Invariant Imbedding, Random Walk and Scattering—II. Discrete Versions

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1. Introduction. In the first paper of this series, [1], we sketched the application of the techniques of invariant imbedding to various random walk processes and questions of scattering theory. In this paper, we wish to fill in the details and to show how the same ideas enable us to treat multi-dimensional, time-dependent, and energy-dependent processes.

A remarkable formal equivalence holds in all of these cases, with the result that the same equations—in different variables naturally—occur over and over.

As in other applications of invariant imbedding, we obtain both new analytic and new computational approaches. In some cases, we can transform problems of formidable proportions along classical lines to reasonable size.

Here we consider only the discrete versions of these processes. Passage to the limit in a number of the formulas we present will yield some interesting analytic representations, which we shall discuss in the next paper in this series. Finally, an analysis of the stratification concept we use repeatedly enables us to present extensive generalizations in quite abstract setting. These will be discussed in the fourth paper of the series.

I. ONE-DIMENSIONAL TIME-INDEPENDENT PROCESSES

2. Inhomogeneous Random Walk with Absorbing Barriers. Consider the finite one-dimensional lattice consisting of the integer values between a and b along the real line, as indicated below:

A particle jumps from lattice-point to lattice-point in accordance with the following law. When at k , there is a probability $q(k)$ of moving one unit to the right and a probability $p(k) = 1 - q(k)$ of moving one unit to the left.

We wish to determine the probability that the particle hits the barrier at a before it hits the barrier at b . The process ends as soon as the particle lands at a or b —hence the name “absorbing barriers.”

This is an extension of the classical “gambler’s ruin” process in which $p(k) = q(k) = \frac{1}{2}$ for $a < k < b$. This case can be treated in a very elegant fashion by means of WALB’s “fundamental identity,” [2], and [3], [4] for generalizations. Another extension is the “game of survival,” [5].

The classical approach proceeds as follows, [6]. Let

- (1) $u(k)$ = the probability that a particle starting at k lands at a before it lands at b .

Then, clearly, for $a < k < b$,

$$(2) \quad u(k) = p(k)u(k-1) + q(k)u(k+1),$$

with the two-point boundary conditions

$$(3) \quad u(a) = 1, \quad u(b) = 0.$$

In what follows we shall attack these problems by a quite different method. The nonlinear recurrence relations we derive were first obtained by T. E. HARRIS [10], in which he pointed out the analogy between these results and certain formulas in the theory of branching processes. Our point of view is, however, quite different.

3. Invariant Imbedding Approach. The observation that the desired probability $u(k)$ is a function of the endpoints a and b , as well as of k , keys our approach. Hence we should write $u(k, a, b)$ to signify this dependence.

Invariant imbedding capitalizes upon this dependence. In place of considering a and b to be constants, we consider them to be parameters of equal importance with k . In place of treating an individual random walk process, we investigate simultaneously an entire family of processes. The individual process is then analyzed in terms of its relation to contiguous members of the family. In this way, we construct a bridge between particular elements of the family of quite simple analytic structure and the processes of interest to us.

Let us then keep one endpoint fixed, say b , and regard u as a function of a and k . To make this dependence explicit, let us introduce the new function

- (1) $f(a, x) =$ the probability that a particle starting at x , $a \leq x \leq b$, will reach a before reaching b .

To obtain the required relation between contiguous elements, we use the simplest geometric fact that a particle starting at x , and moving one unit in either direction at each stage, must hit $a + 1$ before it can hit a . Having reached $a + 1$, the particle must reach a before b , if this is to be the case for the original process.

Translating these remarks into algebraic relations, we have

$$(2) \quad f(a, x) = f(a + 1, x)f(a, a + 1).$$

Iterating this relation, we have

$$(3) \quad f(a, x) = f(a, a + 1)f(a + 1, a + 2) \cdots f(x - 1, x),$$

an interesting representation for solutions of a Jacobi system of equations of the type appearing in (2.2).

