On Nonlinear Ordinary Differential Equations Containing a Parameter

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Communicated by A. ERDÉLYI

0. Introduction. We shall present a method of solving a system of differential equations of the type

$$(A) \quad \epsilon^\sigma \frac{dy}{dx} = f(x, y, \epsilon),$$

where σ , ϵ , x and y are a positive integer, a complex parameter, a complex variable and an n -dimensional vector, respectively. One norm of a vector y is given by

$$||y|| = \max \{ |y_j|; j = 1, 2, \dots, n \},$$

where y_j , $j = 1, \dots, n$, are the components of y . Each component of the n -dimensional vector f is assumed to be holomorphic in all its arguments in a region defined by

$$(B) \quad |x| \leq \delta_0, \quad ||y|| \leq \delta'_0, \quad 0 < |\epsilon| \leq \rho_0, \quad |\arg \epsilon| \leq \alpha_0,$$

where δ_0 , δ'_0 , ρ_0 and α_0 are all positive constants. In addition, we shall suppose that f admits for

$$(C) \quad |x| \leq \delta_0, \quad ||y|| \leq \delta'_0$$

a uniformly asymptotic expansion

$$(D) \quad f(x, y, \epsilon) \simeq \sum_{k=0}^{\infty} \hat{f}_k(x, y) \epsilon^k$$

as ϵ tends to zero through the region

$$(E) \quad 0 < |\epsilon| \leq \rho_0, \quad |\arg \epsilon| \leq \alpha_0.$$

The coefficients $\hat{f}_k$ are assumed to be holomorphic in (C).

*The author is a Fulbright and Smith-Mundt grantee. This work was also partially supported by the National Science Foundation, and is based on the author's U.C.L.A. dissertation. The author wishes to express his appreciation to Professor EARL A. CODDINGTON for his guidance and encouragement.

Let

$$(F) \quad f(x, y, \epsilon) = f_0(x, \epsilon) + A(x, \epsilon)y + \sum_{|p| \geq 2} f_p(x, \epsilon)y^p$$

be the expansion of f in powers of $y_1, \dots, y_n$, where p is a set of n nonnegative integers $p_1, \dots, p_n$, $A(x, \epsilon)$ an n by n matrix, f_0 and f_p n -dimensional vectors, and

$$(G) \quad \begin{cases} y^p = y_1^{p_1} y_2^{p_2} \cdots y_n^{p_n}, \\ |p| = p_1 + p_2 + \cdots + p_n. \end{cases}$$

The coefficients $f_0(x, \epsilon)$, $f_p(x, \epsilon)$, and $A(x, \epsilon)$ are holomorphic in all their arguments for

$$(H) \quad |x| \leq \delta_0, \quad 0 < |\epsilon| \leq \rho_0, \quad |\arg \epsilon| \leq \alpha_0.$$

On the other hand, since f admits for (C) a uniformly asymptotic expansion (D) as ϵ tends to zero through the region (E), the coefficients of the series (F) also admit for

$$(I) \quad |x| \leq \delta_0$$

uniformly asymptotic expansions

$$(J) \quad \begin{cases} f_0(x, \epsilon) \simeq \sum_{k=0}^{\infty} f_{0k}(x) \epsilon^k, \\ f_p(x, \epsilon) \simeq \sum_{k=0}^{\infty} f_{pk}(x) \epsilon^k, \\ A(x, \epsilon) \simeq \sum_{k=0}^{\infty} A_k(x) \epsilon^k \end{cases}$$

as ϵ tends to zero through the region (E). The coefficients of these expansions are holomorphic for (I).

We shall suppose that the function f_{00} is identically equal to zero for (I):

$$(K) \quad f_{00}(x) \equiv 0.$$

Roughly speaking, this implies that $f(x, y, \epsilon)$ satisfies the condition

$$0 = f(x, 0, 0).$$

In other words, the degenerate equation

$$(L) \quad 0 = f(x, y, 0)$$

admits a solution

$$(M) \quad y \equiv 0.$$

This assumption is, roughly speaking, equivalent to the assumption that the degenerate equation (L) admits a solution holomorphic in x for (I).

Under suitable assumptions which we shall give later on, if the independent variable x is confined to a certain angular region of the form

$$(N) \quad 0 < |x| \leq \delta, \quad l \leq \arg x \leq l',$$

where δ , l , and l' are constants, δ being positive, the general solution of the system (A) uniformly approaches the solution (M) of the degenerate equation (L) as ϵ tends to zero through a region of the form

$$(E') \quad 0 < |\epsilon| \leq \rho, \quad |\arg \epsilon| \leq \alpha,$$

where ρ , α are positive constants.

The main purpose of this paper is to show how to construct the general solution of the system (A) for such a region as is indicated above.

Suppose the matrix $A_0(0)$ is nonsingular. Then we can determine a formal solution

$$y = q(x, \epsilon) = \sum_{k=1}^{\infty} p_k(x) \epsilon^k$$

of the system (A) by solving the sequence of equations

$$\begin{aligned} A_0(x)p_1(x) &= -f_{01}(x), \\ A_0(x)p_k(x) &= -\left\{f_{0k}(x) + H_k(x) - \frac{dp_{k-\sigma}(x)}{dx}\right\} \quad (k = 2, 3, \dots) \end{aligned}$$

successively, where $H_k(x)$ are defined by

$$f(x, q(x, \epsilon), \epsilon) - f_0(x, \epsilon) - A_0(x)q(x, \epsilon) = \sum_{k=2}^{\infty} H_k(x) \epsilon^k.$$

The sequence of equations given above is equivalent to the equation

$$\epsilon^{\sigma} \frac{dq(x, \epsilon)}{dx} = f(x, q(x, \epsilon), \epsilon).$$

Thus, by the use of a lemma which was proved in one of our previous papers [13], we obtain the following result.

Theorem 1. *If the matrix $A_0(0)$ is nonsingular, then there exists a solution of the system (A)*

$$(O) \quad y = p(x, \epsilon)$$

such that p is holomorphic in all its arguments for a region of the form

$$(H') \quad |x| \leq \delta', \quad 0 < |\epsilon| \leq \rho', \quad |\arg \epsilon| \leq \alpha',$$

where δ' , ρ' , and α' are positive constants, and such that p admits for $|x| \leq \delta'$ a uniformly asymptotic expansion

$$(P) \quad p(x, \epsilon) \simeq \sum_{k=1}^{\infty} p_k(x) \epsilon^k$$

as ϵ tends to zero through the region $0 < |\epsilon| \leq \rho'$, $|\arg \epsilon| \leq \alpha'$.

By a transformation of the form

$$(Q) \quad y = u + p(x, \epsilon)$$

the system (A) is reduced to a system

$$(A') \quad \epsilon^\sigma \frac{du}{dx} = g(x, u, \epsilon).$$

where the right-hand member g satisfies conditions similar to those satisfied by f , and the expansion of g in powers of $u_1, \dots, u_n$ does not contain the constant term. Thus g is of the form

$$(R) \quad g(x, u, \epsilon) = A(x, \epsilon)u + \sum_{|p| \geq 2} g_p(x, \epsilon)u^p.$$

The matrix A is different from that of the expansion (F) of f , but the difference is $O(\epsilon)$. Suppose that $A_0(0)$ has the Jordan canonical form with the characteristic values $\mu_1, \dots, \mu_n$.

Let us rewrite the equations (A') in the following form:

$$(A'') \quad \epsilon^\sigma \frac{du_j}{dx} = \sum_{k=1}^n a_{jk}(x, \epsilon)u_k + \sum_{|p| \geq 2} g_{jp}(x, \epsilon)u^p \quad (j = 1, \dots, n).$$

where a_{jk} and g_{jp} are the components of A and g_p respectively. Due to a result which we have proved in one of our previous papers [13], we can suppose without loss of generality that the components a_{jk} of the matrix $A(x, \epsilon)$ satisfy the conditions

$$(S) \quad a_{jk}(x, \epsilon) \equiv 0 \quad (\mu_j \neq \mu_k).$$

Let

$$(T) \quad u = P(x, z, \epsilon)$$

be a transformation of the vector u such that $P(x, z, \epsilon)$ can be represented by a uniformly and absolutely convergent series of the form

$$(U) \quad P(x, z, \epsilon) = z + \sum_{|p| \geq 2} P_p(x, \epsilon)z^p$$

in a region of the form

$$(B') \quad |x| \leq \delta'', \quad ||z|| \leq \delta''', \quad 0 < |\epsilon| \leq \rho'', \quad |\arg \epsilon| \leq \alpha'',$$

with the coefficients P_p holomorphic for

$$(H'') \quad |x| \leq \delta'', \quad 0 < |\epsilon| \leq \rho'', \quad |\arg \epsilon| \leq \alpha''$$

and admitting for $|x| \leq \delta''$ uniformly asymptotic expansions

$$(V) \quad P_p(x, \epsilon) \simeq \sum_{k=0}^{\infty} P_{pk}(x) \epsilon^k$$

as ϵ tends to zero through the region $0 < |\epsilon| \leq \rho''$, $|\arg \epsilon| \leq \alpha''$, where δ'' , ρ'' , α'' , and δ''' are positive constants.

We shall prove the following theorem.

