*Analytic Continuation of Random Series*¹

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The present work is divided in two parts. In the first part we study some analytic aspects of the ergodic theory of random series. To be more precise, we consider a non-atomic measure space $(X, \mathfrak{B}, \mu)$ of finite total measure and an ergodic transformation τ on X and for every $f \in L^1(X, \mathfrak{B}, \mu)$ we define the power series

$$F(f; x, z) = \sum_{n=0}^{\infty} f(\tau^n(x))z^n, \quad x \in X, \quad |z| < 1.$$

We are concerned especially with the problem of analytic continuation of $F(f; x, z)$ for almost every $x \in X$. No separability assumptions on the underlying measure space or invertibility assumptions on the ergodic transformation τ are made (except when explicitly stated otherwise).

In the second part of the work we consider an abstract infinite-dimensional Hilbert space $\mathcal{H}$ and the set $\mathfrak{U}$ of all unitary operators on $\mathcal{H}$. We study various equivalent aspects of the fact that the spectrum of a given operator $U \in \mathfrak{U}$ coincides with the whole unit circle.

We shall denote by C the unit circle $\{z \mid |z| = 1\}$.

PART I

1. Preliminaries. Let $(X, \mathfrak{B}, \mu)$ be a measure space. In what follows we shall assume that:

1° $\mu(X) = 1$.

2° The measure space $(X, \mathfrak{B}, \mu)$ is non-atomic.

Let τ be a measurable, measure-preserving mapping of X into itself and let $f \in L^p(X, \mathfrak{B}, \mu)$ ($1 \leq p \leq \infty$). Define the function

$$(1) \quad F(f; x, z) = \sum_{n=0}^{\infty} f(\tau^n(x))z^n, \quad x \in X, \quad |z| < 1.$$

¹The results of this paper are taken from the author's doctoral dissertation written at Yale University under the direction of Professor SHIZUO KAKUTANI. We would like to express our thanks to Professor KAKUTANI for many helpful discussions and valuable suggestions.

We shall study below various properties concerning the analytic continuation of the power series (1) for almost every $x \in X$. The main results will be proved under the following additional assumption:

3° τ is ergodic (but not necessarily invertible).

We recall that a measurable, measure-preserving mapping τ of X into itself is ergodic if and only if, for $E \in \mathfrak{B}$, $\mu(E\Delta\tau^{-1}(E)) = 0$ implies either $\mu(E) = 0$ or $\mu(E) = 1$. For the definitions and results from ergodic theory which are used throughout Part I, we refer to [4], [7].

2. Analytic continuation almost everywhere.

Theorem 1. *Let τ be a measurable, measure-preserving mapping of X into itself. τ is ergodic if and only if for every $f \in L^1(X, \mathfrak{B}, \mu)$, $f \not\equiv 0$, there exists a set $N_f \in \mathfrak{B}$, $\mu(N_f) = 0$, such that for every fixed $x \in \mathbb{C}N_f$, the power series $F(f; x, z)$ defined by (1) has radius of convergence one.*

Proof. Assume that τ is ergodic and let $f \in L^1(X, \mathfrak{B}, \mu)$, $f \not\equiv 0$. By the individual ergodic theorem, $\lim_{n \rightarrow \infty} f(\tau^n(x))/n = 0$ almost everywhere and therefore $\limsup_{n \rightarrow \infty} |f(\tau^n(x))|^{1/n} \leq 1$ almost everywhere. On the other hand, since $f \not\equiv 0$ there exists a constant $\delta > 0$ such that the set $E_\delta = \{x \mid |f(x)| \geq \delta\}$ has strictly positive measure; since τ is ergodic, to almost every point $x \in X$ corresponds a sequence of positive integers $(m_k)_{1 \leq k < \infty}$ such that $\tau^{m_k}(x) \in E_\delta$ for $k = 1, 2, \dots$. It follows that for almost every $x \in X$, $\limsup_{n \rightarrow \infty} |f(\tau^n(x))|^{1/n} \geq 1$. Hence we have almost everywhere $\limsup_{n \rightarrow \infty} |f(\tau^n(x))|^{1/n} = 1$.

Conversely, assume that τ is not ergodic. There exists then a set $B \in \mathfrak{B}$, $0 < \mu(B) < 1$, such that $\mu(B\Delta\tau^{-1}(B)) = 0$; if φ_B is the characteristic function of the set B , then obviously $\limsup_{n \rightarrow \infty} |\varphi_B(\tau^n(x))|^{1/n} = 0$ on a set of strictly positive measure.

Remark. Theorem 1 gives a characterization of ergodic transformations in terms of “almost everywhere properties” of the power series defined by (1). Similar characterizations can be given also for other categories of transformations. Assume, for instance, that τ is a measurable, measure-preserving, invertible transformation; it can be shown that τ is weakly mixing if and only if for every $f \in L^1(X, \mathfrak{B}, \mu)$ there exists a set $N_f \in \mathfrak{B}$, $\mu(N_f) = 0$, such that for every fixed $x \in \mathbb{C}N_f$

$$\lim_{r \rightarrow 1} (1 - r)F(f; x, re^{i\theta}) = 0$$

if $\theta \equiv 0 \pmod{2\pi}$ and

$$\lim_{r \rightarrow 1} (1 - r)F(f; x, r) = c$$

where c is a constant independent of x . The proof of this statement has been suggested by [5]; it is essentially based on the generalized ergodic theorem of WIENER & WINTNER [9] and on the mean ergodic theorem.

In the rest of this paragraph we shall assume that condition 3° of §1 is also satisfied.