4. The Function $f(a, a + 1)$. To answer our original question, that of the value of $f(a, x)$, it remains to determine the function of one variable

$$(1) \quad g(a) = f(a, a + 1),$$

defined for $a \leq b - 1$. Clearly $g(b - 1) = 0$.

Reverting to the description of the original process, we obtain the relation

$$(2) \quad f(a, a + 1) = p(a + 1) + q(a + 1)f(a, a + 2).$$

Using (3.2), we have

$$(3) \quad f(a, a + 2) = f(a, a + 1)f(a + 1, a + 2).$$

Combining these two expressions, we obtain the recurrence relation

$$(4) \quad g(a) = \frac{p(a + 1)}{1 - q(a + 1)g(a + 1)}.$$

Since, as noted above, $g(b - 1) = 0$, we have a simple inductive determination of $u(a)$ for $a \leq b - 1$.

The foregoing result can be derived in a way which emphasizes its physical

significance and its connection with previous work on neutron diffusion. Toward this end, let us consider the following scattering problem. A rod extending from a to b has the property that if a particle is at the position k , there is probability $p(k)$ that it will be scattered to $k - 1$ and probability $q(k) = 1 - p(k)$ that it will be scattered to $k + 1$. A particle is placed at the end a , and we wish to determine the probability that it will be "back-scattered" from a , over all time, rather than be "forward-scattered" through the end b .

We imbed this process within the class of processes with "trigger" particles placed at the end m of rods extending from m to b , with $m = b, b - 1, \dots$, and then write a functional equation interconnecting these processes.

Let us define the function

- (5) $g(a) = f(a; a + 1)$ = the probability that over all time a particle at $a + 1$ will be back-scattered to a by the rod extending from $a + 1$ to b , rather than be forward-scattered at b from the rod.

Observe next that with the particle initially at the position $a + 1$, there is probability $p(a + 1)$ that it will be scattered directly to the point a . On the other hand, if it is scattered initially to the right to $(a + 2)$, then by definition there is probability $g(a + 1)$ that it will eventually be back-scattered from the rod $(a + 2, b)$ to the point $a + 1$, from which it may be scattered to the point a . Should, however, it once again be scattered to the right to the point $(a + 2)$, there is once again probability $g(a + 1)$ that it will eventually reach the point $a + 1$, and so on. In this way we see that

$$(6) \quad g(a) = p(a + 1) + q(a + 1)g(a + 1)p(a + 1) \\ + q(a + 1)g(a + 1)q(a + 1)g(a + 1)p(a + 1) + \dots,$$

which, upon summing the geometric series on the right hand side, leads to the equation

$$(7) \quad g(a) = \frac{p(a + 1)}{1 - q(a + 1)g(a)}.$$

This is equation (4.4).

The method that we have used in deriving equation (4.4) or (4.7) corresponds abstractly to the method used by AMBARZUMIAN, [7], in discussing diffuse reflection from a foggy medium, and by BELLMAN, KALABA & WING, [8], in handling some neutron transport processes.

Similarly the discussion in §3 may be reinterpreted to yield the flux scattered from the end a of a rod as a result of an internal source of particles; cf [9].

5. Expected Sojourn. Let us now introduce the function

- (1) $w(a, x)$ = the conditional expected time required for the particle to reach a before reaching b , starting at x .

Then the same reasoning as above yields the relation

$$(2) \quad w(a, x) = w(a + 1, x) + w(a, a + 1).$$

To obtain an analytic expression for $w(a, a + 1)$, we combine the two expressions

$$(3) \quad \begin{aligned} w(a, a + 1) &= p(a + 1) + q(a + 1)[w(a, a + 2) + 1], \\ w(a, a + 2) &= w(a + 1, a + 2) + w(a, a + 1). \end{aligned}$$

The result is

$$(4) \quad w(a, a + 1) = \frac{1}{p(a + 1)} + \frac{q(a + 1)}{p(a + 1)} w(a + 1, a + 2).$$

Iteration yields the infinite series

$$(5) \quad \begin{aligned} w(a, a + 1) &= \frac{1}{p(a + 1)} + \frac{q(a + 1)}{p(a + 2)p(a + 1)} \\ &\quad + \frac{q(a + 1)q(a + 2)}{p(a + 3)p(a + 2)p(a + 1)} + \cdots, \end{aligned}$$

with the convention that the series terminates if b is finite.