Theorem 2. *Suppose that the numbers μ_j , $j = 1, \dots, n$, lie in the interior of a halfplane bounded by a straight line through the origin. Then there exists a transformation of the type (T) by which the system (A') is reduced to a system of the form*

$$(A''') \quad \epsilon^\sigma \frac{dz_i}{dx} = g_i(x, z, \epsilon) = \sum_{k=1}^n a_{ik}(x, \epsilon) z_k + \sum_{|p| \geq 2} g_{ip}(x, \epsilon) z^p \quad (j = 1, \dots, n)$$

with the coefficients g_{ip} satisfying the conditions

$$(S') \quad g_{ip}(x, \epsilon) \equiv 0 \quad (\mu_{ip} \neq 0),$$

where

$$(W) \quad \mu_{ip} = \mu_i - \sum_{k=1}^n p_k \mu_k.$$

We can suppose without loss of generality that the real parts of the μ_i are all positive. Let

$$(X) \quad \begin{aligned} &0 < \operatorname{Re} \mu_1 = \operatorname{Re} \mu_2 = \dots = \operatorname{Re} \mu_{m_1} \\ &< \operatorname{Re} \mu_{1+m_1} = \operatorname{Re} \mu_{2+m_1} = \dots = \operatorname{Re} \mu_{m_2} \\ &< \dots \quad \dots \quad \dots \quad \dots \\ &< \operatorname{Re} \mu_{1+m_v} = \operatorname{Re} \mu_{2+m_v} = \dots = \operatorname{Re} \mu_{m_v+1} \\ &< \dots \quad \dots \quad \dots \quad \dots \\ &< \operatorname{Re} \mu_{1+m_k} = \operatorname{Re} \mu_{2+m_k} = \dots = \operatorname{Re} \mu_{m_k+1} \quad (m_{1+k} = n). \end{aligned}$$

Then by use of the conditions (S) and (S') we can show that the right-hand members of the equations (A''') are all polynomials in $z_1, \dots, z_n$ of the form

$$g_i(x, z, \epsilon) = \begin{cases} \sum_{\mu_k = \mu_i} a_{ik}(x, \epsilon) z_k & (j = 1, \dots, m_1), \\ \sum_{\mu_k = \mu_i} a_{ik}(x, \epsilon) z_k + G_j(x, z_1, \dots, z_{m_v}, \epsilon) & (j = 1 + m_v, \dots, m_{v+1}). \end{cases}$$

Therefore the general solution of the system (A''') can be obtained by solving linear ordinary differential equations successively.

For example, if $x = 0$ is not a transition point [17] for the system of linear differential equations

$$(Y) \quad \epsilon^\sigma \frac{dy}{dx} = A(x, \epsilon)y,$$

we can solve the system (A''') by making use of the results due to M. HUKUHARA [3] and H. L. TURRITTIN [16].

In particular, if the μ_i are distinct from each other, the system (A''') can be solved simply by successive integration. In addition, if $\mu_{ip} \neq 0$ for all indices (j, p) , the system (A''') has the diagonal form:

$$\epsilon^\sigma \frac{dz_i}{dx} = \mu_i(x, \epsilon) z_i \quad (j = 1, \dots, n).$$

This case was treated by W. WASOW [18].

On the other hand, even if $x = 0$ is a transition point for the system (Y), we can solve the system (A''') for those cases where, for example, the results due to R. E. LANGER [6, 7, 8, 9, 10] are applicable.

After obtaining the general solution of the system (A'''), we can construct the general solution of the system (A') by substituting the solution of (A''') into the transformation (T). In doing so, it is necessary to estimate the magnitude of the solution of the system (A'''), and we shall find a region of the type (N) where the solution is uniformly bounded and tends to zero as ϵ tends to zero through a region of the type (E'). The general solution of the original system (A) is given by

$$(Z) \quad y = p(x, \epsilon) + P(x, \phi(x, C, \epsilon), \epsilon),$$

where ϕ is the general solution of the system (A''') and C an arbitrary constant vector. Thus we attain our main objective.

The results of this paper are extensions of the results concerning the theory of singular perturbations of linear ordinary differential equations which are due to M. HUKUHARA [3], W. J. TRJITZINSKY [14] and H. L. TURRITTIN [16], as well as the results due to W. WASOW [18] which we have indicated above.

On the other hand, there are points of resemblance between our results and the results concerning the problem of irregular singular points which are due to M. HUKUHARA [4], M. IWANO [5], J. MALMQUIST [11] and W. J. TRJITZINSKY [15]. However, our method is essentially different from theirs. In fact, a method similar to theirs can be applied to the present problem, but only for the case where $x = 0$ is not a transition point for the system (Y), whereas our method is equally applicable to the case when $x = 0$ is a transition point.

Our results also can be regarded as extensions of the classical theory of critical points of nonlinear ordinary differential equations which is mainly due to H. DULAC, J. HORN, A. LIAPUNOFF, E. PICARD, and H. POINCARÉ [1, 2].

1. Asymptotic expansion of solutions of nonhomogeneous linear ordinary differential equations. We shall start by giving lemmas concerning a system of nonhomogeneous linear equations

$$(1.1) \quad \epsilon^\sigma \frac{dy}{dx} = B(x, \epsilon)y + b(x, \epsilon),$$

where σ and $B(x, \epsilon)$ are a positive integer and an n by n matrix, respectively, whereas y and $b(x, \epsilon)$ are n -dimensional vectors. The components of b and B are assumed to be holomorphic in all variables for

$$(1.2) \quad |x| \leq \delta_1, \quad 0 < |\epsilon| \leq \rho_1, \quad |\arg \epsilon| \leq \alpha_1,$$

and they are also assumed to admit uniformly asymptotic expansions

$$(1.3) \quad b(x, \epsilon) \simeq \sum_{k=0}^{\infty} b_k(x) \epsilon^k$$

and

$$(1.4) \quad B(x, \epsilon) \simeq \sum_{k=0}^{\infty} B_k(x) \epsilon^k$$

as ϵ tends to zero, where δ_1 , ρ_1 , and α_1 are positive constants. The coefficients b_k and B_k are holomorphic in x for $|x| \leq \delta_1$.

We suppose that the matrix $B_0(0)$ has the form

$$(1.5) \quad B_0(0) = \lambda E + D,$$

where λ is a constant distinct from zero, E the unit-matrix of order n , and D a constant matrix whose components on or above the principal diagonal are all equal to zero. The magnitude of the components of the matrix D can be assumed to be sufficiently small.

Another norm of a vector y is given by

$$(1.6) \quad |||y||| = \sum_{i=1}^n |y_i|.$$

The corresponding norm of a matrix A is given by

$$(1.7) \quad |||A||| = \sup \{ |||Ay|||; |||y||| = 1 \}.$$

Put

$$(1.8) \quad \omega_0 = \arg \lambda.$$

There exists a constant Ω such that

$$(1.9) \quad |\omega_0 - \Omega| < \frac{1}{2}\pi.$$

Put

$$(1.10) \quad x^* = \delta_2 e^{-i\Omega},$$

where δ_2 is a positive constant, and let α and β be two positive constants such that

$$(1.11) \quad K \equiv |\omega_0 - \Omega| + \sigma\alpha + \beta < \frac{1}{2}\pi.$$

Define two domains $\mathfrak{D}_1(\delta_2)$ and $\mathfrak{D}_2(\rho_2)$ by

$$(1.12) \quad \mathfrak{D}_1(\delta_2): |x| < \delta_2, \quad |\arg(x - x^*) - \pi + \Omega| < \beta,$$

and

$$(1.13) \quad \mathfrak{D}_2(\rho_2): 0 < |\epsilon| \leq \rho_2, \quad |\arg \epsilon| \leq \alpha,$$

where ρ_2 is a positive constant.

If $x \in \mathfrak{D}_1(\delta_2)$ and $\epsilon \in \mathfrak{D}_2(\rho_2)$, we have

$$(1.14) \quad \pi - K < \arg [\lambda(x - x^*)/\epsilon^\sigma] < \pi + K.$$

In fact, if we put

$$(1.15) \quad \theta = \arg(x - x^*), \quad \omega = \arg \epsilon,$$

we have

$$(1.16) \quad \arg [\lambda(x - x^*)/\epsilon^\sigma] = \omega_0 + \theta - \sigma\omega.$$

On the other hand,

$$(1.17) \quad \pi - \Omega - \beta < \theta < \pi + \beta - \Omega$$

and

$$(1.18) \quad -\sigma\alpha \leq -\sigma\omega \leq \sigma\alpha.$$

Then

$$(1.19) \quad \pi + \omega_0 - \Omega - (\beta + \sigma\alpha) < \omega_0 + \theta - \sigma\omega < \pi + \omega_0 - \Omega + \beta + \sigma\alpha.$$

This implies

$$(1.20) \quad \pi - (|\omega_0 - \Omega| + \beta + \sigma\alpha) < \omega_0 + \theta - \sigma\omega < \pi + (|\omega_0 - \Omega| + \beta + \sigma\alpha).$$

Hence we obtain the inequality (1.14).

We shall assume δ_2 and ρ_2 to be so small that

$$(1.21) \quad |||B(x, \epsilon) - \lambda E||| \leq \frac{1}{2} |\lambda| \cos K < \frac{1}{2} |\lambda|$$

for

$$(1.22) \quad x \in \mathfrak{D}_1(\delta_2), \quad \epsilon \in \mathfrak{D}_2(\rho_2).$$

Let $\Phi(x, \epsilon)$ be an n by n matrix which satisfies the equation

$$(1.23) \quad \epsilon^\sigma \frac{d\Phi(x, \epsilon)}{dx} = B(x, \epsilon)\Phi(x, \epsilon)$$

and the condition

$$(1.24) \quad \Phi(x^*, \epsilon) = E.$$

Put

$$(1.25) \quad p(x, \epsilon) = \frac{1}{\epsilon^\sigma} \Phi(x, \epsilon) \int_{x^*}^x \Phi^{-1}(t, \epsilon) b(t, \epsilon) dt.$$

The vector p is a solution of the system (1.1).