We shall introduce now the following definitions:

Definition 1. A point $z_0 \in C$ is called a.e. regular (a.e. singular) with respect to $f \in L^p(X, \mathfrak{B}, \mu)$ ($1 \leq p \leq \infty$) if there exists a set $B_r \in \mathfrak{B}$, $\mu(B_r) = 1$ ($B_s \in \mathfrak{B}$, $\mu(B_s) = 1$), such that for every fixed $x \in B_r$ ($x \in B_s$) the power series $F(f; x, z)$ defined by (1) is regular at z_0 (singular at z_0).

Definition 2. A function $f \in L^p(X, \mathfrak{B}, \mu)$ ($1 \leq p \leq \infty$) is said to satisfy the property $P_{(\epsilon, \delta, n)}$ at the point $z_0 \in C$ ($0 < \epsilon < 1$, $0 < \delta < 1$, $n = 1, 2, \dots$) if there exists a set $A_\epsilon \in \mathfrak{B}$, $\mu(A_\epsilon) \leq \epsilon$, such that for $x \in CA_\epsilon$ and $j \geq n$

$$|F^{(j)}(f; x, \tfrac{1}{2}z_0)/j!|^{1/j} \leq 2 - \delta.^2$$

We shall denote by $D_{(\epsilon, \delta, n)}^p(z_0)$ the set

$$\{f \mid f \in L^p(X, \mathfrak{B}, \mu), f \text{ satisfies } P_{(\epsilon, \delta, n)} \text{ at } z_0\}.$$

Proposition 1. Let $f \in L^p(X, \mathfrak{B}, \mu)$ ($1 \leq p \leq \infty$) and let $z_0 \in C$. We have: (i) z_0 is either a.e. regular or a.e. singular with respect to f ; (ii) z_0 is a.e. regular with respect to f if and only if for every ϵ , $0 < \epsilon < 1$, there exist a constant δ , $0 < \delta < 1$, and a natural number $n \geq 1$ such that $f \in D_{(\epsilon, \delta, n)}^p(z_0)$.

Proof. (i) We may assume that $f \not\equiv 0$. Let

$$A = \{x \mid \limsup_{n \rightarrow \infty} |f(\tau^n(x))|^{1/n} = 1\}.$$

By Theorem 1, $\mu(A) = 1$; using the relation

$$(2) \quad F(f; \tau(x), z) = \frac{F(f; x, z) - f(x)}{z}$$

we can easily see that $\tau^{-1}(A) = A$. For $x \in A$ and $|z - \frac{1}{2}z_0| < \frac{1}{2}$ we have obviously

$$F(f; x, z) = \sum_{j=0}^{\infty} (F^{(j)}(f; x, \tfrac{1}{2}z_0)/j!)(z - \tfrac{1}{2}z_0)^j.$$

Let now

$$B = \{x \mid x \in A, \limsup_{n \rightarrow \infty} |F^{(j)}(f; x, \tfrac{1}{2}z_0)/j!|^{1/j} < 2\}$$

Obviously $B \in \mathfrak{B}$; using again (2) we deduce that $\tau^{-1}(B) = B$. By the ergodicity of τ only two cases are possible: (a) $\mu(B) = 1$, in which case z_0 is a.e. regular with respect to f ; (b) $\mu(B) = 0$, in which case z_0 is a.e. singular with respect to f . Thus (i) is proved.

(ii) Suppose first that $f \in D_{(\epsilon, \delta, n)}^p(z_0)$ ($0 < \epsilon < 1$, $0 < \delta < 1$, n a natural number

²We use the notation $F^{(j)}(f; x, \tfrac{1}{2}z_0) = [d^j F(f; x, z)/dz^j]_{z=\frac{1}{2}z_0}$.

≥ 1). There exists then a set $A_f \in \mathfrak{B}$, $\mu(A_f) \leq \epsilon$, such that for $x \in \mathbb{C}A_f$,

$$\limsup_{j \rightarrow \infty} |F^{(j)}(f; x, \tfrac{1}{2}z_0)/j!|^{1/j} \leq 2 - \delta.$$

Since $\mu(\mathbb{C}A_f) > 0$, it follows by (i) that z_0 is a.e. regular with respect to f .

Conversely, assume that z_0 is a.e. regular with respect to f . Let A and B be the sets introduced in the proof of (i) and for $0 < \delta < 1$ let

$$B_\delta = \{x \mid x \in A, \limsup_{j \rightarrow \infty} |F^{(j)}(f; x, \tfrac{1}{2}z_0)/j!|^{1/j} \leq 2 - \delta\}.$$

Since $\mu(B) = 1$ by assumption, there exists a constant δ , $0 < \delta < 1$, such that $\mu(B_\delta) > 0$. Again making use of (2) we deduce that $\tau^{-1}(B_\delta) = B_\delta$ and hence $\mu(B_\delta) = 1$. Let now ϵ be given, $0 < \epsilon < 1$. By applying Egoroff's theorem, we can find a set $A_f \in \mathfrak{B}$, $\mu(A_f) \leq \epsilon$, and a natural number n_0 such that for $x \in \mathbb{C}A_f$ and $j \geq n_0$

$$\begin{aligned} |F^{(j)}(f; x, \tfrac{1}{2}z_0)/j!|^{1/j} &\leq \limsup_{j \rightarrow \infty} |F^{(j)}(f; x, \tfrac{1}{2}z_0)/j!|^{1/j} + \tfrac{1}{2}\delta \\ &\leq 2 - \delta + \tfrac{1}{2}\delta = 2 - \tfrac{1}{2}\delta. \end{aligned}$$

This inequality is equivalent to $f \in D_{(\epsilon, \frac{1}{2}\delta, n_0)}^p(z_0)$. Hence the proof of Proposition 1 is complete.

