*Intrinsic Form of the Characteristic Relations for a Perfect Compressible Fluid in General Relativity and Non-Steady Newtonian Mechanics**

N. COBURN

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1. Introduction. By use of the relation between the vector along a world-line, the normal to a characteristic surface, and a vector along the bicharacteristic corresponding to this normal, the equations of motion and continuity of an isentropic, perfect, non-self-gravitating fluid in general relativity [1] are obtained in intrinsic form. That is, these equations are written in a form involving the sound speed a , the speed of light c , the derivatives of the sound speed along the normal and bicharacteristic directions, and the derivatives of the normal and bicharacteristic. By introducing two unit vectors, both of which are space-like and both of which are orthogonal to the normal and bicharacteristic at each point, the equations of motion and continuity can be replaced by four equations which involve the above noted derivatives of the sound speed as well as its derivatives in the directions of the two unit vectors, the curvatures of the curves along the normal, bicharacteristic, and the remaining two directions, and the curvature of the characteristic surfaces. *These are the intrinsic forms of the equations of motion and continuity.*

The remainder of the paper, except Section 6, is concerned with the case when the characteristic surfaces form a family of ∞^1 parallel hyperplanes and one of the two unit vectors introduced earlier generates a family of ∞^3 parallel straight lines. The existence of the hyperplanes and straight lines implies that the space considered is one of special relativity. Further, the curvatures of the above congruences can now be expressed in terms of some of the partial derivatives of the metric tensor. These last derivatives can in turn be expressed in terms of the derivatives of one function ψ . Thus the equations of motion

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and continuity furnish four relations for the partial derivatives of the sound speed and the function ψ .

Our next step is to reduce these last equations to Newtonian form. This is done by use of a procedure developed in previous papers [1, 2]. Finally, a class of two-dimensional, non-steady flows is obtained. It is shown that the expressions for velocity and density satisfy the Newtonian equations of motion and continuity. The stream-lines are a family of parabolas at any given time.

Finally, we extend the theory of our previous paper [1] and that of the present paper to self-gravitating fluids. In fact, we shall show that the characteristic relation (2.18) of our previous paper remains valid for non-isentropic, self-gravitating fluids. Also, the theory of Sections 1-7 of that paper remains valid for non-isentropic, self-gravitating fluids, provided that the conservation of mass equation is expressed in terms of characteristic variables. Furthermore, the theory of the present paper for intrinsic forms of the characteristic relations is valid for isentropic, self-gravitating fluids which satisfy a degenerate equation of state.

2. Intrinsic Form of the Basic Relations. Let $x^i, j = 1, 2, 3, 4$, denote a curvilinear coordinate system in the four-space of general relativity. Further, we introduce the metric tensor g^{ik} , with contravariant components and λ^i , the components of a unit time-like vector which is tangent to a world line. The covariant derivative will be denoted by ∇_i . Finally, we introduce the scalars

$$(2.1) \quad \begin{aligned} \rho &= \text{the density,} \\ p &= \text{the pressure,} \\ a &= \text{the local sound speed,} \\ c &= \text{the speed of light,} \end{aligned}$$

where $c^2 > 3a^2$, $a > 0$, and the scalars

$$(2.2) \quad B^2 = \frac{c^2 - a^2}{c^2 - 2a^2}, \quad C^2 = \frac{a^2}{c^2 - 2a^2}.$$

The equations of motion and continuity of a perfect compressible fluid are [1]

$$(2.3) \quad \nabla_i (c^2 \rho \lambda^i \lambda^k - p g^{ik}) = 0.$$

In the isentropic polytropic case, we find

$$(2.4) \quad p = \frac{K^*}{\gamma} \rho^\gamma$$

where K^*, γ are constants. Since, by definition,

$$(2.5) \quad a^2 = \frac{dp}{d\rho},$$

Equation (2.3) becomes

$$(2.6) \quad c^2(\lambda^i \lambda^k \nabla_i \rho + \rho \lambda^k \nabla_i \lambda^i + \rho \lambda^i \nabla_i \lambda^k) - a^2 g^{ik} \nabla_i \rho = 0.$$

From our previous results [1], we may write

$$(2.7) \quad \lambda^i = B t^i + C n^i$$

where n^i is any unit space-like vector of the normal cone and t_i is the corresponding unit time-like vector of the bicharacteristic cone. In order to write (2.6) in simple form, we introduce the dependent variables θ, b where

$$(2.8) \quad \theta = (\gamma - 1) \ln \rho, \quad b = \frac{a^2}{c^2}.$$

From (2.4), we find by differentiation and use of (2.5), (2.8)

$$(2.9) \quad \theta = \ln \left(\frac{b}{b_0} \right)$$

where b_0 is a constant. By differentiating (2.9), we obtain

$$(2.10) \quad \frac{db}{d\theta} = b.$$

Further, from (2.2) and (2.8), we see that

$$(2.11) \quad B^2 = \frac{1-b}{1-2b}, \quad C^2 = \frac{b}{1-2b}.$$

By substituting (2.7) into (2.6) and using (2.8), (2.9), (2.10), we find

$$(2.12) \quad \begin{aligned} & \left[(B^2 t^k + BC n^k) \frac{\partial b}{\partial r} + (BC t^k + C^2 n^k) \frac{\partial b}{\partial n} - b g^{ik} \frac{\partial b}{\partial x^i} \right] \\ & + (\gamma - 1) b \left[B^2 (t^k \nabla_i t^i + t^i \nabla_i t^k) + BC (t^k \nabla_i n^i + n^k \nabla_i t^i \right. \\ & + t^i \nabla_i n^k + n^i \nabla_i t^k) + C^2 (n^k \nabla_i n^i + n^i \nabla_i n^k) \\ & \left. + t^k \frac{\partial B^2}{\partial r} + n^k \frac{\partial C^2}{\partial n} + t^k \frac{\partial}{\partial n} (BC) + n^k \frac{\partial}{\partial r} (BC) \right] = 0 \end{aligned}$$

where

$$(2.13) \quad \frac{\partial}{\partial r} \equiv t^i \nabla_i, \quad \frac{\partial}{\partial n} \equiv n^i \nabla_i.$$