Lemma 1. *The solution p defined by (1.25) admits for each compact subset of $\mathfrak{D}_1(\delta_2)$ a uniformly asymptotic expansion*

$$(1.26) \quad p(x, \epsilon) \simeq \sum_{k=0}^{\infty} p_k(x) \epsilon^k$$

as ϵ tends to zero through $\mathfrak{D}_2(\rho_2)$, where the coefficients p_k are holomorphic in x for $\mathfrak{D}_1(\delta_2)$. We have also the estimate

$$(1.27) \quad |||p(x, \epsilon)||| \leq \frac{2}{|\lambda| \cos K} \sup_{t \in \mathfrak{D}_1(\delta_2)} |||b(t, \epsilon)|||$$

for $x \in \mathfrak{D}_1(\delta_2)$, $\epsilon \in \mathfrak{D}_2(\rho_2)$.

Proof. Since the matrix

$$(1.28) \quad \Psi(x, \epsilon) = \exp \{-\lambda(x - x^*)/\epsilon^\sigma\} \Phi(x, \epsilon)$$

satisfies the equation

$$(1.29) \quad \epsilon^\sigma \frac{d\Psi(x, \epsilon)}{dx} = (B(x, \epsilon) - \lambda E)\Psi(x, \epsilon),$$

we have the estimate

$$(1.30) \quad |||\Psi(x, \epsilon)||| \leq \exp \{|\lambda| (\cos K) |x - x^*|/2 |\epsilon|^\sigma\}$$

for (1.22). In fact, if we put

$$(1.31) \quad T(\xi, \epsilon) = 1 + \frac{|\lambda| \cos K}{2 |\epsilon|^\sigma} \int_0^\xi |||\Psi(x^* + \xi e^{i\theta}, \epsilon)||| d\xi,$$

where

$$(1.32) \quad \xi = |x - x^*|, \quad \theta = \arg(x - x^*),$$

we can derive the inequality

$$(1.33) \quad |||\Psi(x, \epsilon)||| \leq T(\xi, \epsilon),$$

or

$$(1.34) \quad \frac{dT(\xi, \epsilon)}{d\xi} \leq \frac{|\lambda| \cos K}{2 |\epsilon|^\sigma} T(\xi, \epsilon)$$

for (1.22), from (1.21) and (1.29). This implies (1.30).

We can then obtain the estimate

$$(1.35) \quad |||\Phi(x, \epsilon)||| \leq \exp \{-|\lambda| |x - x^*| (\cos K)/2 |\epsilon|^\sigma\}$$

for (1.22), because (1.14) implies

$$(1.36) \quad \cos(\arg[\lambda(x - x^*)/\epsilon^\sigma]) < -\cos K.$$

We can also obtain the estimate

$$(1.37) \quad |||\Phi(x, \epsilon)\Phi^{-1}(t, \epsilon)||| \leq \exp \{ -|\lambda| |x - t| (\cos K)/2 |\epsilon|^\sigma \}$$

if t is on the segment joining x to x^* .

Since

$$(1.38) \quad |||p(x, \epsilon)||| \leq \frac{1}{|\epsilon|^\sigma} \sup_{t \in \mathfrak{D}_1(\delta_2)} |||b(t, \epsilon)||| \int_0^\xi \exp \{ -|\lambda| s (\cos K)/2 |\epsilon|^\sigma \} ds,$$

we obtain the estimate (1.27).

We shall next show that $B_0(x)$ is nonsingular for $x \in \overline{\mathfrak{D}_1(\delta_2)^*}$. In fact, the inequality (1.12) implies

$$(1.39) \quad |||B_0(x) - \lambda E||| < \frac{1}{2} |\lambda|$$

for $x \in \mathfrak{D}_1(\delta_2)$. Then if $B_0(x)y = 0$, we have

$$|\lambda| |||y||| \leq \frac{1}{2} |\lambda| |||y|||.$$

Hence $y = 0$.

Let

$$(1.40) \quad q(x, \epsilon) = \sum_{k=0}^{\infty} p_k(x) \epsilon^k$$

be a formal power series in ϵ whose coefficients p_k are n -dimensional vectors holomorphic in x for $x \in \mathfrak{D}_1(\delta_2)$. Then the relation

$$(1.41) \quad \epsilon^\sigma \frac{dq(x, \epsilon)}{dx} = B(x, \epsilon)q(x, \epsilon) + b(x, \epsilon)$$

is equivalent to the sequence of equations

$$(1.42) \quad \begin{aligned} B_0(x)p_0(x) &= -b_0(x), \\ B_0(x)p_k(x) &= -b_k(x) - \sum_{h=1}^k B_h(x)p_{k-h}(x) + dp_{k-\sigma}(x)/dx \quad (k \geq 1), \end{aligned}$$

where $p_k(x) = 0$ for $k < 0$. Since $B_0(x)$ is nonsingular, we can determine the series (1.40) so as to satisfy the relation (1.14). Thus we obtain a formal solution of the system (1.1).

Put

$$(1.43) \quad v(x, \epsilon) = p(x, \epsilon) - q^N(x, \epsilon),$$

where

$$(1.44) \quad q^N(x, \epsilon) = \sum_{k=0}^N p_k(x) \epsilon^k.$$

* $\overline{\mathfrak{D}_1(\delta_2)}$ is the closure of $\mathfrak{D}_1(\delta_2)$.

Then we have

$$(1.45) \quad \epsilon^\sigma \frac{dv(x, \epsilon)}{dx} = B(x, \epsilon)v(x, \epsilon) + c(x, \epsilon),$$

where

$$(1.46) \quad c(x, \epsilon) = B(x, \epsilon)q^N(x, \epsilon) + b(x, \epsilon) - \epsilon^\sigma \frac{dq^N(x, \epsilon)}{dx}.$$

From (1.45) we can derive the formula

$$(1.47) \quad v(x, \epsilon) = \Phi(x, \epsilon) \left\{ -q^N(x^*, \epsilon) + \frac{1}{\epsilon^\sigma} \int_{x^*}^x \Phi^{-1}(t, \epsilon) c(t, \epsilon) dt \right\}.$$

By applying the estimate (1.27) to (1.47), we have

$$(1.48) \quad \left\| \left\| \epsilon^{-\sigma} \Phi(x, \epsilon) \int_{x^*}^x \Phi^{-1}(t, \epsilon) c(t, \epsilon) dt \right\| \right\| \leq M \sup_{t \in \mathfrak{D}_1(\delta_2)} \|c(t, \epsilon)\|$$

for (1.22), where M is a certain constant.

On the other hand, q being a formal solution of the system (1.1), we have

$$(1.49) \quad \| \| c(x, \epsilon) \| \| \leq M' |\epsilon|^{N+1}$$

for (1.22), where M' is a certain constant.

On each compact subset of $\mathfrak{D}_1(\delta_2)$, $\Phi(x, \epsilon)$ is $O(|\epsilon|^{N+1})$, because we have obtained the estimate (1.35). Formulas (1.47), (1.48) and (1.49) thus imply

$$(1.50) \quad \| \| v(x, \epsilon) \| \| = O(|\epsilon|^{N+1})$$

for each compact subset of $\mathfrak{D}_1(\delta_2)$. Lemma 1 is thereby proved.

Lemma 2. *If b is holomorphic and bounded for (1.22) and admits for each compact subset of $\mathfrak{D}_1(\delta_2)$ a uniformly asymptotic expansion, then we have the same conclusion as that obtained in Lemma 1.*

Proof. In fact, in order to derive the estimate (1.27) for (1.22), we did not make use of the asymptotic expansion of b . Then under the new assumptions given above, $p(x, \epsilon)$ defined by (1.25) satisfies the inequality (1.27). Then $p(x, \epsilon)$ is bounded for (1.22).

Let $\mathfrak{C}$ be a compact subset of $\mathfrak{D}_1(\delta_2)$. Then there exists a point x^{**} of $\mathfrak{D}_1(\delta_2)$ with absolute value $|x^{**}| = \delta_3$ such that the region defined by

$$(1.51) \quad |x| \leq \delta_3, \quad |\arg(x - x^{**}) - \pi + \Omega| \leq \beta,$$

is a compact subset of $\mathfrak{D}_1(\delta_2)$ and contains the set $\mathfrak{C}$ in its interior.

Let $\tilde{\Phi}(x, \epsilon)$ be a matrix satisfying the equation

$$(1.52) \quad \epsilon^\sigma \frac{d\tilde{\Phi}(x, \epsilon)}{dx} = B(x, \epsilon)\tilde{\Phi}(x, \epsilon)$$

and the condition

$$(1.53) \quad \tilde{\Phi}(x^{**}, \epsilon) = E.$$

Then $p(x, \epsilon)$ can be given the form

$$(1.54) \quad p(x, \epsilon) = \tilde{\Phi}(x, \epsilon) \left\{ p(x^{**}, \epsilon) + \frac{1}{\epsilon^\sigma} \int_{x^{**}}^x \tilde{\Phi}^{-1}(t, \epsilon) b(t, \epsilon) dt \right\}.$$

Since $p(x^{**}, \epsilon)$ is bounded for $\mathfrak{D}_2(\rho_2)$, we can prove Lemma 2 by applying the results in Lemma 1 to the expression (1.54) of $p(x, \epsilon)$.

2. Formal transformation. Now we shall prove Theorem 2. Therefore we shall consider the system (A') (or (A'')). We can suppose without loss of generality that the conditions (S) are satisfied. At first we shall not assume that the characteristic values $\mu_1, \dots, \mu_n$ of the matrix $A_0(0)$ lie in the interior of a half-plane bounded by a straight line through the origin.