Proposition 2. Let $0 < \epsilon < 1$, $0 < \delta < 1$, n a natural number ≥ 1 , $z_0 \in C$ and $1 \leq p \leq \infty$. We have: (i) The set $D_{(\epsilon, \delta, n)}^p(z_0)$ is closed in $L^p(X, \mathfrak{B}, \mu)$. (ii) If there exists a function g_p in $L^p(X, \mathfrak{B}, \mu)$ such that z_0 is a.e. singular with respect to g_p , then $D_{(\epsilon, \delta, n)}^p(z_0)$ is nowhere dense in $L^p(X, \mathfrak{B}, \mu)$.

Proof. (i) Let h be an arbitrary element in $L^p(X, \mathfrak{B}, \mu)$. It is easy to see that for $j = 0, 1, \dots$

$$\|F^{(j)}(h; (\cdot), \tfrac{1}{2}z_0)\|_p \leq C(j) \|h\|_p$$

where $C(j)$ is a constant independent of h . Let now $(f_m)_{1 \leq m < \infty}$ be a Cauchy sequence of elements belonging to $D_{(\epsilon, \delta, n)}^p(z_0)$. There exists $f \in L^p(X, \mathfrak{B}, \mu)$ such that $\lim_{m \rightarrow \infty} \|f - f_m\|_p = 0$. We shall show that $f \in D_{(\epsilon, \delta, n)}^p(z_0)$. Since for $j = 0, 1, \dots$ the mapping $h \rightarrow F^{(j)}(h; (\cdot), \tfrac{1}{2}z_0)$ is linear and continuous on $L^p(X, \mathfrak{B}, \mu)$, we may suppose without loss of generality that there exists a set $N \in \mathfrak{B}$, $\mu(N) = 0$, such that for $x \in \mathbb{C}N$ and $j = 0, 1, 2, \dots$

$$(3) \quad \lim_{m \rightarrow \infty} |F^{(j)}(f_m; x, \tfrac{1}{2}z_0)/j!| = |F^{(j)}(f; x, \tfrac{1}{2}z_0)/j!|$$

(eventually we make use of the diagonal process).

Since, for $m = 1, 2, \dots$, $f_m \in D_{(\epsilon, \delta, n)}^p(z_0)$, there exists a set $A_m = A_{f_m} \in \mathfrak{B}$, $\mu(A_m) \leq \epsilon$, such that for $x \in \mathbb{C}A_m$ and $j \geq n$

$$|F^{(j)}(f_m; x, \tfrac{1}{2}z_0)/j!|^{1/j} \leq 2 - \delta.$$

Define now the set A_f by the equation

$$\mathbb{C}A_f = \mathbb{C}N \cap \limsup_{m \rightarrow \infty} \mathbb{C}A_m.$$

Since $\mu(\mathbf{CA}_m) \geq 1 - \epsilon$ for $m = 1, 2, \dots$, it follows that $\mu(\mathbf{CA}_j) \geq 1 - \epsilon$; on the other hand, using (3) and the definition of the set A_j , we deduce immediately that for $x \in \mathbf{CA}_j$ and $j \geq n$

$$|F^{(i)}(f; x, \tfrac{1}{2}z_0)/j!|^{1/i} \leq 2 - \delta.$$

The last inequality is equivalent to $f \in D_{(\epsilon, \delta, n)}^p(z_0)$. Hence the assertion (i) is proved.

(ii) Since $D_{(\epsilon, \delta, n)}^p(z_0)$ is closed in $L^p(X, \mathfrak{B}, \mu)$, it will be enough to show that the interior of $D_{(\epsilon, \delta, n)}^p(z_0)$ is the void set. Suppose, on the contrary, that there exists an element f_0 belonging to the interior of the set $D_{(\epsilon, \delta, n)}^p(z_0)$. In particular, we can find then a scalar $\lambda \neq 0$ such that $f_0 + \lambda g_p \in D_{(\epsilon, \delta, n)}^p(z_0)$. Using the obvious identity

$$F(g_p; x, z) = \frac{F(f_0 + \lambda g_p; x, z) - F(f_0; x, z)}{\lambda},$$

we deduce that z_0 is a.e. regular with respect to g_p . This contradicts the assumption. Therefore (ii) is proved.

Proposition 3. *Let $1 \leq p \leq \infty$ be fixed, let Q be a countable, everywhere dense subset of C , and assume that for every $z \in Q$ there exists a function $g_z \in L^p(X, \mathfrak{B}, \mu)$ such that z is a. e. singular with respect to g_z . The set of all functions $f \in L^p(X, \mathfrak{B}, \mu)$ such that the unit circle C is the natural boundary of the power series $F(f; x, z)$ defined by (1), for almost every $x \in X$, is a set of second category in $L^p(X, \mathfrak{B}, \mu)$, namely an everywhere dense G_δ .*

Proof. Let $0 < \epsilon < 1$ and define the set

$$R_p = \mathbf{c}\left(\bigcup_{z \in Q} \bigcup_{n, k=1}^{\infty} D_{(\epsilon, 1/k, n)}^p(z)\right)$$

By Proposition 2, (ii), R_p is an everywhere dense G_δ in $L^p(X, \mathfrak{B}, \mu)$. Therefore it remains to show that R_p coincides with the set of all functions $f \in L^p(X, \mathfrak{B}, \mu)$ such that C is the natural boundary of $F(f; x, z)$ for almost every $x \in X$.

