if $0 < m < 1$, t_1 is positive and can be made arbitrarily small by choosing k small enough, irrespective of the value of m within this range. Further, we see from (2.6) that if m satisfies this inequality, then

$$p < 0 \quad \text{for } t < t_1 \quad \text{and} \quad p > 0 \quad \text{for } t > t_1.$$

Thus the period of negative pressure can be made as small as we please.

For very large values of t , *i.e.* for

$$t \gg 1 \gg k,$$

$e^{-k/t}$ is very nearly 1. If also we put

$$m = 1 - \epsilon,$$

we have for large t

$$(2.8) \quad \begin{cases} e^{\sigma} \sim (1 + at^{1-\epsilon})^{4/3}, \\ 8\pi p \sim \frac{4}{3}(1 + at^{1-\epsilon})^{-1}a\epsilon(1 - \epsilon)t^{-1-\epsilon}, \\ 8\pi\rho \sim \frac{4}{3}(1 + at^{1-\epsilon})^{-2}a^2(1 - \epsilon)^2t^{-2\epsilon}. \end{cases}$$

Let us compare this model with that of Einstein and de Sitter (TOLMAN, [5], page 415) for which

$$(2.9) \quad \begin{cases} e^{\sigma} = (1 + at)^{4/3}, \\ 8\pi p = 0, \\ 8\pi\rho = \frac{4}{3}a^2(1 + at)^{-2}. \end{cases}$$

The Einstein-de Sitter model is in agreement with the observations of cosmology.¹ Now by choosing ϵ sufficiently small (2.8) can for any one given value of t be made to approximate as closely as we wish to (2.9). Since the cosmological observations are all made at one value of t , namely the present, it follows that

¹ The agreement is, in fact, no more than passable. In the homogeneous cosmological models one may define an acceleration parameter equal to $-q_0H^2$ where H is Hubble's constant and q_0 is another constant determined by observation which seems to lie between 1.5 and 5.5 (McVITTIE, [4]). The Einstein-de Sitter model requires $q_0 = \frac{1}{2}$, and the density of matter predicted by it is also rather high. However, in view of the uncertainties in the present observations it seems reasonable to give the model the benefit of the doubt for the time being.

a sufficiently small choice of ϵ will make (2.8) agree with the observations. Hence we conclude that the model represented by (2.1) and (2.2) can be made to agree with observation by appropriate choice of constants. Thus *we have constructed a model of the universe, in agreement with observation, which arises by a smooth transition from flat space-time.*

3. A further example. It might perhaps be thought that the problem of evolution for a perfect fluid would be uniquely determined if the pressure were required to be positive throughout. This is not so, however, as will now be demonstrated by constructing a model showing an indeterminacy in its evolution and satisfying

$$(3.1) \quad p > 0.$$

We shall impose also the requirement

$$(3.2) \quad p \leq \frac{1}{3}\rho;$$

this is done for physical reasons, since all known matter satisfies it.

First let us take a metric

$$(3.3) \quad ds^2 = -e^{g(t)}(dx^2 + dy^2 + dz^2) + dt^2,$$

so that p and ρ are given by (2.4) and (2.5), and choose g so that there exist constants α and β such that

$$(3.4) \quad 8\pi p > \alpha^2 > 0,$$

$$(3.5) \quad 8\pi(\frac{1}{3}\rho - p) > \beta^2 > 0,$$

throughout the history of the model from its singular state (if any) until some given time $T(> 1)$. We shall also arrange that any singular state at, say, $t = t_0$ occurs before $t = 0$.

From (2.4) and (2.5) it follows that conditions (3.4) and (3.5) are equivalent to

$$(3.6) \quad -(\ddot{g} + \frac{3}{4}\dot{g}^2) > \alpha^2 > 0,$$

$$(3.7) \quad \ddot{g} + \dot{g}^2 > \beta^2 > 0.$$

Examples of models satisfying the conditions are those given by

$$e^g = (1 + at)^n,$$

where

$$(3.8) \quad a > 0, \quad 1 < n < \frac{4}{3}.$$

For these models we have

$$\begin{aligned} -(\ddot{g} + \frac{3}{4}\dot{g}^2) &= n(1 - \frac{3}{4}n)a^2(1 + at)^{-2} > n(1 - \frac{3}{4}n)a^2(1 + aT)^{-2}, \\ \ddot{g} + \dot{g}^2 &= n(n - 1)a^2(1 + at)^{-2} > n(n - 1)a^2(1 + aT)^{-2}; \end{aligned}$$

so, clearly, if n satisfies (3.8), α and β exist such that (3.6) and (3.7) are fulfilled. Further, the singular state (at $t = -a^{-1}$) occurs before $t = 0$.

The model exhibiting the indeterminacy in evolution is given by

$$(3.9) \quad \begin{cases} ds^2 = -e^\sigma(dx^2 + dy^2 + dz^2) + dt^2 & \text{for } t_0 < t \leq 0 \text{ and for } 1 \leq t \leq T \\ ds^2 = -e^\sigma(1 + bf)(dx^2 + dy^2 + dz^2) + dt^2 & \text{for } 0 < t < 1, \end{cases}$$

where e^σ is the function appearing in (3.3), where

$$(3.10) \quad f = \exp \{-kt^{-1} + k(t-1)^{-1}\},$$

and where b and k are constants ($k > 0$). Let p^* , ρ^* denote the pressure and density in the model (3.9); then the metric tensor, p^* , ρ^* and all their derivatives are continuous throughout $t_0 < t \leq T$, and p^* and ρ^* satisfy (3.1) and (3.2) outside the range $0 < t < 1$ because of the choice of g . We shall show that they satisfy the conditions also within this range.

Write

$$(3.11) \quad e^{\sigma*} = e^\sigma(1 + bf);$$

then the desired conditions on p^* and ρ^* within $0 < t < 1$, viz.,

$$(3.12) \quad p^* > 0, \quad p^* \leq \frac{1}{3}\rho^*,$$

are equivalent to

$$(3.13) \quad -(\ddot{g}^* + \frac{3}{4}\dot{g}^{*2}) > 0,$$

$$(3.14) \quad \ddot{g}^* + \dot{g}^{*2} \geq 0.$$

Now using (3.10), (3.11), (3.6) and (3.7), we find

$$\begin{aligned} -(\ddot{g}^* + \frac{3}{4}\dot{g}^{*2}) &> \alpha^2 - b(1 + bf)^{-1}[\ddot{f} + \frac{3}{2}\dot{f}\dot{g} - \frac{1}{4}bf^2(1 + bf)^{-1}], \\ \ddot{g}^* + \dot{g}^{*2} &> \beta^2 + b(1 + bf)^{-1}[\ddot{f} + 2\dot{f}\dot{g}]. \end{aligned}$$

Since α and β are independent of b , and f and its derivatives and also $\dot{g}$ are bounded in $0 < t < 1$, it is clear that b may be chosen so that (3.13) and (3.14), and therefore (3.12), are satisfied. Hence (3.1) and (3.2) are satisfied throughout the period $t_1 < t \leq T$.

Finally, we note that (3.9) specifies a class of models, obtained by varying k , all of which satisfy the continuity conditions, as well as the conditions on the pressure. One of these is, of course, the original metric (3.3), obtained by putting k and b equal to zero. We conclude that *there are models whose evolution is not uniquely determined, even if the physically desirable conditions (3.1) and (3.2) are imposed.*

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