We are assuming that $A_0(0)$ has the Jordan canonical form. Therefore $A_0(0)$ can be written in the form

$$(2.1) \quad A_0(0) = \begin{bmatrix} \mu_1 & & & & 0 \\ \epsilon_2 & \mu_2 & & & \\ & \epsilon_3 & \mu_3 & & \\ & & \ddots & \ddots & \\ 0 & & & \epsilon_n & \mu_n \end{bmatrix},$$

where we can suppose that the magnitudes of $\epsilon_2, \dots, \epsilon_n$ are sufficiently small.

Let us consider a nonlinear transformation of the type (T) given in the Introduction which can be represented by a formal power series of the type (U). Let

$$(2.2) \quad \epsilon^\sigma \frac{dz}{dx} = g(x, z, \epsilon)$$

be the system of equations derived from (A') by the nonlinear transformation (T). Then we have

$$(2.3) \quad \epsilon^\sigma P_x(x, z, \epsilon) + \epsilon^\sigma P_z(x, z, \epsilon) \frac{dz}{dx} = g(x, P(x, z, \epsilon), \epsilon),$$

where P_z denotes the matrix of partial derivatives $P_z = (\partial P_k / \partial z_i)$.

Put

$$(2.4) \quad P(x, z, \epsilon) = z + Q(x, z, \epsilon).$$

Then we have

$$(2.5) \quad P_x(x, z, \epsilon) = Q_x(x, z, \epsilon)$$

and

$$(2.6) \quad P_z(x, z, \epsilon) = E + Q_z(x, z, \epsilon).$$

Hence

$$(2.7) \quad (E + Q_z(x, z, \epsilon))\epsilon^\sigma \frac{dz}{dx} = g(x, P(x, z, \epsilon), \epsilon) - \epsilon^\sigma Q_x(x, z, \epsilon)$$

or

$$(2.8) \quad g(x, z, \epsilon) = (E + Q_z(x, z, \epsilon))^{-1}(g(x, P(x, z, \epsilon), \epsilon) - \epsilon^\sigma Q_x(x, z, \epsilon)).$$

Let

$$(2.9) \quad Q(x, z, \epsilon) = \sum_{|p| \geq 2} P_p(x, \epsilon) z^p,$$

$$(2.10) \quad g(x, z, \epsilon) = A(x, \epsilon)z + \sum_{|p| \geq 2} g_p(x, \epsilon) z^p,$$

$$(2.11) \quad g(x, u, \epsilon) = A(x, \epsilon)u + \sum_{|p| \geq 2} \hat{g}_p(x, \epsilon) u^p,$$

$$(2.12) \quad G(x, z, \epsilon) = g(x, z, \epsilon) - A(x, \epsilon)z = \sum_{|p| \geq 2} g_p(x, \epsilon) z^p,$$

$$(2.13) \quad \hat{G}(x, u, \epsilon) = g(x, u, \epsilon) - A(x, \epsilon)u = \sum_{|p| \geq 2} \hat{g}_p(x, \epsilon) u^p,$$

and

$$(2.14) \quad \hat{G}(x, P(x, z, \epsilon), \epsilon) = \sum_{|p| \geq 2} \hat{g}_p(x, \epsilon) [P(x, z, \epsilon)]^p = \sum_{|p| \geq 2} h_p(x, \epsilon) z^p.$$

Then we have

$$\begin{aligned} \epsilon^\sigma Q_x(x, z, \epsilon) &= g(x, P(x, z, \epsilon), \epsilon) - P_z(x, z, \epsilon)g(x, z, \epsilon) \\ &= A(x, \epsilon)P(x, z, \epsilon) + \hat{G}(x, P(x, z, \epsilon), \epsilon) \\ &\quad - g(x, z, \epsilon) - Q_z(x, z, \epsilon)g(x, z, \epsilon) \\ (2.15) \quad &= A(x, \epsilon)z + A(x, \epsilon)Q(x, z, \epsilon) \\ &\quad + \hat{G}(x, P(x, z, \epsilon), \epsilon) - A(x, \epsilon)z \\ &\quad - G(x, z, \epsilon) - Q_z(x, z, \epsilon)A(x, \epsilon)z - Q_z(x, z, \epsilon)G(x, z, \epsilon) \\ &= A(x, \epsilon)Q(x, z, \epsilon) + \hat{G}(x, P(x, z, \epsilon), \epsilon) \\ &\quad - G(x, z, \epsilon) - Q_z(x, z, \epsilon)A(x, \epsilon)z - Q_z(x, z, \epsilon)G(x, z, \epsilon). \end{aligned}$$

Let

$$(2.16) \quad p_z = (p_1 z^{-e_1}, \dots, p_n z^{-e_n}),$$

where $p_1, \dots, p_n$ are nonnegative integers and

$$(2.17) \quad e_k = (\delta_{1k}, \delta_{2k}, \dots, \delta_{nk}) \quad (k = 1, \dots, n)$$

with

$$(2.18) \quad \delta_{hk} = \begin{cases} 1 & (h = k), \\ 0 & (h \neq k). \end{cases}$$

Then we have

$$(2.19) \quad Q_z(x, z, \epsilon) = \sum_{|p| \geq 2} z^p P_p(x, \epsilon) p_z.$$

Therefore

$$(2.20) \quad Q_z(x, z, \epsilon) A(x, \epsilon) z = \sum_{|p| \geq 2} z^p P_p(x, \epsilon) p_z A(x, \epsilon) z.$$

Since

$$(2.21) \quad p_z A(x, \epsilon) z = \sum_{h, k} p_k a_{kh}(x, \epsilon) z^{\epsilon h - \epsilon k},$$

we have

$$\begin{aligned} (2.22) \quad Q_z(x, z, \epsilon) A(x, \epsilon) z &= \sum_{|p| \geq 2} z^p P_p(x, \epsilon) \sum_{k, h} p_k a_{kh}(x, \epsilon) z^{\epsilon h - \epsilon k} \\ &= \sum_{\substack{|p| \geq 2 \\ h, k}} p_k P_p(x, \epsilon) a_{kh}(x, \epsilon) z^{p + \epsilon h - \epsilon k} = \sum_{|p| \geq 2} z^p \sum_{q + \epsilon h - \epsilon k = p} q_k P_q(x, \epsilon) a_{kh}(x, \epsilon) \\ &= \sum_{|p| \geq 2} z^p \left(\sum_{h \neq k} (p_k + 1) a_{kh}(x, \epsilon) P_{p + \epsilon k - \epsilon h}(x, \epsilon) + \sum_{k=1}^n p_k a_{kk}(x, \epsilon) P_p(x, \epsilon) \right). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (2.23) \quad Q_z(x, z, \epsilon) G(x, z, \epsilon) &= \sum_{|r| \geq 2} P_r(x, \epsilon) r_z z^r \sum_{|q| \geq 2} g_q(x, \epsilon) z^q \\ &= \sum_{|r|, |q| \geq 2} P_r(x, \epsilon) r_z g_q(x, \epsilon) z^{r+q} = \sum_{|r|, |q| \geq 2} z^{r+q} P_r(x, \epsilon) \sum_{k=1}^n r_k g_{kq}(x, \epsilon) z^{-\epsilon k} \\ &= \sum_{|r|, |q| \geq 2} P_r(x, \epsilon) r_k g_{kq}(x, \epsilon) z^{r+q-\epsilon k} = \sum_{|p| \geq 3} z^p \sum_{r+q-\epsilon k=p} r_k g_{kq}(x, \epsilon) P_r(x, \epsilon), \end{aligned}$$

where $g_{kq}(x, \epsilon)$ are the components of $g_q(x, \epsilon)$.

Thus we obtain a sequence of equations

$$\begin{aligned} (2.24) \quad \epsilon^\sigma \frac{dP_p(x, \epsilon)}{dx} &= A(x, \epsilon) P_p(x, \epsilon) - \sum_{k=1}^n p_k a_{kk}(x, \epsilon) P_p(x, \epsilon) \\ &\quad - \sum_{h \neq k} (p_k + 1) a_{kh}(x, \epsilon) P_{p + \epsilon k - \epsilon h}(x, \epsilon) \\ &\quad + h_p(x, \epsilon) - g_p(x, \epsilon) - \sum_{r+q-\epsilon k=p} r_k g_{kq}(x, \epsilon) P_r(x, \epsilon), \end{aligned}$$

where the last term must be omitted for $|p| = 2$.

Let us rewrite the equations (2.24) in the following form

$$\begin{aligned}
 \epsilon^\sigma \frac{dP_{ip}(x, \epsilon)}{dx} &= \sum_{k=1}^n a_{ik}(x, \epsilon) P_{kp}(x, \epsilon) - \sum_{k=1}^n p_k a_{kk}(x, \epsilon) P_{ip}(x, \epsilon) \\
 (2.25) \quad &- \sum_{h \neq k} (p_k + 1) a_{kh}(x, \epsilon) P_{ip+\epsilon_k-\epsilon_h}(x, \epsilon) \\
 &+ h_{ip}(x, \epsilon) - g_{ip}(x, \epsilon) - \sum_{r+q-\epsilon_k=p} r_k g_{kq}(x, \epsilon) P_{ir}(x, \epsilon),
 \end{aligned}$$

where $P_{ip}(x, \epsilon)$, $h_{ip}(x, \epsilon)$ and $g_{ip}(x, \epsilon)$ are the components of $P_p(x, \epsilon)$, $h_p(x, \epsilon)$ and $g_p(x, \epsilon)$, respectively. It should be noted that the last term of the right-hand member of (2.25) must be omitted for $|p| = 2$.