Let $f \in R_p$ and $z_0 \in Q$; then $f \notin D_{(\epsilon, 1/k, n)}^p(z_0)$ for any $k = 1, 2, \dots$ and $n = 1, 2, \dots$; consequently $f \notin D_{(\epsilon, \delta, n)}^p(z_0)$ for any $\delta, 0 < \delta < 1$, and $n = 1, 2, \dots$. Hence by Proposition 1, z_0 is a.e. singular with respect to f . So for every $z_0 \in Q$ there exists a set $N_{z_0} \in \mathfrak{B}$, $\mu(N_{z_0}) = 0$, such that for $x \in \mathbf{CN}_{z_0}$ the power series $F(f; x, z)$ is singular at z_0 . Let $N = \bigcup_{z \in Q} N_z$; then $\mu(N) = 0$ and it is easily seen that for every fixed $x \in \mathbf{CN}$ the power series $F(f; x, z)$ cannot be continued analytically outside C . Conversely, assume that the function $f \in L^p(X, \mathfrak{B}, \mu)$ is such that the unit circle C is the natural boundary of $F(f; x, z)$ for almost every $x \in X$. Using again Proposition 1, we see that $f \notin D_{(\epsilon, 1/k, n)}^p(z_0)$ for any $z_0 \in Q$ and $k, n = 1, 2, \dots$. Hence $f \in R_p$ and the proof of Proposition 3 is complete.

Remark. The proof of (ii), Proposition 2 and the construction of the set R_p introduced in Proposition 3 have been suggested by the reading of [8].

3. Existence theorems. A complex-valued function f defined on X is a simple function if it is $\mathfrak{B}$ -measurable and if $f(X)$ is a finite set. We shall denote by $\mathfrak{E}(\mathfrak{B})$ the set of all simple functions f such that f is not equal to a constant almost everywhere. We shall denote by $\mathfrak{E}_r(\mathfrak{B})$ the set of all simple functions f having the following property: There exists a complex number α (depending on f) such that $\mu(f^{-1}(\{\alpha\}))$ is an irrational number. It is easy to see that $\mathfrak{E}_r(\mathfrak{B}) \subset \mathfrak{E}(\mathfrak{B})$ and that $\mathfrak{E}_r(\mathfrak{B})$ is dense in $\mathfrak{E}(\mathfrak{B})$ with respect to the topology of $L^\infty(X, \mathfrak{B}, \mu)$ (we make use of the non-atomicity of the measure space $(X, \mathfrak{B}, \mu)$).

Let $E \in \mathfrak{B}$ be such that $0 < \mu(E) < 1$ and let us define the function

$$(4) \quad \Phi_E(x) = \sum_{n=0}^{\infty} \frac{\varphi_E(\tau^n(x))}{2^{n+1}}$$

(φ_E is the characteristic function of the set E). Since Φ_E is obviously $\mathfrak{B}$ -measurable, the set

$$(5) \quad P = \{x \mid \Phi_E(x) \text{ is a rational number}\}$$

belongs to $\mathfrak{B}$. Let us remark now that $x \in P$ if and only if the sequence $(\varphi_E(\tau^n(x)))_{0 \leq n < \infty}$ is periodic, *i.e.* if and only if there exist natural numbers $N \geq 0$, $\omega \geq 1$ (depending on x) such that $\varphi_E(\tau^{n+\omega}(x)) = \varphi_E(\tau^n(x))$ for $n \geq N$. It is then easily seen that $\tau^{-1}(P) = P$. By the ergodicity of τ , only two cases are possible:

$$(A) \quad \mu(P) = 0,$$

$$(B) \quad \mu(P) = 1.$$

Theorem 2. Let τ be a measurable, measure-preserving mapping of X into itself.

(i) If all the iterates τ^n , $n = 1, 2, \dots$, are ergodic, then for every $f \in \mathfrak{E}(\mathfrak{B})$ the unit circle C is the natural boundary of the power series $F(f; x, z)$ defined by (1), for almost every $x \in X$.³ (ii) If τ is ergodic, then for every $f \in \mathfrak{E}_r(\mathfrak{B})$ the unit circle C is the natural boundary of the power series $F(f; x, z)$ defined by (1), for almost every $x \in X$.⁴

Proof. (i) Let $f \in \mathfrak{E}(\mathfrak{B})$; there exists then a constant $\alpha \neq 0$ such that $0 < \mu(f^{-1}(\{\alpha\})) < 1$. Let us write $E = f^{-1}(\{\alpha\})$ and consider the function Φ_E and the set P defined by formulas (4) and (5) respectively. We shall show that for the set P the possibility (B) cannot occur.

Suppose, on the contrary, that $\mu(P) = 1$. For every rational number $q \in [0, 1]$ let

$$P(q) = \{x \mid \Phi_E(x) = q\}.$$

³In general, the iterates of an ergodic transformation need not all be ergodic. (*Added in proof.* It can be shown that the converse of the assertion (i) is also true: If for every $f \in \mathfrak{E}(\mathfrak{B})$ the unit circle C is the natural boundary of the power series $F(f; x, z)$ defined by (1), for almost every $x \in X$, then the iterates τ^n , $n = 1, 2, \dots$, are all ergodic.)

⁴The author is indebted to Professor KAKUTANI for various suggestions which led to the present formulation of (ii), Theorem 2.