In order to analyze (2.12), we introduce the two unit orthogonal vector fields $p^i, 'p^i$, both of which are orthogonal to both t^i, n^i . The orientation of $p^i, 'p^i, t^i, n^i$ will be specified by requiring that, in the local three-dimensional space orthogonal to n^i , the ordered triad $t^i, p^i, 'p^i$ forms a right hand system. From the mutual orthogonality of $p^i, 'p^i, n^i, t^i$, we obtain

$$(2.14) \quad t^i n_i = p^i 'p_i = p^i n_i = p^i t_i = 'p^i n_i = 'p^i t_i = 0.$$

Further, since t^i is a unit time-like vector, n^i is a unit space-like vector, and p^i , $'p^i$ are unit vectors, we may write

$$(2.15) \quad t^i t_i = 1, \quad n^i n_i = -1, \quad p^i p_i = \sigma, \quad 'p^i 'p_i = ' \sigma$$

where σ , $'\sigma$ are $+1$ or -1 , according as p^i , $'p^i$ are time-like or space-like. Every orthogonal 4-tuple (four orthogonal vectors) in the space-time of general or special relativity consists of three space-like vectors and one time-like vector [3]. Hence both σ and $'\sigma$ are -1 , or

$$(2.16) \quad p^i p_i = 'p^i 'p_i = -1.$$

If we form the scalar product of (2.12) with t^i , n^i , p^i , $'p^i$ and use (2.14), (2.15), we obtain

$$(2.17) \quad \left[B^2 \frac{\partial b}{\partial r} + BC \frac{\partial b}{\partial n} - b \frac{\partial b}{\partial r} \right] + (\gamma - 1)b \left[B^2 \nabla_i t^i \right. \\ \left. + BC(\nabla_i n^i + t^k t^i \nabla_i n_k) + C^2 t^k n^i \nabla_i n_k + \frac{\partial B^2}{\partial r} + \frac{\partial}{\partial n}(BC) \right] = 0,$$

$$(2.18) \quad \left[BC \frac{\partial b}{\partial r} + C^2 \frac{\partial b}{\partial n} + b \frac{\partial b}{\partial n} \right] + (\gamma - 1)b \left[-B^2 n^k t^i \nabla_i t_k \right. \\ \left. + BC(\nabla_i t^i - n^k n^i \nabla_i t_k) + C^2 \nabla_i n^i + \frac{\partial C^2}{\partial n} + \frac{\partial(BC)}{\partial r} \right] = 0,$$

$$(2.19) \quad -\frac{\partial b}{\partial p} + (\gamma - 1)[BC(p^k t^i \nabla_i n_k + p^k n^i \nabla_i t_k) \\ + B^2 p^k t^i \nabla_i t_k + C^2 p^k n^i \nabla_i n_k] = 0,$$

$$(2.20) \quad -\frac{\partial b}{\partial s} + (\gamma - 1)[BC('p^k t^i \nabla_i n_k + 'p^k n^i \nabla_i t_k) \\ + B^2 'p^k t^i \nabla_i t_k + C^2 'p^k n^i \nabla_i n_k] = 0$$

where

$$(2.21) \quad \frac{\partial}{\partial p} \equiv p^i \nabla_i, \quad \frac{\partial}{\partial s} \equiv 'p^i \nabla_i.$$

Now we consider the curvature terms in (2.17)–(2.20). Since n^i is orthogonal to ∞^1 three-dimensional characteristic surfaces, it follows that [4]

$$(2.22) \quad \nabla_i n_k = n_i u_k + h_{ik}$$

where u_k is the curvature vector of the n_k congruence and h_{ik} is the second fundamental tensor of the characteristic surfaces. Further, the mean curvature M^* of these surfaces is

$$(2.23) \quad M^* = \nabla_i n^i = g^{ik} h_{ik}$$

where g^{ik} is the metric tensor. This tensor may be decomposed into

$$(2.24) \quad g^{ik} = t^i t^k - n^i n^k - p^i p^k - 'p^i 'p^k.$$

Now we introduce the curvature vectors w_i , m_i , z_i of the p_i , t_i , $'p_i$ congruences, respectively:

$$(2.25) \quad w_i = p^k \nabla_k p_i, \quad m_i = t^k \nabla_k t_i, \quad z_i = {}'p^k \nabla_k {}'p_i.$$

It is easily seen that

$$(2.26) \quad \begin{aligned} t^i t^k \nabla_i t_k &= 0, & n^i n^k \nabla_i t_k &= -t^k u_k, \\ p^i p^k \nabla_i t_k &= -t^k w_k, & {}'p^i {}'p^k \nabla_i t_k &= -t^k z_k. \end{aligned}$$

Hence, by use of (2.24), (2.26), the curvature term $\nabla_i t^i = g^{ik} \nabla_i t_k$ becomes

$$(2.27) \quad \nabla_i t^i = t^k (u_k + w_k + z_k).$$

By use of (2.22), (2.23), (2.27), we can evaluate all the curvature terms of (2.17), (2.18). In addition, Equations (2.19), (2.20) contain the curvature terms

$$(2.28) \quad K = p^k (t^i \nabla_i n_k + n^i \nabla_i t_k) = -(t^i n^k + n^i t^k) \nabla_i p_k,$$

$$(2.29) \quad {}'K = {}'p^k (t^i \nabla_i n_k + n^i \nabla_i t_k) = -(t^i n^k + n^i t^k) \nabla_i {}'p_k.$$