Put

$$(2.26) \quad \mu_{ip} = \mu_i - \sum_{k=1}^n p_k \mu_k,$$

and let us classify indices (j, p) according to the relations

$$(2.27) \quad \mu_{ip} = \mu_{kq}, \quad |p| = |q|.$$

Precisely speaking two indices (j, p) and (k, q) belong to the same group if and only if they satisfy the conditions (2.27).

We shall consider the case $|p| = 2$. In this case the equations (2.25) have the form

$$\begin{aligned}
 \epsilon^\sigma \frac{dP_{ip}(x, \epsilon)}{dx} &= \sum_{k=1}^n a_{ik}(x, \epsilon) P_{kp}(x, \epsilon) - \sum_{k=1}^n p_k a_{kk}(x, \epsilon) P_{ip}(x, \epsilon) \\
 (2.25-2) \quad &- \sum_{h \neq k} (p_k + 1) a_{kh}(x, \epsilon) P_{ip+\epsilon_k-\epsilon_h}(x, \epsilon) + h_{ip}(x, \epsilon) - g_{ip}(x, \epsilon).
 \end{aligned}$$

By the definition of $h_p(x, \epsilon)$, if $|p| = 2$,

$$(2.28) \quad h_p(x, \epsilon) = g_p(x, \epsilon).$$

Let us write the equations (2.25-2) in the following form:

$$(2.25-2') \quad \epsilon^\sigma \frac{dP_{ip}(x, \epsilon)}{dx} = \sum_{(h, q)} S_{ip, hq}(x, \epsilon) P_{hq}(x, \epsilon) + g_{ip}(x, \epsilon) - g_{ip}(x, \epsilon).$$

According to the relations (2.27), we can classify those indices (j, p) for which $|p| = 2$ into groups. Denote those groups by $N_1, N_2, \dots, N_l$. By virtue of the conditions (S), if $a_{ik}(x, \epsilon) \neq 0$, we have $\mu_i = \mu_k$. Then $\mu_{kp} = \mu_{ip}$. We can also prove the fact that $\mu_{ip+\epsilon_k-\epsilon_h} = \mu_{ip}$ if $a_{kh}(x, \epsilon) \neq 0$. Hence

$$(2.29) \quad S_{ip, hq}(x, \epsilon) = 0$$

if (j, p) and (h, q) do not belong to the same group. In each group N_ν , we shall arrange indices (j, p) in the linear order by stipulating that (j, p) comes before (k, q) if the first nonzero element of $(j, p) - (k, q)$ is negative.

Let $U_\nu(x, \epsilon)$ be a vector whose components are those $P_{ip}(x, \epsilon)$ whose indices (j, p) belong to the group N_ν . The components of $U_\nu(x, \epsilon)$ are assumed to be arranged in the linear order specified above.

Let $S_\nu(x, \epsilon)$ be a square matrix whose components are those $S_{i\nu, kq}(x, \epsilon)$ for which (j, p) and (k, q) belong to the group N_ν . The components of $S_\nu(x, \epsilon)$ are also assumed to be arranged according to the linear order specified above.

Then the equations (2.25-2') can be given the form

$$(2.25-2-\nu) \quad \epsilon^\sigma \frac{dU_\nu(x, \epsilon)}{dx} = S_\nu(x, \epsilon) U_\nu(x, \epsilon) + b_\nu(x, \epsilon) \quad (\nu = 1, \dots, l),$$

where $b_\nu(x, \epsilon)$ is a vector whose components are those $g_{i\nu}(x, \epsilon) - g_{i\nu}(x, \epsilon)$ for which (j, p) belongs to the group N_ν .

Since all indices (j, p) belonging to the same group N_ν have the value of $\mu_{i\nu}$ in common, we shall denote by $\mu(\nu)$ this common value.

Let

$$(2.30) \quad S_\nu(x, \epsilon) \simeq \sum_{k=0}^{\infty} S_{\nu, k}(x) \epsilon^k.$$

Then the matrix $S_{\nu, 0}(0)$ has the form

$$(2.31) \quad S_{\nu, 0}(0) = \mu(\nu) E^{(\nu)} + D^{(\nu)},$$

where $E^{(\nu)}$ is the unit-matrix and $D^{(\nu)}$ is a constant matrix whose components on or above the principal diagonal are equal to zero. In fact, we have

$$(2.32) \quad S_{i\nu, i\nu}(x, \epsilon) = a_{i i}(x, \epsilon) - \sum_{k=1}^n p_k a_{k k}(x, \epsilon) = \mu_{i\nu} + O(|x| + |\epsilon|).$$

On the other hand

$$(2.33) \quad S_{i\nu, kq}(x, \epsilon) = O(|x| + |\epsilon|)$$

if (j, p) comes before (k, q) , because $a_{kh}(x, \epsilon)$ satisfies the condition

$$(2.34) \quad a_{kh}(x, \epsilon) = O(|x| + |\epsilon|)$$

if $k < h$.

Now we can determine $P_{i\nu}(x, \epsilon)$ and $g_{i\nu}(x, \epsilon)$ by using Lemma 1. For (j, p) such that $\mu_{i\nu} \neq 0$ put

$$(2.35) \quad g_{i\nu}(x, \epsilon) = 0,$$

and for (j, p) such that $\mu_{i\nu} = 0$ put

$$(2.36) \quad P_{i\nu}(x, \epsilon) = 0.$$

Then, if $(j, p) \in N_\nu$ and $\mu_{i\nu} = \mu(\nu) \neq 0$, we can determine $P_{i\nu}(x, \epsilon)$ by solving the equations (2.25-2- ν). In doing so, we can make use of Lemma 1. On the other hand, if $(j, p) \in N_\nu$ and $\mu_{i\nu} = \mu(\nu) = 0$, we can determine $g_{i\nu}(x, \epsilon)$ by the relations

$$(2.37) \quad b_\nu(x, \epsilon) = 0$$

or

$$(2.38) \quad g_{i\nu}(x, \epsilon) = \hat{g}_{i\nu}(x, \epsilon).$$

In general, $h_p(x, \epsilon)$ and the last term of the right-hand member of the equation (2.24) contain only those $P_q(x, \epsilon)$ for which $|q| < |p|$. Therefore the same method as given above can be applied and we obtain the following lemma.

Lemma 3. *We can determine the formal transformation (T) so that the coefficients $P_p(x, \epsilon)$ admit uniformly asymptotic expansions in certain domains and so that the coefficients $g_p(x, \epsilon)$ of the equations (2.2) satisfy the conditions*

$$(2.39) \quad g_{ip}(x, \epsilon) = 0 \quad (\mu_{ip} \neq 0).$$

Remark. The domain for which $P_{ip}(x, \epsilon)$ admits a uniformly asymptotic expansion generally depends on the index (j, p) .

3. Domain of convergence. Let us consider the equations (2.25) for $|p| = m$. Let us write the equations (2.25) in the following form:

$$(3.1-m) \quad \epsilon^\sigma \frac{dP_{ip}(x, \epsilon)}{dx} = \sum_{(k, q)} S_{ip, kq}(x, \epsilon) P_{kq}(x, \epsilon) + b_{ip}(x, \epsilon) - g_{ip}(x, \epsilon),$$

where

$$(3.2) \quad b_{ip}(x, \epsilon) = h_{ip}(x, \epsilon) - \sum_{r+q-\epsilon k=p} r_k g_{kq}(x, \epsilon) P_{ir}(x, \epsilon).$$

According to the relations (2.27), we can classify those indices (j, p) for which $|p| = m$ into groups. Denote those groups by $N_1, N_2, \dots, N_t$.

As we have seen in the preceding section, we can prove the fact that

$$(3.3) \quad S_{ip, kq}(x, \epsilon) = 0$$

if (j, p) and (h, q) do not belong to the same group.

In each group N_ν , we shall arrange indices (j, p) in the linear order specified in the preceding section.

Let $U_\nu(x, \epsilon)$ be a vector whose components are those $P_{ip}(x, \epsilon)$ such that (j, p) belongs to the group N_ν . The components of $U_\nu(x, \epsilon)$ are assumed to be arranged in the linear order specified above.

Let $S_\nu(x, \epsilon)$ be a square matrix whose components are those $S_{ip, kq}(x, \epsilon)$ for which (j, p) and (k, q) belong to the group N_ν . The components of $S_\nu(x, \epsilon)$ are also assumed to be arranged according to the linear order specified above.

Then the equations (3.1-m) can be given the form

$$(3.1-m-\nu) \quad \epsilon^\sigma \frac{dU_\nu(x, \epsilon)}{dx} = S_\nu(x, \epsilon) U_\nu(x, \epsilon) + b_\nu(x, \epsilon) - g_\nu(x, \epsilon) \quad (\nu = 1, \dots, l),$$

where $b_\nu(x, \epsilon)$ and $g_\nu(x, \epsilon)$ are vectors whose components are those $b_{ip}(x, \epsilon)$ and $g_{ip}(x, \epsilon)$ for which (j, p) belongs to the group N_ν , respectively.

Since all indices (j, p) belonging to the same group N_ν have the value of μ_{ip} in common, we shall denote by $\mu(\nu)$ this common value. Then, if we write $S_\nu(x, \epsilon)$ in the form

$$(3.4) \quad S_\nu(x, \epsilon) \simeq \sum_{k=0}^{\infty} S_{\nu, k}(x) \epsilon^k,$$

the matrix $S_{\nu 0}(0)$ has the form

$$(3.5) \quad S_{\nu 0}(0) = \mu(\nu)E^{(\nu)} + D^{(\nu)},$$

where $E^{(\nu)}$ is the unit-matrix and $D^{(\nu)}$ a constant matrix whose components on or above the principal diagonal are equal to zero.