We can find a rational number $q_0 \in [0, 1]$ such that $\mu(P(q_0)) > 0$. Let us assume that q_0 is not a dyadic rational (the case when q_0 is a dyadic rational can be treated in a similar manner). Then q_0 has a unique dyadic representation

$$q_0 = \sum_{k=1}^{\infty} \frac{\epsilon_k(q_0)}{2^k}, \quad \epsilon_k(q_0) = 0 \text{ or } 1,$$

and there exist natural numbers $N \geq 0, \omega \geq 1$ such that $\epsilon_{k+\omega}(q_0) = \epsilon_k(q_0)$ for $k \geq N+1$. We may suppose without loss of generality that $\epsilon_{N+1}(q_0) = 1$. It follows that for every $x \in P(q_0)$

$$\varphi_E(\tau^{N+n\omega}(x)) = \varphi_E(\tau^N(x)) = \epsilon_{N+1}(q_0) = 1 \quad \text{for } n = 1, 2, \dots,$$

or equivalently

$$x \in P(q_0) \text{ implies } \tau^{n\omega}(x) \in \tau^{-N}(E) \text{ for } n = 0, 1, 2, \dots$$

The last relation contradicts the ergodicity of τ^ω . Therefore the assumption (B) leads to a contradiction and we must have $\mu(P) = 0$.

Consider now the power series $F(f; x, z) = \sum_{n=0}^{\infty} f(\tau^n(x))z_n$. Its coefficients are taken from the finite set $f(X)$. The above reasoning shows that for every fixed $x \in \mathbf{CP}$ the sequence $(\varphi_E(\tau^n(x)))_{0 \leq n < \infty}$ is not periodic. It follows that for every $x \in \mathbf{CP}$ the value $f(x) = \alpha$ appears non-periodically in the sequence $(f(\tau^n(x)))_{0 \leq n < \infty}$; hence, *a fortiori*, for every $x \in \mathbf{CP}$ the sequence $(f(\tau^n(x)))_{0 \leq n < \infty}$ is not periodic. Using a classical theorem in the complex variable theory due to CARLSON-JENTZSCH-PÓLYA-SZEGÖ (see [1], vol. II, p. 315), we deduce that for every $x \in \mathbf{CP}$ the unit circle C is the natural boundary of the power series $F(f; x, z)$. Since $\mu(\mathbf{CP}) = 1$, the proof of (i) is complete.

(ii) Let $f \in \mathcal{E}_r(\mathcal{B})$; there exists then a constant α such that $\mu(f^{-1}(\{\alpha\}))$ is an irrational number and we may suppose without loss of generality that $\alpha \neq 0$. As in the proof of (i) we write $E = f^{-1}(\{\alpha\})$ (we have obviously $0 < \mu(E) < 1$) and we consider the function Φ_E and the set P defined by formulas (4) and (5) respectively. Using a different device, we shall show again that $\mu(P) = 0$.

Suppose, on the contrary, that $\mu(P) = 1$. By the individual ergodic theorem, there exists a set $A \in \mathcal{B}$, $\mu(A) = 0$, such that for $x \in \mathbf{CA}$

$$(6) \quad \lim_{m \rightarrow \infty} \frac{\varphi_E(x) + \varphi_E(\tau(x)) + \dots + \varphi_E(\tau^{m-1}(x))}{m} = \mu(E).$$

Consider the set $P \cap \mathbf{CA}$; obviously $\mu(P \cap \mathbf{CA}) = 1$. Let $x \in P \cap \mathbf{CA}$. Since $x \in P$, there exist natural numbers $N \geq 0, \omega \geq 1$ such that, for $n \geq N$, $\varphi_E(\tau^{n+\omega}(x)) = \varphi_E(\tau^n(x))$. Let

$$(7) \quad p = \sum_{j=0}^{N-1} \varphi_E(\tau^j(x)), \quad q = \sum_{j=N}^{N+\omega-1} \varphi_E(\tau^j(x)).$$

Since $x \in \mathbf{CA}$, we can apply (6) and we obtain in particular,

$$\lim_{m \rightarrow \infty} \frac{\varphi_E(x) + \dots + \varphi_E(\tau^{N+m\omega-1}(x))}{N + m\omega} = \mu(E),$$

and, on the other hand, using (7)

$$\lim_{m \rightarrow \infty} \frac{\varphi_E(x) + \cdots + \varphi_E(\tau^{N+m\omega-1}(x))}{N+m\omega} = \lim_{m \rightarrow \infty} \frac{p+mq}{N+m\omega} = \frac{q}{\omega}.$$

The last two relations are contradictory since $\mu(E)$ is by assumption an irrational number. Hence the statement made above is proved: $\mu(P) = 0$.

The argument is now concluded as in the proof of (i): We consider the power series $F(f; x, z) = \sum_{n=0}^{\infty} f(\tau^n(x))z^n$, whose coefficients are taken from the finite set $f(X)$, we remark that for every $x \in \mathbf{CP}$ the sequence $(f(\tau^n(x)))_{0 \leq n < \infty}$ is not periodic, and we apply the Carlson-Jentzsch-Pólya-Szegő theorem. Therefore (ii) is also proved.

Theorem 3. *Let τ be an ergodic transformation of the measure space $(X, \mathfrak{B}, \mu)$ and let $1 \leq p \leq \infty$ be fixed. The set R_p of all functions $f \in L^p(X, \mathfrak{B}, \mu)$ such that the unit circle C is the natural boundary of the power series $F(f; x, z)$, defined by (1) for almost every $x \in X$, is a set of second category in $L^p(X, \mathfrak{B}, \mu)$, namely an everywhere dense G_δ .*

Proof. Theorem 3 follows from (ii), Theorem 2 and Proposition 3.