First, we express the derivatives of b , B^2 , BC , C^2 in (2.17)–(2.20) in terms of b , $\ln a^2$. By use of (2.8), (2.11), we find

$$(2.30) \quad \frac{\partial b}{\partial r} = b \frac{\partial \ln a^2}{\partial r}, \quad \frac{\partial b}{\partial n} = b \frac{\partial \ln a^2}{\partial n}, \quad \text{etc.},$$

$$(2.31) \quad \frac{\partial B^2}{\partial r} = \frac{b}{(1-2b)^2} \frac{\partial \ln b}{\partial r},$$

$$(2.32) \quad \frac{\partial(BC)}{\partial r} = \frac{b}{2(1-2b)^2(b-b^2)^{\frac{1}{2}}} \frac{\partial \ln b}{\partial r},$$

$$(2.33) \quad \frac{\partial C^2}{\partial r} = \frac{b}{(1-2b)^2} \frac{\partial \ln b}{\partial r}, \quad \text{etc.}$$

By substituting (2.22)–(2.33) into (2.17)–(2.20), we obtain the *intrinsic form of the characteristic relations for an isentropic, polytropic gas in general relativity*:

$$(2.34) \quad \begin{aligned} & [1 + (\gamma - 5)b + 6b^2 - 4b^3] \frac{\partial}{\partial r} \ln b \\ & + \frac{b}{2} [b(1-b)]^{-\frac{1}{2}} [\gamma + 1 - 6b + 4b^2] \frac{\partial}{\partial n} \ln b \\ & + (\gamma - 1)(1-2b)[t^k u_k + (1-b)t^k(w_k + z_k) \\ & + b^{\frac{1}{2}}(1-b)^{\frac{1}{2}}(M^* - m^i n_i)] = 0, \end{aligned}$$

$$(2.35) \quad \begin{aligned} & b[\gamma + 1 - 6b + 4b^2] \frac{\partial}{\partial r} \ln b + 2b[\gamma + 1 \\ & - 6b + 4b^2][b(1-b)]^{\frac{1}{2}} \frac{\partial}{\partial n} \ln b \\ & + 2(\gamma - 1)(1-2b)[b(1-b)]^{\frac{1}{2}}[(b-1)m_i n^i \\ & + b^{\frac{1}{2}}(1-b)^{\frac{1}{2}}(2t^k u_k + t^k w_k + t^k z_k) + b g^{ik} h_{ik}] = 0, \end{aligned}$$

$$(2.36) \quad -b(1-2b) \frac{\partial}{\partial p} \ln b + (\gamma-1)[(b(1-b))^{\frac{1}{2}} K + (1-b)m_i p^i + bu_i p^i] = 0,$$

$$(2.37) \quad -b(1-2b) \frac{\partial}{\partial s} \ln b + (\gamma-1)[(b(1-b))^{\frac{1}{2}} 'K + (1-b)m_i 'p^i + bu_i 'p^i] = 0,$$

3. A Class of Coordinate Systems in the Space-Time of Special Relativity Associated with a Special Class of Flows. We shall consider those flows which possess one family of ∞^1 *plane parallel hyperplanes* as characteristic hyper-surfaces. Thus n_i is a *parallel vector field* and (see 2.22)

$$(3.1) \quad u_i = 0, \quad h_{ik} = 0, \quad M^* = 0.$$

We have assumed that the ordered triad $t_i, p_i, 'p_i$ (where t_i is time-like and $p_i, 'p_i$ are space-like) forms a right hand system. Let us assume that $'p_i$ *consists of parallel straight lines* and is a *parallel vector field*. Thus, from (2.25), we find

$$(3.2) \quad z_i = 0.$$

Now we introduce Cartesian orthogonal coordinates of special relativity x, y, z, t so that $'p_i$ lies along $y = \text{variable}$ and n_i lies along $z = \text{variable}$. In any plane orthogonal to $'p_i, n_i$, we introduce two families of parameter curves: $\alpha = \text{variable}$ along $t_i, \epsilon = \text{variable}$ along p_i . Further, we introduce the parameters δ, μ where

$$(3.3) \quad z = \delta, \quad y = \mu.$$

In the four-space of special relativity, the arc-length element is

$$(3.4) \quad \sigma ds^2 = (c dt)^2 - (dx^2 + dy^2 + dz^2).$$

Hence, in any two-dimensional plane $\delta = \text{constant}, \mu = \text{constant}$, the arc-length element is

$$(3.5) \quad \sigma ds^2 = (A d\alpha)^2 - (E d\epsilon)^2,$$

where

$$(3.6) \quad A^2 = \left(\frac{c \partial t}{\partial \alpha} \right)^2 - \left(\frac{\partial x}{\partial \alpha} \right)^2,$$

$$(3.7) \quad E^2 = - \left(\frac{c \partial t}{\partial \epsilon} \right)^2 + \left(\frac{\partial x}{\partial \epsilon} \right)^2.$$

Note that σ is $+1$ for time-like displacements and -1 for space-like displacements. Further, t_i (along $\alpha = \text{variable}$) is orthogonal to p_i (along $\epsilon = \text{variable}$) or

$$(3.8) \quad 0 = c^2 \frac{\partial t}{\partial \alpha} \frac{\partial t}{\partial \epsilon} - \frac{\partial x}{\partial \alpha} \frac{\partial x}{\partial \epsilon}.$$

If ψ, ϕ are two variables which are functions of $\alpha, \epsilon, \delta, \mu$, then we can replace (3.6), (3.7) by

$$(3.9) \quad \frac{c}{\partial \alpha} \frac{\partial t}{\partial \alpha} = A \cosh \psi, \quad \frac{\partial x}{\partial \alpha} = A \sinh \psi,$$

$$(3.10) \quad \frac{c}{\partial \epsilon} \frac{\partial t}{\partial \epsilon} = E \sinh \phi, \quad \frac{\partial x}{\partial \epsilon} = E \cosh \phi.$$

From (3.8), we see that

$$(3.11) \quad \psi = \phi.$$

Thus the components of the unit vectors t^i, p^i in the x, y, z, ct coordinates are

$$(3.12) \quad \begin{aligned} t^i &: (\sinh \psi, 0, 0, \cosh \psi), \\ p^i &: (\cosh \psi, 0, 0, \sinh \psi). \end{aligned}$$