Let us suppose

$$(3.6) \quad \mu(\nu) \neq 0.$$

Put

$$(3.7) \quad \omega_{\nu} = \arg \mu(\nu)$$

and let Ω_{ν} be a positive constant satisfying the condition

$$(3.8) \quad |\omega_{\nu} - \Omega_{\nu}| < \frac{1}{2}\pi.$$

Let α_{ν} and β_{ν} be two positive constants such that

$$(3.9) \quad K_{\nu} \equiv |\omega_{\nu} - \Omega_{\nu}| + \sigma\alpha_{\nu} + \beta_{\nu} < \frac{1}{2}\pi.$$

Define two domains $\mathfrak{D}_1(\delta^{(\nu)})$ and $\mathfrak{D}_2(\rho^{(\nu)})$ by

$$(3.10) \quad \mathfrak{D}_1(\delta^{(\nu)}): |x| < \delta^{(\nu)}, \quad |\arg(x - x_{\nu}^*) - \pi + \Omega_{\nu}| < \beta_{\nu}$$

and

$$(3.11) \quad \mathfrak{D}_2(\rho^{(\nu)}): 0 < |\epsilon| \leq \rho^{(\nu)}, \quad |\arg \epsilon| \leq \alpha_{\nu},$$

where

$$(3.12) \quad x_{\nu}^* = \delta^{(\nu)}e^{-i\Omega_{\nu}},$$

$\delta^{(\nu)}$ and $\rho^{(\nu)}$ being positive constants.

We shall choose $\delta^{(\nu)}$ and $\rho^{(\nu)}$ so small that we have

$$(3.13) \quad |||S_{\nu}(x, \epsilon) - \mu(\nu)E^{(\nu)}||| \leq \frac{1}{2}(|\mu(\nu)| \cos K_{\nu})$$

for

$$(3.14) \quad x \in \mathfrak{D}_1(\delta^{(\nu)}), \quad \epsilon \in \mathfrak{D}_2(\rho^{(\nu)}).$$

It should be noted here that $g_{\nu}(x, \epsilon) = 0$ if $\mu(\nu) \neq 0$, because of Lemma 3.

Then if the components of b_{ν} are holomorphic and bounded for (3.14) and admit for each compact subset of $\mathfrak{D}_1(\delta^{(\nu)})$ a uniformly asymptotic expansion, U_{ν} can be determined so as to admit for each compact subset of $\mathfrak{D}_1(\delta^{(\nu)})$ a uniformly asymptotic expansion and satisfy the inequality

$$(3.15) \quad |||U_{\nu}(x, \epsilon)||| \leq \frac{2}{|\mu(\nu)| \cos K_{\nu}} \sup_{t \in \mathfrak{D}_1(\delta^{(\nu)})} |||b_{\nu}(t, \epsilon)|||$$

for (3.14).

In general, for the matrix

$$C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

the inequalities

$$\sum_{j=1}^n |c_{jk}| \leq M \quad (k = 1, \dots, n)$$

imply

$$|||C||| \leq M.$$

Then if we have

$$(3.16) \quad \left| a_{ji}(x, \epsilon) - \sum_{k=1}^n p_k a_{kk}(x, \epsilon) - \mu_{j\nu} \right| \leq \sum_{k \neq j} |a_{ki}(x, \epsilon)| + \sum_{k \neq k} p_k |a_{kk}(x, \epsilon)| \leq \frac{1}{2}(|\mu_{j\nu}| \cos K_\nu)$$

for all (j, p) belonging to the group N_ν , we have the inequality (3.13).

Hereafter we shall suppose that the characteristic values $\mu_1, \dots, \mu_n$ of the matrix $A_0(0)$ lie in the interior of a halfplane bounded by a straight line through the origin.

Since $\sum_{i=1}^n p_i \mu_i / |p|$ lies in the interior of the convex hull of the points $\mu_1, \dots, \mu_n$, the constant Ω_ν can be chosen independent of ν and m . Therefore α_ν and β_ν can also be chosen independent of ν and m . Let us denote by Ω, α , and β those Ω_ν, α_ν , and β_ν respectively. Further, we can suppose that there exists a positive constant γ such that

$$(3.17) \quad K_\nu \equiv |\omega_\nu - \Omega| + \sigma\alpha + \beta < \frac{1}{2}\pi - \gamma \equiv K.$$

On the other hand, according to H. POINCARÉ, there exists a certain positive constant $R (> 1)$ such that

$$(3.18) \quad \frac{|p|}{|\mu_{j\nu}|} < R$$

if $|p|$ is sufficiently large [12]. In fact, under the assumptions there exists a constant ζ such that

$$(3.19) \quad \operatorname{Re}(e^{i\zeta} \mu_j) > 0 \quad (j = 1, \dots, n).$$

Therefore we have

$$(3.20) \quad |\mu_{j\nu}| \geq \operatorname{Re} \left(e^{i\zeta} \sum_{h=1}^n \mu_h p_h \right) - |\mu_j| \geq |p| \min_{h=1}^n \operatorname{Re}(e^{i\zeta} \mu_h) - |\mu_j|.$$

This implies (3.18) for sufficiently large $|p|$.

We incidentally have proved that $\mu_{i\nu} \neq 0$ for sufficiently large $|p|$. This implies that

$$(3.21) \quad g_\nu(x, \epsilon) = 0$$

if $|p|$ is sufficiently large.

We shall suppose $|p| = m$ so large that we have $\mu_{i\nu} \neq 0$ and the inequalities (3.18). Then there exists a certain positive constant R' sufficiently large so that the inequality

$$(3.22) \quad \sum_{j=1}^n |a_{ji}(x, \epsilon) - \mu_j| + \sum_{h \neq k} |a_{hk}(x, \epsilon)| \leq (\cos K)/R'$$

implies the inequalities (3.16). We can choose R' independent of ν and m because of the inequalities (3.18).

We shall choose $\delta^{(\nu)}$ and $\rho^{(\nu)}$ so small that we have the estimate (3.22) for (3.14). Thus we can determine the domain (3.14) independent of ν and m , and we have there the estimate (3.15).

4. Majorant series. In the preceding section we have constructed a domain independent of ν and m , and of the type

$$(4.1) \quad |x| < \delta, \quad |\arg(x - x^*) - \pi + \Omega| < \beta, \\ 0 < |\epsilon| \leq \rho, \quad |\arg \epsilon| \leq \alpha,$$

where

$$(4.2) \quad x^* = \delta e^{-i\Omega}.$$

For every ν we have (3.17), and for the domain (4.1) we have the estimate

$$(4.3) \quad |||U_\nu(x, \epsilon)||| \leq \frac{2}{|\mu(\nu)| \cos K_\nu} \sup |||b_\nu(t, \epsilon)|||$$

if $|p| = m$ is sufficiently large, where sup must be taken over the region

$$(4.4) \quad |t| < \delta, \quad |\arg(t - x^*) - \pi + \Omega| < \beta.$$

This estimate implies the inequalities

$$(4.5) \quad \sum_{N_\nu} |P_{i\nu}(x, \epsilon)| \leq \frac{2}{|\mu(\nu)| \cos K_\nu} \sup \sum_{N_\nu} \{ |h_{i\nu}(t, \epsilon)| \\ + \sum_{r+q-e_k=p} r_k |g_{kq}(t, \epsilon) P_{ir}(t, \epsilon)| \} \\ < \frac{R''}{\cos K} \sup \{ \sum_{N_\nu} |h_{i\nu}(t, \epsilon)| + \sum_{N_\nu} \sum_{r+q-e_k=p} |g_{kq}(t, \epsilon)| |P_{ir}(t, \epsilon)| \}$$

for (3.1), where R'' is a certain positive constant independent of ν and m and sup also must be taken over the region (4.4).

Then if we put

$$(4.6) \quad \tilde{P}_m(x, \epsilon) = \sum_m^{\sim} |P_{ip}(x, \epsilon)|,$$

$$(4.7) \quad \tilde{h}_m(x, \epsilon) = \sum_m^{\sim} |h_{ip}(x, \epsilon)|,$$

and

$$(4.8) \quad \tilde{g}_m(x, \epsilon) = \sum_m^{\sim} |g_{ip}(x, \epsilon)|,$$

where

$$\sum_m^{\sim} = \sum_{\substack{i=1, \dots, n \\ |p|=m}},$$

we have the inequalities

$$(4.9) \quad \tilde{P}_m(x, \epsilon) < \frac{R''n}{\cos K} \sup \{ \tilde{h}_m(t, \epsilon) + \sum_{l+h=m+1} \tilde{P}_l(t, \epsilon) \tilde{g}_h(t, \epsilon) \}$$

for (4.1), if $|p| = m$ is sufficiently large.