6. Characteristic Functions. As usual, whenever an expected value can be determined, the same techniques yield relations for generating functions. Let

$$(1) \quad y(a, x) = \text{Exp}(e^{iz}),$$

where $z = z(a, x)$ is the random variable equal to the time spent by the particle in going from x to a , before ever hitting b .

Since

$$(2) \quad z(a, x) = z(a + 1, x) + z(a, a + 1),$$

we have the functional equation

$$(3) \quad y(a, x) = y(a + 1, x)y(a, a + 1).$$

The same reasoning as above yields the relations

$$(4) \quad \begin{aligned} y(a, a + 1) &= p(a + 1)e^{is} + q(a + 1)e^{is}y(a, a + 2), \\ y(a, a + 2) &= y(a + 1, a + 2)y(a, a + 1). \end{aligned}$$

From these we obtain the recurrence relation

$$(5) \quad y(a, a + 1) = \frac{p(a + 1)e^{is}}{1 - q(a + 1)e^{is}y(a + 1, a + 2)}.$$

Using this relation, we can obtain higher moments.

7. More General Random Walk Processes. We can apply similar techniques

to treat random walk processes in which we allow steps of several units at each stage. We shall discuss this elsewhere in connection with "games of survival."

II. MULTI-DIMENSIONAL SCATTERING AND ENERGY DEPENDENCE

8. Introduction. In this part of the paper, we wish to consider two processes of particular interest. The first concerns a one-dimensional random walk in which the energy of the particle changes as a result of its wandering, while the second pertains to a two-dimensional random walk process in which the direction of motion changes as a result of each collision.

Both of these can, in discrete form, be considered to be particular cases of a process of the following type:

A particle in state i , $i = 1, 2, \dots, N$, can occupy any of the lattice points x between a and b . Let

- (1) $p_{ij}(x)$ = the probability that a particle at x in state i will go one unit to the left, and arrive in state j , $i, j = 1, 2, \dots, N$,
 $q_{ij}(x)$ = the probability that a particle at x in state i will go one unit to the right, and arrive in state j , $i, j = 1, 2, \dots, N$.

Let us then define the N^2 functions

- (2) $u_{ij}(a, x)$ = the probability that a particle starting at x in state i will hit a in state j before reaching b in any state.

Proceeding as in Part I, we obtain the relation

$$(3) \quad u_{ij}(a, x) = \sum_k u_{ik}(a+1, x)u_{kj}(a, a+1).$$

Consequently, if we introduce the matrix function

$$(4) \quad U(a, x) = (u_{ij}(a, x)),$$

we derive the basic relation

$$(5) \quad U(a, x) = U(a+1, x)U(a, a+1),$$

the analogue of (3.2).

9. Determination of $U(a, a+1)$. Proceeding as before, we have

$$(1) \quad u_{ij}(a, a+1) = p_{ij}(a+1) + \sum_k q_{ik}(a+1)u_{kj}(a, a+2),$$

or, if $V(a) = (u_{ij}(a, a+1))$, $P(a) = (p_{ij}(a))$, $Q(a) = (q_{ij}(a))$,

$$(2) \quad V(a) = P(a+1) + Q(a+1)U(a, a+2).$$

Using (8.5), we obtain the relation

$$(3) \quad U(a, a+2) = U(a+1, a+2)U(a, a+1).$$

Using this in (2), we have

$$(4) \quad V(a) = P(a+1) + Q(a+1)V(a+1)V(a),$$

or

$$(5) \quad V(a) = [I - Q(a+1)V(a+1)]^{-1}P(a+1).$$

Since $V(b) = 0$, once again we have a direct iterative technique for determining $V(a)$, and thus $U(a, x)$.