4. Consequences. Let τ be an ergodic transformation of the measure space $(X, \mathfrak{B}, \mu)$, let $1 \leq p \leq \infty$ be fixed, and let U be the isometric operator induced by τ in $L^p(X, \mathfrak{B}, \mu)$: $Uf = f \circ \tau$ for every $f \in L^p(X, \mathfrak{B}, \mu)$. For every $f \in L^p(X, \mathfrak{B}, \mu)$ consider the vector-valued analytic function $F(f; z)$ defined by

$$(8) \quad F(f; z) = \sum_{n=0}^{\infty} U^n f \cdot z^n, \quad |z| < 1.$$

It obviously has radius of convergence equal to one, provided $f \neq 0$.

Corollary 1. *Suppose that τ is ergodic, let $1 \leq p \leq \infty$ be fixed, and let U be the isometric operator induced by τ in $L^p(X, \mathfrak{B}, \mu)$. There exists a set R_p of second category in $L^p(X, \mathfrak{B}, \mu)$ such that for every $f \in R_p$ the unit circle C is the natural boundary of the vector-valued analytic function $F(f; z)$ defined by (8).*

Proof. Let $f \in L^p(X, \mathfrak{B}, \mu)$; we shall show that if the vector-valued analytic function $F(f; z)$ defined by (8) is regular at $z_0 \in C$, then z_0 is a.e. regular with respect to f (Definition 1). Corollary 1 will then be an immediate consequence of Theorem 3. We may suppose that $1 \leq p < \infty$ (for $p = \infty$ the assertion to be proved is trivial). Since $F(f; z)$ is regular at z_0 , there exist a constant δ , $0 < \delta < 1$, and a natural number $n \geq 1$ such that for $j \geq n$

$$\|F^{(j)}(f; \tfrac{1}{2}z_0)/j!\|_p^{1/j} \leq 2 - \delta.$$

Let now

$$B_j = \{x \mid \|F^{(j)}(f; x, \tfrac{1}{2}z_0)/j!\|_p^{1/j} > 2 - \tfrac{1}{2}\delta\}$$

for $j \geq n$. A direct computation shows that for $j \geq n$

$$\|F^{(j)}(f; \tfrac{1}{2}z_0)/j!\|_p \geq (2 - \tfrac{1}{2}\delta)^j \cdot \mu(B_j)^{1/p}.$$

Hence if we set $a = ((2 - \delta)/(2 - \tfrac{1}{2}\delta))^p$, then $0 < a < 1$ and, for $j \geq n$, $\mu(B_j) \leq a^j$. Let now ϵ be given, $0 < \epsilon < 1$, and let n_ϵ be a natural number sufficiently large so that $\sum_{i=n_\epsilon}^\infty a^i \leq \epsilon$. Define the set $A_\epsilon = \bigcup_{i=n_\epsilon}^\infty B_i$. Obviously $\mu(A_\epsilon) \leq \epsilon$ and for $x \in \mathbb{C}A_\epsilon$ and $j \geq n_\epsilon$ we have

$$|F^{(j)}(f; x, \tfrac{1}{2}z_0)/j!|^{1/j} \leq 2 - \tfrac{1}{2}\delta.$$

It follows that $f \in D_{(\epsilon, \frac{1}{2}\delta, n_\epsilon)}^p(z_0)$, and therefore by Proposition 1, (ii) the assertion is proved.

Corollary 2. Suppose that τ is an invertible, ergodic transformation and let U be the unitary operator induced by τ in $L^2(X, \mathfrak{B}, \mu)$. The spectrum of U coincides with the whole unit circle: $\sigma(U) = C$.

Proof. Let us remark that for $|z| < 1$ the resolvent of U^{-1} has the following analytic expansion in the uniform operator topology:

$$(U^{-1} - zI)^{-1} = \sum_{n=0}^{\infty} U^{n+1} \cdot z^n.$$

Let us also remark that $\sigma(U) = \sigma(U^{-1})^{-1}$. The desired result is then an immediate consequence of Corollary 1.

Remarks. (1) The result of Corollary 2 can also be deduced using the Conjugacy Lemma of HALMOS (see [4], p. 77). (2) Under the assumptions of Corollary 2, if U is the invertible isometric operator induced by τ in $L^p(X, \mathfrak{B}, \mu)$ ($1 \leq p \leq \infty$), then $\sigma(U) = C$. The argument is the same.

PART II

1. Preliminaries. Let $\mathfrak{H}$ be an infinite-dimensional Hilbert space and let $\mathfrak{U}$ be the set of all unitary operators on $\mathfrak{H}$. For $U \in \mathfrak{U}$, $f \in \mathfrak{H}$ and $|z| < 1$ we shall consider the vector-valued analytic function

$$(9) \quad F(U; f, z) = \sum_{n=0}^{\infty} U^{n+1} f \cdot z^n.$$

Our purpose is to prove the following theorem:

Theorem 4. For an operator $U \in \mathfrak{U}$ the following three assertions are equivalent: (i) The conjugate class of U with respect to $\mathfrak{U}$, i.e. the set $\{V^{-1}UV \mid V \in \mathfrak{U}\}$, is dense in $\mathfrak{U}$ for the strong operator topology. (ii) The spectrum of U coincides with the whole unit circle $\sigma(U) = C$. (iii) The set R of all elements $f \in \mathfrak{H}$ such that the unit circle C is the natural boundary of the vector-valued analytic function $F(U; f, z)$ defined by (9) is a set of second category in $\mathfrak{H}$, namely an everywhere dense G_δ .