Note that the metric tensor in the x, y, z, ct system is (where $x^1 = x, x^2 = y, x^3 = z, x^4 = ct$)

$$(3.13) \quad g_{44} = 1 \quad g_{11} = g_{22} = g_{33} = -1, \quad g_{ii} = 0, \quad i \neq j.$$

From (3.12), (3.13), we see that t^i is a unit time-like vector and p^i is a unit space-like vector. Further, the components of $'p^i, n^i$ are

$$(3.14) \quad \begin{aligned} 'p^i &: (0, 1, 0, 0), \\ n^i &: (0, 0, 1, 0). \end{aligned}$$

We have assumed that the ordered triple $t^i, p^i, 'p^i$ forms a right hand set in any three-dimensional hyperplane ($z = \text{constant}$) orthogonal to n^i . Further, the $\alpha = \text{variable}$ curves (with tangent vector t^i) and the $\epsilon = \text{variable}$ curves (with tangent vector p^i) lie in the two-dimensional planes $z = \text{constant}, y = \text{constant}$, which are orthogonally to $n^i, 'p^i$. We assume that the sense of the principal normal to these curves is such that the ordered triple consisting of the tangent vector, the principal normal vector, and the vector $'p^i$ form a right hand system. The curvature may be positive or negative. By use of the Frenet formulas for plane curves in the x, t plane which possesses an indefinite metric [5] and the fact that the principal normals of the p_k, t_k congruences lie along $t_k, -p_k$, respectively, we find (see 2.25)

$$(3.15) \quad \begin{aligned} p^i \nabla_i p_k &= w_k = -'k t_k, \\ t^i \nabla_i t_k &= m_k = -k p_k, \end{aligned}$$

where $'k$ and k are the curvatures of the curves $\epsilon = \text{variable}$ and $\alpha = \text{variable}$, respectively. From (3.12), (3.13), we find

$$(3.16) \quad \begin{aligned} t_i &: (-\sinh \psi, 0, 0, \cosh \psi), \\ p_i &: (-\cosh \psi, 0, 0, \sinh \psi). \end{aligned}$$

By use of (3.5), we see that

$$(3.17) \quad p^i \nabla_i p_k = \frac{1}{E} \frac{\partial p_k}{\partial \epsilon}, \quad t^i \nabla_i t_k = \frac{1}{A} \frac{\partial t_k}{\partial \alpha}.$$

Forming the scalar product of the first equation of (3.15) with t^k , we find by use of (3.17), the first equation of (3.12), and the second equation of (3.16)

$$(3.18) \quad -'k = w_k t^k = \frac{1}{E} \frac{\partial \psi}{\partial \epsilon}.$$

Similarly, by forming the scalar product of the second equation of (3.15) with p^k , we find

$$(3.19) \quad k = m_k p^k = -\frac{1}{A} \frac{\partial \psi}{\partial \alpha}.$$

The integrability conditions of (3.9), (3.10) are

$$(3.20) \quad \frac{\partial \psi}{\partial \alpha} = \frac{1}{E} \frac{\partial A}{\partial \epsilon}, \quad \frac{\partial \psi}{\partial \epsilon} = \frac{1}{A} \frac{\partial E}{\partial \alpha}.$$

Substituting (3.20) into (3.18), (3.19), we obtain

$$(3.21) \quad w_k t^k = -'k = \frac{1}{AE} \frac{\partial E}{\partial \alpha},$$

$$m_k p^k = k = -\frac{1}{AE} \frac{\partial A}{\partial \epsilon}.$$

In our future work, we shall consider the $\alpha, \epsilon, \delta, \mu$ coordinate system where

$$z = \delta, \quad y = \mu$$

and α, ϵ are the coordinate variables along t^i, p^i , respectively, in any x, t coordinate plane. This implies that the curves along t^i, p^i are coincident in all planes $z = \text{constant}, y = \text{constant}$.

Now we determine (2.34)–(2.37) for the present case when the characteristic surfaces are ∞^1 parallel hyperplanes $z = \text{constant}$. We note that t^i lies along $\alpha = \text{variable}$, p^i lies along $\epsilon = \text{variable}$, n^i lies along $z = \delta = \text{variable}$, $'p^i$ lies along $y = \mu = \text{variable}$. Thus, by use of (3.5), (3.16), we find

$$(3.22) \quad \frac{\partial}{\partial r} = \frac{1}{A} \frac{\partial}{\partial \alpha}, \quad \frac{\partial}{\partial p} = \frac{1}{E} \frac{\partial}{\partial \epsilon},$$

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial z} = \frac{\partial}{\partial \delta}, \quad \frac{\partial}{\partial s} = \frac{\partial}{\partial y} = \frac{\partial}{\partial \mu}.$$

If we substitute (2.28), (2.29), (3.1), (3.2), (3.12)–(3.15), (3.18), (3.19) into (2.34)–(2.37), the latter become

$$(3.23) \quad [1 + (\gamma - 5)b + 6b^2 - 4b^3] \frac{1}{A} \frac{\partial \ln b}{\partial \alpha} + \frac{b}{2} [b(1 - b)]^{-\frac{1}{2}}$$

$$\cdot [\gamma + 1 - 6b + 4b^2] \frac{\partial \ln b}{\partial \delta} + (\gamma - 1)(1 - 2b)(1 - b) \frac{1}{E} \frac{\partial \psi}{\partial \epsilon} = 0,$$

$$(3.24) \quad [\gamma + 1 - 6b + 4b^2] \frac{1}{A} \frac{\partial \ln b}{\partial \alpha} + 2[\gamma + 1 - 6b + 4b^2][b(1 - b)]^{\frac{1}{2}} \cdot \frac{\partial \ln b}{\partial \delta} + 2(\gamma - 1)(1 - 2b)(1 - b) \frac{1}{E} \frac{\partial \psi}{\partial \epsilon} = 0,$$

$$(3.25) \quad b(1 - 2b) \frac{1}{E} \frac{\partial \ln b}{\partial \epsilon} + (\gamma - 1)[b(1 - b)]^{\frac{1}{2}} \frac{\partial \psi}{\partial \delta} + (\gamma - 1)(1 - b) \frac{1}{A} \frac{\partial \psi}{\partial \alpha} = 0,$$

$$(3.26) \quad \frac{\partial \ln b}{\partial \mu} = 0.$$

4. Transition to Newtonian Mechanics. In this section, we discuss two topics: (1) the term “order” of a quantity and (2) the Newtonian analogues of Section 3. Our procedure is similar to that of a previous paper [1], whose general ideas we outline below.