10. Discussion. It is clear that in the same way we can obtain analogues of the results of §5 and §6.

Turning from the analytic aspects, let us examine the computational aspects. Approaching the problem along conventional lines, we obtain a system of linear equations of the form

$$(1) \quad w_{ij}(x) = \sum_k p_{ik} w_{kj}(x-1) + \sum_k q_{ik}(x) w_{kj}(x+1).$$

Keeping j fixed, we have a system of order $N(b-a)$. If $N = 10$ and $b-a = 100$, this is of order 1000, a respectable and even formidable number, even in the light of modern devices.

Using the technique described above, the solution is made to depend upon the inversion and repeated multiplication of 10×10 matrices, a complicated, but far more feasible process.

III. TIME-DEPENDENT PROCESSES

11. Introduction. Let us now consider the one-dimensional random walk process in which each stage consumes one time unit. Let

$$(1) \quad u(a, x, t) = \sum_{s=0}^t u(a+1, x, s)u(a, a+1, t-s).$$

As we shall see, we derive equations completely analogous to those exhibited above for the generating function

$$(2) \quad U(a, x) = \sum_{t=0}^{\infty} u(a, x, t)r^t.$$

These represent generalizations of the equations given in Part I, since they reduce to those for $r = 1$.

12. Analytic Results. Starting with the previously given fundamental relation

$$(1) \quad u(a, x, t) = \sum_{s=0}^t u(a+1, x, s)u(a, a+1, s).$$

We introduce the generating function

$$(2) \quad U(a, x) = \sum_{t=0}^{\infty} u(a, x, t)r^t,$$

and once again derive the relation

$$(3) \quad U(a, x) = U(a + 1, x)U(a, a + 1).$$

Furthermore,

$$(4) \quad u(a, a + 1, t) = p(a + 1) \delta(1, t) + q(a + 1)u(a, a + 2, t - 1),$$

for $t \geq 1$, where

$$(5) \quad \delta(1, t) = \begin{cases} 1, & t = 1, \\ 0, & t \neq 1. \end{cases}$$

Thus, multiplying (4) by r^t and summing over t ,

$$(6) \quad U(a, a + 1) = p(a + 1)r + q(a + 1)rU(a, a + 2).$$

From here on the argument proceeds as before. The final result is

$$(7) \quad U(a, a + 1) = \frac{p(a + 1)r}{1 - q(a + 1)rU(a + 1, a + 2)}.$$

Equation (6) can be derived directly making use of the properties of the process and of generating functions.

IV. TWO DIMENSIONAL CASE

13. Introduction. Let us now consider a discrete grid in the annulus between the concentric circles with radii a and b . The grid is formed by the intersections of concentric circles with the radii $x = a, a + 1, a + 2, \dots, b - 1, b$, with the rays emanating from the origin at the angles $\theta = 0, \alpha, \dots, N\alpha = 2\pi$. The same basic equations will arise if we once again introduce generating functions. To simplify analysis we shall assume circular symmetry.

14. Sketch of Analytic Results. Let

$$(1) \quad u(a, x, \theta) = \text{the probability that a particle on circle } x \text{ will hit circle } a \text{ at angle } \theta \text{ further clockwise before hitting anywhere on circle } b.$$

The basic equation is now

$$(2) \quad u(a, x, \theta) = \sum_{\psi} u(a + 1, x, \psi)u(a, a + 1, \theta - \psi).$$

To specify the random walk process, we allow the particle at the grid-point (x, θ) to go on the next step to one of the four neighboring points

$$(3) \quad (x + 1, \theta), \quad (x - 1, \theta), \quad (x, \theta + \alpha), \quad (x, \theta - \alpha).$$

Having done this, the analysis proceeds as before.

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