2. Proof of Theorem 4. In what follows we shall denote by $E(\cdot)$ the spectral resolution of U .

(i)→(ii). Let W be a unitary operator with the following two properties: (a) $\sigma(W) = C$; (b) W has a continuous spectrum, *i.e.* if $E_\infty(\cdot)$ is the spectral resolution of W , then $E_\infty(\{\lambda\}) = 0$ for every $\lambda \in C$. Such an operator can be obtained, for instance, as follows: Consider an orthonormal basis $(e_{n,\iota})_{n \in \mathbb{Z}, \iota \in I}$ of $\mathcal{H}$ and define W by the relations $We_{n,\iota} = e_{n+1,\iota}$ for $n \in \mathbb{Z}, \iota \in I$.⁵

Assume that $\sigma(U) \neq C$. By (i) there exists a filtering family $(V_\alpha^{-1}UV_\alpha)_{\alpha \in J}$ such that $\lim_{\alpha \in J} V_\alpha^{-1}UV_\alpha = W$ strongly. For every $\alpha \in J$ let $E_\alpha(\cdot)$ be the spectral resolution of $V_\alpha^{-1}UV_\alpha$. By a classical theorem, since W has continuous spectrum, it follows that $(E_\alpha(A))_{\alpha \in J}$ converges strongly to $E_\infty(A)$ for every open arc $A \subset C$. Since $\sigma(V_\alpha^{-1}UV_\alpha) = \sigma(U) \neq C$ for every $\alpha \in J$, there exists an open, non-void arc $A_0 \subset C$ such that $E_\alpha(A_0) = 0$ for every $\alpha \in J$. It follows that $E_\infty(A_0) = 0$ and this obviously contradicts the assumption (a). Hence the implication (i)→(ii) is proved.

(ii)→(iii). Let Q be a countable, everywhere dense subset of C . For every $z_0 \in Q, n, k = 1, 2, \dots$, let $D_{(k,n)}(z_0)$ be the set

$$\{f \mid f \in \mathcal{H}, \sup_{j \geq n} \|F^{(j)}(U; f, \tfrac{1}{2}z_0)/j!\|^{1/j} \leq 2 - 1/k\}.$$

For every $z_0 \in Q$ and $k = 1, 2, \dots$, let

$$L_k(z_0) = \{z \mid |z - \tfrac{1}{2}z_0| < 1/(2 - 1/k)\}.$$

It is easy to verify that for each $f \in D_{(k,n)}(z_0)$ the vector-valued analytic function $F(U; f, z)$ defined by (9) admits an analytic extension in $L_k(z_0)$. Since for $j = 1, 2, \dots$ the mapping $f \rightarrow \|F^{(j)}(U; f, \tfrac{1}{2}z_0)/j!\|$ is continuous, we deduce immediately that $D_{(k,n)}(z_0)$ is closed. We shall show that $D_{(k,n)}(z_0)$ is nowhere dense in $\mathcal{H}$.

For each $f \in \mathcal{H}$, let $\sigma(f)$ be the spectrum of f ([2], pp. 327–328). We recall that: (c) $\sigma(f) \neq \emptyset$ if $f \neq 0$. (d) For every closed subset Ω of C ,

$$E(\Omega)\mathcal{H} = \{f \mid f \in \mathcal{H}, \sigma(f) \subset \Omega\}.$$

Since $\sigma(U) = C$, $E(C \cap \overline{L_{2k}(z_0)}) \neq 0$. Hence by (c) and (d) there exists a non-zero element $g \in \mathcal{H}$ such that $\emptyset \neq \sigma(g) \subset C \cap \overline{L_{2k}(z_0)}$.

Suppose that $D_{(k,n)}(z_0)$ is not nowhere dense; there exists then an element f_0 belonging to the interior of $D_{(k,n)}(z_0)$. It follows that we can find a scalar $\lambda \neq 0$ such that $f_0 + \lambda g \in D_{(k,n)}(z_0)$. Since

$$F(U; g, z) = \frac{F(U; f_0 + \lambda g, z) - F(U; f_0, z)}{\lambda},$$

we deduce that $F(U; g, z)$ admits an analytic extension in $L_k(z_0)$ in the sense of [2], p. 326. On the other hand, we know that $\emptyset \neq \sigma(g) \subset C \cap \overline{L_{2k}(z_0)} \subset L_k(z_0)$;

⁵ $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ and I denotes here a set of indices.

hence we have reached a contradiction and therefore the assertion is proved.⁶

Let now $R = \mathbf{C} \left(\bigcup_{z \in Q} \bigcup_{n,k=1}^{\infty} D_{(k,n)}(z) \right)$. It is clear that R is a set of second category in $\mathcal{H}$ and it is easy to verify that R coincides with the set of all elements $f \in \mathcal{H}$ such that C is the natural boundary of the vector-valued analytic function $F(U; f, z)$. Therefore the implication (ii) $\rightarrow$ (iii) is proved.

(iii) $\rightarrow$ (ii) is obvious since $\sigma(U) = \sigma(U^{-1})^{-1}$ and since, for $f \in \mathcal{H}$ and $|z| < 1$, $F(U; f, z) = (U^{-1} - zI)^{-1}f$.

(ii) $\rightarrow$ (i). We remark first that the unitary operators of the form

$$(10) \quad S = \lambda_1 P_1 + \cdots + \lambda_n P_n,$$

where $\lambda_1, \dots, \lambda_n \in \mathbf{C}$ and $P_1 \neq 0, \dots, P_n \neq 0$ are hermitian orthogonal projections such that $P_1 + \cdots + P_n = I$, are dense in $\mathcal{U}$ with respect to the uniform operator topology. Therefore, in order to prove (i) it will be enough to show that given S of the form (10) and given the neighborhood

$$(11) \quad N(\epsilon; g_1, \dots, g_p; S) = \{T \mid \|Tg_t - Sg_t\| \leq \epsilon, 1 \leq t \leq p\}$$

we can find a unitary operator V_ϵ such that $V_\epsilon^{-1}UV_\epsilon$ belongs to

$$N(\epsilon; g_1, \dots, g_p; S).$$

Let us also remark that: (e) If Ω is an open non-void subset of C , then $E(\Omega)\mathcal{H}$ is infinite-dimensional (because $E(G)\mathcal{H} \neq \{0\}$ for every open non-void set G).