We define “order” of a quantity in terms of the *maximum range* of values through which the quantity may vary. Thus, if the *absolute* value of the quantity Q varies through the possible *range* zero to 10^k (say $k = 5$ in the ft. sec. system of units), then Q is said to be of *order one*; if the absolute value of Q has the permissible range zero to $(10^k/c)^n$, where n is a positive integer, then Q is said to be of *order $1/c^n$* ; if $n \geq 2$, Q is often said to be of order zero (rather than of order $1/c^n$).

We make two basic assumptions. First, in any sum or product involving terms of order one at most, a term of order zero may be replaced by zero. Secondly, we use an order assumption on the normals which replaces the assumption of our previous paper [1] that a/c is of order zero. This order condition is discussed fully in our paper [2] and need not concern us here.

In order to make the transition from the relativistic equations of Section 3 to their Newtonian analogues, we shall need four results of our previous paper [1]. In order to state these results in our present notation, we note that

$$(4.1) \quad x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^4 = ct$$

are the Cartesian orthogonal coordinates of a system of special relativity, with metric tensor (see 3.13)

$$(4.2) \quad g_{11} = g_{22} = g_{33} = -1, \quad g_{44} = 1, \quad g_{ii} = 0, \quad i \neq j, \\ g^{11} = g^{22} = g^{33} = -1, \quad g^{44} = 1, \quad g^{ii} = 0, \quad i \neq j.$$

The Latin indices run from one through four ($i, j = 1, 2, 3, 4$). The metric tensor of the corresponding system of Newtonian coordinates is (the prime indicates a tensor in Newtonian mechanics)

$$(4.3) \quad 'g_{11} = 'g_{22} = 'g_{33} = 1, \quad 'g_{\alpha\beta} = 0, \quad \alpha \neq \beta.$$

The Greek indices run from one through three ($\alpha, \beta = 1, 2, 3$). Thus, if corre-

sponding relativistic and Newtonian quantities are related by an arrow, then

$$(4.4) \quad g_{ij} \rightarrow -'g_{\alpha\beta}, \quad i = \alpha, \quad j = \beta.$$

The four results of our previous paper [1] become, in the coordinate system (4.1),

$$(4.5) \quad \lambda^i \rightarrow \frac{v^\alpha}{c}, \quad \lambda^4 \rightarrow 1, \quad j = \alpha,$$

$$(4.6) \quad \nabla_i \rightarrow '\nabla_\alpha, \quad j = \alpha,$$

$$(4.7) \quad A \rightarrow c \frac{\partial t}{\partial \alpha}, \quad E \rightarrow \frac{\partial x}{\partial \epsilon},$$

$$(4.8) \quad v^\alpha = \frac{dx^\alpha}{dt} - aN^\alpha, \quad T^\alpha = \frac{dx^\alpha}{dt}$$

where N^α , T^α are the Newtonian normal and bicharacteristic vectors. In our previous paper [1], these results are numbered (7.4), (7.13), (7.25), (7.24), (7.23), respectively (or see 3.11, 3.15 of [2]). Note that the vector t^i (along $\alpha = \text{variable}$) is time-like and the vector p^i (along $\epsilon = \text{variable}$) is space-like.

Let us choose the parameters α , ϵ so that

$$\frac{\partial t}{\partial \alpha}, \quad \frac{\partial x}{\partial \epsilon}$$

are of order one. In view of (4.7), we may introduce scalars $\bar{A}$, $\bar{E}$ of order one such that

$$(4.9) \quad A \rightarrow c\bar{A}, \quad E \rightarrow \bar{E}.$$

By use of the fact that $\partial t/\partial \alpha$ is of order one and (4.7), (4.9), we find from (3.9) that

$$(4.10) \quad \frac{\partial t}{\partial \alpha} = \bar{A}, \quad \cosh \psi \rightarrow 1.$$

Let us assume that $\bar{\psi}$ is of order one and

$$(4.11) \quad \sinh \psi \rightarrow \frac{\bar{\psi}}{c}.$$

We note that (4.10) implies

$$\sinh^2 \psi \rightarrow 0$$

which is consistent with (4.11). From (4.9), (4.11), and the second relation of (3.9), we find

$$(4.12) \quad \frac{\partial x}{\partial \alpha} = \bar{\psi}\bar{A}.$$

Further, (3.10) becomes (neglecting terms of order zero)

$$(4.13) \quad \frac{\partial t}{\partial \epsilon} = 0, \quad \frac{\partial x}{\partial \epsilon} = \bar{E}.$$

The Newtonian analogues of the integrability conditions for the α , ϵ , δ , μ variables to form a coordinate system are

$$(4.14) \quad \frac{\partial \bar{A}}{\partial \epsilon} = 0, \quad \frac{\partial \bar{\psi}}{\partial \epsilon} = \frac{1}{\bar{A}} \frac{\partial \bar{E}}{\partial \alpha}.$$

These equations follow directly from (4.10), (4.12), (4.13) or by multiplying both equations of (3.20) by $\cosh \psi$ and using (4.9), (4.11), and the second relation of (4.10).