Put

$$\check{G}_i(x, u, \epsilon) = \sum_{|p| \geq 2} |\hat{g}_{ip}(x, \epsilon)| u^p,$$

$$\check{G}(x, u, \epsilon) = \sum_{i=1}^n \check{G}_i(x, u, \epsilon),$$

$$\check{g}(x, v, \epsilon) = \check{G}(x, \tilde{u}, \epsilon) \quad \text{with} \quad \tilde{u} = \begin{bmatrix} v \\ v \\ \vdots \\ \vdots \\ v \end{bmatrix},$$

$$\bar{P}_i(x, z, \epsilon) = z_i + \sum_{|p| \geq 2} |P_{ip}(x, \epsilon)| z^p,$$

$$\bar{P}(x, z, \epsilon) = \sum_{i=1}^n \bar{P}_i(x, z, \epsilon),$$

$$\bar{p}(x, v, \epsilon) = \bar{P}(x, \tilde{z}, \epsilon) \quad \text{with} \quad \tilde{z} = \begin{bmatrix} v \\ v \\ \vdots \\ \vdots \\ v \end{bmatrix},$$

$$\bar{h}_i(x, z, \epsilon) = \sum_{|p| \geq 2} |h_{ip}(x, \epsilon)| z^p,$$

$$\begin{aligned}
\bar{h}(x, z, \epsilon) &= \sum_{i=1}^n \bar{h}_i(x, z, \epsilon), \\
\check{h}(x, v, \epsilon) &= \bar{h}(x, \check{z}, \epsilon), \\
\bar{G}_i(x, z, \epsilon) &= \sum_{|p| \geq 2} |g_{ip}(x, \epsilon)| z^p, \\
\bar{G}(x, z, \epsilon) &= \sum_{i=1}^n \bar{G}_i(x, z, \epsilon), \\
\bar{g}(x, v, \epsilon) &= \bar{G}(x, \check{z}, \epsilon).
\end{aligned}$$

Then we have

$$\begin{aligned}
\bar{p}(x, v, \epsilon) &= nv + \sum_{m=2}^{\infty} \bar{P}_m(x, \epsilon) v^m, \\
\bar{g}(x, v, \epsilon) &= \sum_{m=2}^{\infty} \bar{g}_m(x, \epsilon) v^m,
\end{aligned}$$

and

$$\check{h}(x, v, \epsilon) = \sum_{m=2}^{\infty} \check{h}_m(x, \epsilon) v^m.$$

Put

$$(4.10) \quad \check{g}(x, \bar{p}(x, v, \epsilon), \epsilon) = \sum_{m=2}^{\infty} \check{K}_m(x, \epsilon) v^m.$$

The functions $h_{ip}(x, \epsilon)$ were defined by

$$\hat{G}_i(x, P(x, z, \epsilon), \epsilon) = \sum_{|p| \geq 2} h_{ip}(x, \epsilon) z^p,$$

where

$$\hat{G}(x, u, \epsilon) = \sum_{|p| \geq 2} \hat{g}_p(x, \epsilon) u^p.$$

Then, since

$$\hat{G}_i(x, u, \epsilon) \ll \check{G}_i(x, u, \epsilon)$$

and

$$P_i(x, z, \epsilon) \ll \bar{P}(x, z, \epsilon),$$

we have

$$\hat{G}_i(x, P(x, z, \epsilon), \epsilon) \ll \check{G}_i(x, \bar{P}(x, z, \epsilon), \epsilon),$$

where

$$\bar{P}(x, z, \epsilon) = \begin{bmatrix} \bar{P}(x, z, \epsilon) \\ \bar{P}(x, z, \epsilon) \\ \vdots \\ \bar{P}(x, z, \epsilon) \end{bmatrix}.$$

Therefore we obtain

$$\tilde{h}(x, z, \epsilon) \ll \check{g}(x, \bar{P}(x, z, \epsilon), \epsilon).$$

Hence

$$\check{h}(x, v, \epsilon) \ll \check{g}(x, \bar{p}(x, v, \epsilon), \epsilon)$$

or

$$(4.11) \quad \tilde{h}_m(x, \epsilon) \leq \check{K}_m(x, \epsilon).$$

Let $\Phi[v] = \sum_{m=2}^{\infty} \Phi_m v^m$ and $\Psi[v] = \sum_{m=2}^{\infty} \Psi_m v^m$ be majorant series for $\check{g}(x, v, \epsilon)$ and $\bar{g}(x, v, \epsilon)$, respectively, in the domain (4.1) and for sufficiently small v .

It should be noted that, if $|p|$ is sufficiently large, $g_p(x, \epsilon) = 0$ because $\mu_{ip} \neq 0$. Therefore, we can construct convergent majorant series for both $\check{g}(x, v, \epsilon)$ and $\bar{g}(x, v, \epsilon)$ because $\check{g}$ is a convergent series and $\bar{g}$ a polynomial.

We shall consider a functional equation in X :

$$(4.12) \quad v(n + X) = nv + M\{X\Psi[v] + \Phi[v(n + X)]\}$$

or

$$(4.12') \quad X = v^{-1}M\{X\Psi[v] + \Phi[v(n + X)]\},$$

where M is a positive constant.

This equation admits a holomorphic solution

$$(4.13) \quad X = V(v) = \sum_{m=1}^{\infty} V_m v^m$$

with the formulas

$$(4.14) \quad V_m = M\left\{\sum_{l+h=m+1} V_l \Psi_h + L_{m+1}\right\},$$

where

$$(4.15) \quad \Phi[v(n + V(v))] = \sum_{m=2}^{\infty} L_m v^m.$$

Put

$$(4.16) \quad W(v) = v(n + V(v)) = nv + \sum_{m=2}^{\infty} W_m v^m.$$

Since $W_m = V_{m-1}$, we have

$$(4.17) \quad W_m = M\left\{\sum_{l+h=m+1} W_l \Psi_h + L_m\right\}.$$

If M is sufficiently large, we have $\bar{P}_{m'} \leq W_{m'}$ for $m' \leq m_0$, where m_0 is a fixed sufficiently large integer. Then we have

$$\tilde{h}_{m_0+1} \leq \check{K}_{m_0+1} \leq L_{m_0+1}$$

and

$$\sum_{l+h=m_0+2} \tilde{P}_l \tilde{g}_h \leq \sum_{l+h=m_0+2} W_l \Psi_h .$$

Therefore by making use of the estimate (4.9) and the formula (4.17) we have

$$(4.18) \quad \tilde{P}_{m_0+1} \leq W_{m_0+1} .$$

Thus by induction we can obtain

$$(4.19) \quad \bar{p}(x, v, \epsilon) \ll W(v)$$

if M is sufficiently large.

Theorem 2 is thereby proved.

5. Estimation of the magnitude of solutions of the equations (A'''). Let us consider the system (A''') given in the Introduction whose coefficients satisfy the conditions (S) and (S').

We are assuming that the matrix $A_0(0)$ has the Jordan canonical form and its characteristic values $\mu_1, \dots, \mu_n$ lie in the interior of a halfplane bounded by a straight line through the origin.

We can suppose without loss of generality that the real parts of the μ_i are all positive. Otherwise, it is sufficient to rotate the independent variable x around the origin by multiplying it by $e^{i\zeta}$ with a suitable ζ .

Suppose that the real parts of the μ_i are arranged in increasing order as given in (X) of the Introduction. Then by use of the conditions (S) and (S') we can show that the right-hand members of the equations (A''') are all polynomials in $z_1, \dots, z_n$ of the form

$$(5.1) \quad g_j(x, z, \epsilon) = \begin{cases} \sum_{\mu_k = \mu_j} a_{jk}(x, \epsilon) z_k & (j = 1, \dots, m_1), \\ \sum_{\mu_k = \mu_j} a_{jk}(x, \epsilon) z_k + G_j(x, z_1, \dots, z_{m_\nu}, \epsilon) & (j = 1 + m_\nu, \dots, m_{\nu+1}). \end{cases}$$

To prove this fact, it is necessary to notice that a relation

$$(5.2) \quad \mu_{j\nu} = \mu_j - \sum_{k=1}^n p_k \mu_k = 0$$

implies

$$(5.2') \quad \operatorname{Re} \mu_j = \sum_{k=1}^n p_k \operatorname{Re} \mu_k .$$

Therefore if $\mu_{j\nu} = 0$, $|p| \geq 2$, and $m_\nu < j \leq m_{\nu+1}$, we have $p_k = 0$ for $k > m_\nu$ because $\operatorname{Re} \mu_k \geq \operatorname{Re} \mu_j$ for such k 's. Thus because of the conditions (S') we obtain the expression (5.1).

It should be noted that $\mu_j = \mu_k$ implies $\operatorname{Re} \mu_j = \operatorname{Re} \mu_k$. Then if $m_\nu < j \leq m_{\nu+1}$ and $\mu_k = \mu_j$, we have $m_\nu < k \leq m_{\nu+1}$.

Let

$$(5.3) \quad z = \phi(x, C, \epsilon)$$

be a solution of the system (A''') satisfying the initial condition

$$(5.4) \quad \phi(0, C, \epsilon) = C,$$

where C is an n -dimensional vector whose components C_i are arbitrary constants independent of ϵ .

Put

$$(5.5) \quad \omega_j = \arg \mu_j \quad (j = 1, \dots, n).$$

Since $\operatorname{Re} \mu_j > 0$ for every j , there exists a certain positive constant K such that $0 < K < \frac{1}{2}\pi$ and

$$(5.6) \quad |\omega_j| < K \quad (j = 1, \dots, n).$$

Let us choose two positive constants α and β so small that we have

$$(5.7) \quad -K + (\sigma\alpha + \beta) < \omega_j < K - (\sigma\alpha + \beta) \quad (j = 1, \dots, n).$$

Lemma 4. *Let $\phi_j(x, C, \epsilon)$ be the components of the vector (5.3). Then we have the estimate*

$$(5.8) \quad |\phi_j(x, C, \epsilon)| \leq M_j(C) \exp \{ -|\mu_j| |x| (\cos K)/2 |\epsilon|^\sigma \} \quad (j = 1, \dots, n)$$

for

$$(5.9) \quad |x| \leq \delta, \quad |\arg x - \pi| \leq \beta, \quad 0 < |\epsilon| \leq \rho, \quad |\arg \epsilon| \leq \alpha,$$

where $M_j(C)$ are positive constants depending on C which tend to zero as $\|C\|$ tends to zero.