(f) If Ω is a Borel subset of C of diameter less than δ , then, for every $g \in E(\Omega)\mathcal{H}$ and $\lambda \in \Omega$, $\|(U - \lambda I)g\| \leq \delta \|g\|$ (see [3], p. 68).

By (10) we can write $g_t = h_{t1} + \cdots + h_{tn}$ ($h_{ti} \in P_i\mathcal{H}$, $1 \leq j \leq n$) for $1 \leq t \leq p$, where $g_1, \dots, g_p$ are the vectors introduced in (11). Let $\mathfrak{M}_i = [\{h_{1i}, \dots, h_{pi}\}]$ for $1 \leq j \leq n$.⁷ It is then clear that $\dim \mathfrak{M}_i \leq p$ for $1 \leq j \leq n$ and that $\mathfrak{M}_{i'} \perp \mathfrak{M}_{i''}$ for $j' \neq j''$, $1 \leq j', j'' \leq n$.

Let now $M = \sup_{1 \leq i \leq p} \|g_i\|$ and let δ be a constant satisfying the inequality

$$(12) \quad 0 < \delta < \min(\epsilon/nM, \frac{1}{2} \min\{|\lambda' - \lambda''| \mid \lambda' \neq \lambda'', \lambda', \lambda'' \in \sigma(S)\})$$

For $1 \leq j \leq n$, let Ω_j be an open arc of C containing λ_j , of diameter δ . From (e) and (12) we deduce that $E(\Omega_j)\mathcal{H}$ is infinite-dimensional for $1 \leq j \leq n$ and that $E(\Omega_{j'})\mathcal{H} \perp E(\Omega_{j''})\mathcal{H}$ for $j' \neq j''$, $1 \leq j', j'' \leq n$.

Using the above mentioned properties of $\mathfrak{M}_1, \dots, \mathfrak{M}_n$ and $E(\Omega_1)\mathcal{H}, \dots, E(\Omega_n)\mathcal{H}$, we can define a unitary operator V_ϵ such that $V_\epsilon \mathfrak{M}_i \subset E(\Omega_i)\mathcal{H}$ for $1 \leq j \leq n$. Let $1 \leq t \leq p$ be fixed; we have

$$(13) \quad \|V_\epsilon^{-1}UV_\epsilon g_t - Sg_t\| \leq \sum_{i=1}^n \|V_\epsilon^{-1}UV_\epsilon h_{ti} - Sh_{ti}\|.$$

⁶Instead of using the concept of the spectrum of an element, we may use the fact that an operator-valued function is holomorphic for the uniform topology if it is holomorphic for the strong topology (see [6], p. 93).

⁷For a set $A \subset \mathcal{H}$ we denote by $[A]$ the closed linear manifold spanned by A .

Since $Sh_{tj} = \lambda_j h_{tj}$ for $1 \leq j \leq n$, we obtain

$$(14) \quad \begin{aligned} \|V_\epsilon^{-1}UV_\epsilon h_{tj} - Sh_{tj}\| &= \|V_\epsilon^{-1}UV_\epsilon h_{tj} - \lambda_j h_{tj}\| \\ &= \|V_\epsilon^{-1}UV_\epsilon h_{tj} - V_\epsilon^{-1}V_\epsilon(\lambda_j h_{tj})\| = \|UV_\epsilon h_{tj} - \lambda_j V_\epsilon h_{tj}\|. \end{aligned}$$

Using the fact that $V_\epsilon h_{tj} \in E(\Omega_j)\mathcal{H}$ for $1 \leq j \leq n$ and the remark (f) we see that

$$(15) \quad \|UV_\epsilon h_{tj} - \lambda_j V_\epsilon h_{tj}\| \leq \delta \|h_{tj}\| \leq \delta \|g_t\| \leq \delta M.$$

By combining (13), (14), (15) and taking into account (12), we deduce that

$$(16) \quad \|V_\epsilon^{-1}UV_\epsilon g_t - Sg_t\| \leq \delta nM < \epsilon$$

for $1 \leq t \leq p$, or equivalently that $V_\epsilon^{-1}UV_\epsilon \in N(\epsilon; g_1, \dots, g_p; S)$. This completes the proof of Theorem 4.

Remarks. (1) In connection with the implication (ii)→(i) see also the Conjugacy Lemma, [4], p. 77. (2) In connection with the implication (ii)→(iii) see also [8]. (3) Theorem 4 can be generalized in various directions. We shall mention only the following extension of the implication (ii)→(i): Let N be a normal operator on $\mathcal{H}$, let $E(\cdot)$ be its spectral resolution, let $\Delta = \sigma(N)$, and assume that, for every $\lambda \in \Delta$ and neighborhood W_λ of λ , $E(W_\lambda)\mathcal{H}$ is infinite-dimensional. Then the set $\{V^{-1}NV \mid V \in \mathcal{U}\}$ is dense, with respect to the strong operator topology, in the set $\mathfrak{N}_\Delta$ of all normal operators A on $\mathcal{H}$ such that $\sigma(A) \subset \Delta$.

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