If we replace A by $c\bar{A}$, E by $\bar{E}$, ψ by $\bar{\psi}/c$ in (3.23)–(3.26), and note that $b = a^2/c^2$ is of order zero (see (2.8)), then (3.23)–(3.26) become

$$(4.15) \quad \frac{2}{\bar{A}} \frac{\partial a}{\partial \alpha} + (\gamma + 1)a \frac{\partial a}{\partial \delta} + \frac{(\gamma - 1)a}{\bar{E}} \frac{\partial \bar{\psi}}{\partial \epsilon} = 0,$$

$$(4.16) \quad \frac{(\gamma + 1)}{\bar{A}} \frac{\partial a}{\partial \alpha} + 2(\gamma + 1)a \frac{\partial a}{\partial \delta} + \frac{(\gamma - 1)a}{\bar{E}} \frac{\partial \bar{\psi}}{\partial \epsilon} = 0,$$

$$(4.17) \quad \frac{2}{\bar{E}} \frac{\partial a}{\partial \epsilon} + (\gamma - 1) \frac{\partial \bar{\psi}}{\partial \delta} + \frac{(\gamma - 1)}{a\bar{A}} \frac{\partial \bar{\psi}}{\partial \alpha} = 0,$$

$$(4.18) \quad \frac{\partial a}{\partial \mu} = 0.$$

The equations (4.15), (4.16) can be replaced by the following pair:

$$(4.19) \quad \frac{\partial a}{\partial \delta} = \frac{(1 - \gamma)}{(\gamma + 1)a\bar{A}} \frac{\partial a}{\partial \alpha},$$

$$(4.20) \quad \frac{3 - \gamma}{a\bar{A}} \frac{\partial a}{\partial \alpha} + \frac{\gamma - 1}{\bar{E}} \frac{\partial \bar{\psi}}{\partial \epsilon} = 0.$$

By replacing $\partial \bar{\psi}/\partial \epsilon$ by $\partial \bar{E}/\partial \alpha$ through use of the second equation of (4.14), we find that (4.20) reduces to

$$(4.21) \quad \frac{3 - \gamma}{a} \frac{\partial a}{\partial \alpha} + \frac{\gamma - 1}{\bar{E}} \frac{\partial \bar{E}}{\partial \alpha} = 0.$$

The integral of (4.21) is

$$(4.22) \quad a^{3-\gamma} \bar{E}^{\gamma-1} = f(\epsilon, \mu, \delta)$$

where f is an arbitrary differentiable function of ϵ, μ, δ . *The equations (4.17), (4.18), (4.19), (4.21) are the intrinsic forms of the equations of motion and continuity for the non-steady motion of a compressible isentropic fluid in Newtonian mechanics possessing ∞^1 parallel hyperplanes as relativistic characteristic surfaces.* In order to determine v^λ , Equation (4.8) must be used. In the present case, the vector N^λ can be obtained from (3.14). Similarly T^λ can be found from (3.12).

5. A Special Class of Non-Steady, Two-Dimensional Flows. We seek solutions of (4.14), (4.17), (4.19), (4.22) of the form

$$(5.1) \quad \begin{aligned} a &= a(\epsilon), & \bar{\psi} &= \bar{\psi}(\alpha), \\ \bar{E} &= \bar{E}(\epsilon), & \bar{A} &= \bar{A}(\alpha). \end{aligned}$$

Thus only (4.17) must be satisfied. This equation leads to

$$(5.2) \quad \frac{da^2}{d\epsilon} = k\bar{E}, \quad \frac{d\bar{\psi}}{d\alpha} = \frac{k\bar{A}}{(1-\gamma)}$$

where k is an arbitrary constant. Hence $\bar{E}$, $\bar{A}$ can be chosen arbitrarily and (5.2) determines a and $\bar{\psi}$. Then, from (4.10), (4.12), (4.13), we find

$$(5.3) \quad t = \int^{\alpha} \bar{A} d\alpha,$$

$$(5.4) \quad x = \int^{\alpha} \bar{\psi} \bar{A} d\alpha + \int^{\epsilon} \bar{E} d\epsilon.$$

By use of (5.2), the relation (5.3) leads to

$$(5.5) \quad t = \frac{(1-\gamma)}{k} \bar{\psi}.$$

Integrating the first term of the right hand side of (5.4), we find by use of the second equation of (5.2)

$$(5.6) \quad \int^{\alpha} \bar{\psi} \bar{A} d\alpha = \frac{(1-\gamma)}{2k} (\bar{\psi})^2.$$

By use of (5.2), (5.5), (5.6), the equation (5.4) furnishes the relation for a^2 :

$$(5.7) \quad a^2 = kx - \frac{k^2 t^2}{2(1-\gamma)}.$$

Since the bicharacteristics are the curves $\alpha = \text{variable}$ (or ϵ, δ, μ all constants), from (5.3)–(5.5), (3.3) we see that the components of the vector T^α (which lies along $\alpha = \text{variable}$) of (4.8) are

$$(5.8) \quad \frac{dx}{dt} = \frac{kt}{1-\gamma}, \quad \frac{dy}{dt} = \frac{dz}{dt} = 0.$$

Again, the characteristic hypersurfaces are orthogonal to the z -direction. Hence the space-normalized components of the normal vector are

$$(5.9) \quad N_\mu : (0, 0, 1).$$

Substituting (5.7)–(5.9) into (4.8), we obtain for the components of the velocity vector

$$(5.10) \quad v^1 = \frac{kt}{1-\gamma}, \quad v^2 = 0, \quad v^3 = -\left[kx - \frac{k^2 t^2}{2(1-\gamma)}\right]^{\frac{1}{2}}.$$