Proof. At first we shall suppose without loss of generality that off-diagonal elements of the matrix $A_0(0)$ are sufficiently small.

Let us consider first the equations (A''') for those j 's such that $\mu_j = \mu_1$:

$$(5.10) \quad \epsilon^\sigma \frac{dz_i}{dx} = \sum_{\mu_k = \mu_1} a_{ik}(x, \epsilon) z_k \quad (\mu_i = \mu_1).$$

Denote by $\hat{A}(x, \epsilon)$ the matrix in the right-hand members of (5.10). If δ and ρ are sufficiently small, we have

$$(5.11) \quad \|\hat{A}(x, \epsilon) - \mu_1 E_N\| \leq \frac{1}{2}(|\mu_1| \cos K)$$

for (5.9), where N is the number of the j 's such that $\mu_j = \mu_1$ and E_N is the unit-matrix of order N .

Let $\hat{\Phi}(x, \epsilon)$ be an N by N matrix satisfying the equation

$$(5.12) \quad \epsilon^\sigma \frac{d\hat{\Phi}(x, \epsilon)}{dx} = \hat{A}(x, \epsilon) \hat{\Phi}(x, \epsilon)$$

and the condition

$$(5.13) \quad \hat{\Phi}(0, \epsilon) = E_N.$$

Then the matrix

$$(5.14) \quad \Psi(x, \epsilon) = \exp \left\{ \frac{-\mu_1 x}{\epsilon^\sigma} \right\} \hat{\Phi}(x, \epsilon)$$

satisfies the equation

$$(5.15) \quad \epsilon^\sigma \frac{d\Psi(x, \epsilon)}{dx} = (\hat{A}(x, \epsilon) - \mu_1 E_N) \Psi(x, \epsilon)$$

and the condition

$$(5.16) \quad \Psi(0, \epsilon) = E_N.$$

Put

$$(5.17) \quad T(\xi, \epsilon) = 1 + \frac{|\mu_1| \cos K}{2 |\epsilon|^\sigma} \int_0^\xi ||\Psi(te^{i\theta}, \epsilon)|| dt,$$

where

$$(5.18) \quad \theta = \arg x, \quad \xi = |x|.$$

Then we have

$$(5.19) \quad \frac{dT(\xi, \epsilon)}{d\xi} = \frac{|\mu_1| \cos K}{2 |\epsilon|^\sigma} ||\Psi(x, \epsilon)||.$$

Since

$$\Psi(x, \epsilon) = E_N + \frac{1}{\epsilon^\sigma} \int_0^x (\hat{A}(u, \epsilon) - \mu_1 E_N) \Psi(u, \epsilon) du,$$

we have

$$(5.20) \quad ||\Psi(x, \epsilon)|| \leq T(\xi, \epsilon).$$

Hence

$$(5.21) \quad \frac{dT(\xi, \epsilon)}{d\xi} \leq \frac{|\mu_1| \cos K}{2 |\epsilon|^\sigma} T(\xi, \epsilon).$$

This implies

$$(5.22) \quad T(\xi, \epsilon) \leq \exp \{ |\mu_1| \xi (\cos K)/2 |\epsilon|^\sigma \},$$

and from (5.20) we obtain

$$(5.23) \quad ||\Psi(x, \epsilon)|| \leq \exp \{ |\mu_1| |x| (\cos K)/2 |\epsilon|^\sigma \}.$$

From (5.14) we obtain

$$\begin{aligned} ||\hat{\Phi}(x, \epsilon)|| &= ||\exp \{ \mu_1 x / \epsilon^\sigma \} \Psi(x, \epsilon)|| \leq |\exp \{ \mu_1 x / \epsilon^\sigma \}| \exp \{ |\mu_1| |x| (\cos K)/2 |\epsilon|^\sigma \} \\ &= \exp \left\{ \frac{|\mu_1| |x|}{|\epsilon|^\sigma} \left[\cos \left(\arg \left[\frac{\mu_1 x}{\epsilon^\sigma} \right] \right) + \frac{1}{2} \cos K \right] \right\}. \end{aligned}$$

Since for (5.9) we have

$$(5.24) \quad \pi - K < \omega_1 + \theta - \sigma\omega < \pi + K,$$

where

$$(5.25) \quad \omega = \arg \epsilon,$$

we obtain

$$(5.26) \quad \cos (\omega_1 + \theta - \sigma\omega) < -\cos K.$$

Then

$$(5.27) \quad ||\hat{\Phi}(x, \epsilon)|| \leq \exp \{ -|\mu_1| |x| (\cos K)/2 |\epsilon|^\sigma \}$$

for (5.9).

Since the general solution of the system (5.10) is given by

$$\hat{\Phi}(x, \epsilon) \hat{C},$$

where $\hat{C}$ is an arbitrary N -dimensional vector, the estimate (5.8) is obtained for those j 's such that $\mu_j = \mu_1$.

By use of similar calculation, we obtain the estimate (5.8) for $j = 1, \dots, m_1$.

Now let us consider the equations (A''') for those j 's such that $\mu_j = \mu_{1+m_1}$:

$$(5.28) \quad \epsilon^\sigma \frac{dz_i}{dx} = \sum a_{ik}(x, \epsilon) z_k + G_i(x, \phi_1(x, C, \epsilon), \dots, \phi_{m_1}(x, C, \epsilon), \epsilon) \quad (\mu_j = \mu_{1+m_1})$$

where the sum extends over those k 's for which $\mu_k = \mu_{1+m_1}$.

Denote by $\tilde{A}(x, \epsilon)$ the matrix in the right-hand members of (5.28) and by $\tilde{G}(x, C, \epsilon)$ the vector whose components are

$$G_i(x, \phi_1(x, C, \epsilon), \dots, \phi_{m_1}(x, C, \epsilon), \epsilon) \quad (\mu_j = \mu_{1+m_1}).$$

Then the equations (5.28) can be given the form

$$(5.29) \quad \epsilon^\sigma \frac{d\tilde{z}}{dx} = \tilde{A}(x, \epsilon) \tilde{z} + \tilde{G}(x, C, \epsilon),$$

where $\tilde{z}$ denotes a vector whose components are z_i ($\mu_j = \mu_{1+m_1}$).

Let q denote p_{m_1} . Since

$$(5.30) \quad \tilde{G}(x, C, \epsilon) = \sum_{|p| \geq 2} \tilde{G}_p(x, \epsilon) \phi_1^{p_1}(x, C, \epsilon) \dots \phi_{m_1}^{p_{m_1}}(x, C, \epsilon)$$

and

$$(5.31) \quad \tilde{G}_p(x, \epsilon) \equiv 0 \quad \left(\mu_{1+m_1} \neq \sum_{k=1}^{m_1} p_k \mu_k \right),$$

we have

$$(5.32) \quad \begin{aligned} ||\tilde{G}(x, C, \epsilon)|| &\leq \sum_{|p| \geq 2} ||\tilde{G}_p(x, \epsilon)|| |\phi_1(x, C, \epsilon)|^{p_1} \cdots |\phi_{m_1}(x, C, \epsilon)|^a \\ &\leq \sum_{|p| \geq 2} ||\tilde{G}_p(x, \epsilon)|| M_1^{p_1} \cdots M_{m_1}^a \exp \left\{ -\frac{|x| \cos K}{2 |\epsilon|^\sigma} \sum_{k=1}^{m_1} p_k |\mu_k| \right\} \\ &\leq M'(C) \exp \{ -|x| |\mu_{1+m_1}| (\cos K)/2 |\epsilon|^\sigma \} \end{aligned}$$

for (5.9), where $M'(C)$ is a positive constant depending on C and tend to zero as C tends to zero.

On the other hand, we have

$$(5.33) \quad ||\tilde{A}(x, \epsilon) - \mu_{1+m_1} E_N|| \leq \frac{1}{2} (|\mu_{1+m_1}| \cos K)$$

for (5.9) if δ and ρ are sufficiently small, where N is the number of the j 's such that $\mu_j = \mu_{1+m_1}$ and E_N the unit-matrix of order N .

We define an N by N matrix $\tilde{\Phi}(x, \epsilon)$ by the conditions

$$(5.34) \quad \epsilon^\sigma \frac{d\tilde{\Phi}(x, \epsilon)}{dx} = \tilde{A}(x, \epsilon) \tilde{\Phi}(x, \epsilon)$$

and

$$(5.35) \quad \tilde{\Phi}(0, \epsilon) = E_N.$$

Then the general solution of the system (5.29) is given by

$$(5.36) \quad \tilde{z} = \tilde{\Phi}(x, \epsilon) \left\{ \tilde{C} + \frac{1}{\epsilon^\sigma} \int_0^x \tilde{\Phi}(u, \epsilon)^{-1} \tilde{G}(u, C, \epsilon) du \right\},$$

where $\tilde{C}$ is an arbitrary N -dimensional vector. Therefore, by use of an estimation similar to that made in §1, we have the estimate (5.8) for the j 's such that $\mu_j = \mu_{1+m_1}$.

By induction, we can complete the proof of Lemma 4.

6. General solution of the system (A). If we insert the general solution of the system (A''') in the transformation (T) obtained in Theorem 2, the derived series are convergent for (5.9) if $||C||$ is sufficiently small. Thus we obtain the general solution of the system (A) in the form given by (Z) in the Introduction.

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