In order to compute the density ρ , we note that the gas obeys the adiabatic law. Hence, by use of (5.7), we find

$$(5.11) \quad a^2 = K^* \rho^{\gamma-1} = kx - \frac{k^2 t^2}{2(1-\gamma)}.$$

By logarithmic differentiation of (5.11), we obtain

$$(5.12) \quad \begin{aligned} \frac{\partial \ln \rho}{\partial x} &= \frac{k}{\gamma-1} \left[kx - \frac{k^2 t^2}{2(1-\gamma)} \right]^{-1}, & \frac{\partial \ln \rho}{\partial y} &= \frac{\partial \ln \rho}{\partial z} = 0, \\ \frac{\partial \ln \rho}{\partial t} &= \frac{k^2 t}{(\gamma-1)^2} \left[kx - \frac{k^2 t^2}{2(1-\gamma)} \right]^{-1}. \end{aligned}$$

Hence the term on the right hand side of the equations of motion,

$$-\frac{1}{\rho} \frac{\partial p}{\partial x^\lambda} = -a^2 \frac{\partial \ln \rho}{\partial x^\lambda} \quad (\lambda = 1, 2, 3),$$

can be evaluated by use of (5.7), (5.12):

$$(5.13) \quad -\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{k}{1-\gamma}, \quad -\frac{1}{\rho} \frac{\partial p}{\partial y} = 0, \quad -\frac{1}{\rho} \frac{\partial p}{\partial z} = 0,$$

Now we shall show that (5.10)–(5.13) satisfy the equations of motion and continuity. The Newtonian equations of motion are

$$(5.14) \quad \frac{\partial v_\lambda}{\partial t} + v^\mu \frac{\partial v_\lambda}{\partial x^\mu} = -\frac{1}{\rho} \frac{\partial p}{\partial x^\lambda}.$$

First we determine the time-derivatives of v_λ . Differentiating (5.10), we obtain

$$(5.15) \quad \begin{aligned} \frac{\partial v_1}{\partial t} &= \frac{k}{(1-\gamma)}, & \frac{\partial v_2}{\partial t} &= 0, \\ \frac{\partial v_3}{\partial t} &= \frac{k^2 t}{2(1-\gamma)} \left[kx - \frac{k^2 t^2}{2(1-\gamma)} \right]^{-\frac{1}{2}}. \end{aligned}$$

Now we form the space derivatives of v_λ and find

$$(5.16) \quad \begin{aligned} \frac{\partial v_1}{\partial x} &= \frac{\partial v_1}{\partial y} = \frac{\partial v_1}{\partial z} = 0, & \frac{\partial v_2}{\partial x} &= \frac{\partial v_2}{\partial y} = \frac{\partial v_2}{\partial z} = 0, \\ \frac{\partial v_3}{\partial x} &= -\frac{1}{2}k \left[kx - \frac{k^2 t^2}{2(1-\gamma)} \right]^{-\frac{1}{2}}, & \frac{\partial v_3}{\partial y} &= \frac{\partial v_3}{\partial z} = 0. \end{aligned}$$

Hence the inertia terms

$$v^\mu \frac{\partial v_\lambda}{\partial x^\mu}$$

are found to be

$$(5.17) \quad \begin{aligned} v^\mu \frac{\partial v_1}{\partial x^\mu} &= v^\mu \frac{\partial v_2}{\partial x^\mu} = 0, \\ v^\mu \frac{\partial v_3}{\partial x^\mu} &= \frac{k^2 t}{2(\gamma - 1)} \left[kx - \frac{k^2 t^2}{2(1 - \gamma)} \right]^{-\frac{1}{2}}. \end{aligned}$$

Finally, from (5.16), the divergence of v_λ vanishes. By substituting (5.13), (5.15), (5.17) into (5.14), we see that *the equations of motion are satisfied*. Since the divergence of v_λ vanishes, the equation of continuity becomes

$$\frac{\partial \ln \rho}{\partial t} + v^\mu \frac{\partial \ln \rho}{\partial x^\mu} = 0.$$

By use of (5.12) and (5.10), we see *this equation is identically satisfied*.

The differential equations of the "streamlines" can be obtained by use of (5.10). We find by integration that the streamlines are given by

$$(5.18) \quad x = \frac{kt^2}{2(1 - \gamma)} + C_1, \quad y = C_2, \quad z = -(kC_1)^{\frac{1}{2}}t + C_3,$$

where C_1, C_2, C_3 are arbitrary constants. The projection of these curves in the x, z -plane is a family of parabolas

$$(5.19) \quad x = \frac{(C_3 - z)^2}{2C_1(1 - \gamma)} + C_1.$$

The components of the vorticity are determined by differentiating (5.10):

$$(5.20) \quad \omega_{12} = \omega_{23} = 0, \quad \omega_{13} = -\frac{k}{4} \left[kx - \frac{k^2 t^2}{2(1 - \gamma)} \right]^{-\frac{1}{2}}.$$

6. General Fluids. In this section we shall show that *the characteristic relation (2.18) of our previous paper [1] remains valid for non-isentropic, self-gravitating fluids, provided that ρ is replaced by η and $a^2 = dp/d\rho$ is replaced by $'a^2$ where (if S is the entropy)*

$$(6.1) \quad 'a^2 = \left(\frac{\partial p}{\partial \eta} \right)_S$$

and

$$(6.2) \quad \eta c^2 = \rho c^2 \left(1 + \frac{e}{c^2} + \frac{p}{\rho c^2} \right).$$

In (6.2), e is the specific internal energy. Thus the stress energy tensor becomes

$$(6.3) \quad T^{jk} = \eta c^2 \lambda^j \lambda^k - p g^{jk},$$

and for non-isentropic flows the conservation of mass relation

$$(6.4) \quad \nabla_i (\rho \lambda^i) = 0$$

must be added to the system. This relation is equivalent to the condition that entropy is constant along a world-line [6]. In order to prove our result, we assume that the metric tensor g^{ik} is continuous across any characteristic manifold. It should be noted that SYNGE & O'BRIEN and LICHNEROWICZ make a similar assumption for shock manifolds [7, 8] and TAUB makes a slightly weaker assumption for shock manifolds [9].

The discontinuity relations for (6.4) and the conservation of stress energy are [10]

$$(6.5) \quad L[\nabla_i \rho] + \rho \frac{\partial \phi}{\partial x^k} [\nabla_i \lambda^k] = 0,$$

$$(6.6) \quad c^2 L \lambda^i [\nabla_i \eta] + c^2 \eta \left(L[\nabla_i \lambda^i] + \lambda^i \frac{\partial \phi}{\partial x^k} [\nabla_i \lambda^k] \right) - g^{ik} \frac{\partial \phi}{\partial x^k} [\nabla_i p] = 0$$

where $\phi(x^i) = \text{constant}$ is the characteristic manifold and

$$(6.7) \quad L = \lambda^k \frac{\partial \phi}{\partial x^k}.$$

If we substitute (6.2) into (6.6) and eliminate $c^2 \rho \lambda^i [\nabla_i \lambda^k]$ ($\partial \phi / \partial x^k$) by use of (6.5), we obtain

$$(6.8) \quad \begin{aligned} & \lambda^i L[\nabla_i (e\rho)] + \lambda^i L[\nabla_i p] - g^{ik} \frac{\partial \phi}{\partial x^k} [\nabla_i p] \\ & + \lambda^i \frac{\partial \phi}{\partial x^k} (e\rho + p) [\nabla_i \lambda^k] + L(c^2 \rho + e\rho + p) [\nabla_i \lambda^i] = 0. \end{aligned}$$

Forming the scalar product of (6.8) with λ_i and noting that λ_i is a unit vector, we find

$$(6.9) \quad L[\nabla_i (e\rho)] + (e\rho + p) \frac{\partial \phi}{\partial x^k} [\nabla_i \lambda^k] = 0.$$

If we eliminate the jump in $\nabla_i \lambda^k$ by use of (6.5), the equation (6.9) becomes

$$(6.10) \quad \rho [\nabla_i e] - \frac{p}{\rho} [\nabla_i \rho] = 0.$$

From the first law of thermodynamics, we obtain the jump relation

$$(6.11) \quad [\nabla_i e] - \frac{p}{\rho} [\nabla_i \rho] = T [\nabla_i S]$$

where T is the temperature. From (6.10), (6.11) we see that

$$(6.12) \quad [\nabla_i S] = 0.$$

Thus, if we write

$$(6.13) \quad \nabla_i p = \left(\frac{\partial p}{\partial \eta} \right)_s \nabla_i \eta + \left(\frac{\partial p}{\partial S} \right)_\eta \nabla_i S,$$

then

$$(6.14) \quad [\nabla_i p] = \left(\frac{\partial p}{\partial \eta} \right)_s [\nabla_i \eta].$$

By substituting this relation into (6.6) and proceeding as in our previous paper [1], we obtain the characteristic relation

$$(6.15) \quad (c^2 - 2 {}'a^2)L^2 + {}'a^2 g^{ik} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^k} = 0$$

where $'a^2$ is given by (6.1).

By following the theory of our previous paper [1], it is easily seen that *Sections 1-7 remain valid for non-isentropic, self-gravitating fluids except that $'a^2$ is a function of η and S* . Further, the equation of conservation of mass (6.4) must be expressed in terms of characteristic variables. This implies that ρ is known as a function η, S . Also *the theory of the present paper for intrinsic forms of the characteristic relations is valid for isentropic, self-gravitation fluids which satisfy a degenerate equation of state* [6]. This follows from the following facts: First, for such fluids

$$(6.16) \quad p = A \rho^\gamma$$

where $\gamma = \frac{4}{3}$. Again, it should be noted that for such fluids the local sound speed a is given by [6]

$$(6.17) \quad \frac{a^2}{c^2} = \frac{\rho^2}{\eta} \frac{d}{d\rho} \left(\frac{\eta}{\rho} \right).$$

From this relation and some other formulas of TAUB's paper [6, see 2.15, 2.17], we find

$$(6.18) \quad \frac{dp}{d\eta} = {}'a^2 = \frac{a^2 c^2}{c^2 + a^2},$$

$$(6.19) \quad \frac{dp}{d\rho} = a^2 \frac{\eta}{\rho}.$$

From (6.18), (6.19), we find

$$(6.20) \quad \frac{d\eta}{\eta} = \frac{d\rho}{\rho} \frac{a^2 + c^2}{c^2},$$

$$(6.21) \quad a^2 = \frac{c^2 {}'a^2}{c^2 - {}'a^2}.$$

Thus, if we define the functions $'b, ' \theta$ by (see 2.8)

$$(6.22) \quad {}' \theta = (\gamma - 1) \ln \eta, \quad {}'b = \frac{{}'a^2}{c^2},$$

then from (6.18), (6.20) we find by use of (2.8), (2.10)

$$(6.23) \quad \frac{d \, 'b}{d \, ' \theta} = \frac{a^2 c^4}{(a^2 + c^2)^3} = 'b(1 - 'b)^2.$$

To obtain the basic formulas (2.34)–(2.37) of Section 2, we replaced $d\theta$ by db/b . Hence, for isentropic self-gravitating fluids, the relation (2.34)–(2.37) remain valid provided

$$\ln b = \int \frac{db}{b}$$

is replaced by

$$\int \frac{d \, 'b}{'b(1 - 'b)^2}$$

and b is replaced by $'b$.

Added in proof. In May, 1959, G. L. SAINI of the Indian Institute of Technology sent me a copy of a manuscript in which he extended the theory of [1] to self-gravitating non-isentropic fluids. SAINI works with S, ρ as the independent thermodynamic variables and determines the various necessary modifications and extensions of [1] explicitly. Also, he has called my attention to a sign error in [1] on pg. 455 and to a missing term in equation (4.24). To correct the sign error, the negative nappe of the normal cone should be used and a minus sign inserted before the multipliers of λ^i in (3.2), (3.3). As consequences of the error in equation (4.24), additional terms should be inserted in equations (4.38), (6.9) and (7.44).

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University of Michigan
Ann Arbor, Michigan