Free Boundary Problems for Parabolic Equations III. Dissolution of a Gas Bubble in Liquid

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Introduction.

We shall refer to [2] and [3] as Part I and Part II respectively. In Part I we considered the problem of melting of solids, the free boundary (solid-liquid line) being a monotone function of time. In Part II we considered the development of one liquid drop surrounded by totally supersaturated or totally undersaturated vapor of its own substance. In this Part III we study the behavior of one gas bubble in liquid (not necessarily of the same substance) in which some of the same gas is dissolved. There are essentially two different situations: (i) the bubble decreases and (ii) the bubble grows. It is proved that (i) takes place even in case the solution of the gas in the liquid is supersaturated, provided the initial radius of the bubble is sufficiently small (depending on the amount of supersaturation). Thus even supersaturated solutions are stable in the sense that bubbles cannot grow in them if they are too small initially. This mathematical fact was first noticed by J. B. KELLER in an unpublished manuscript [5].

In this paper we prove the existence and uniqueness of the solution and also study some of its properties. Making an appropriate transformation, the problem is reduced to the study of the system (1.16)–(1.20) where $s(t)$ is the radius of the bubble and $u(x, t)$, $\varphi(x)$ are related in a simple manner to the density at time t and the initial density of the dissolved gas, respectively. This system in case $m = n = 0$ is equivalent, in form, to the system of equations for the drop (in Part II, §1) provided we define, in Part II, §1, $\tilde{z} = z - 1$ and then define $u = x\tilde{z}$. We further notice that in the case of the drop we have (when we define $u = x\tilde{z}$) $\varphi \leq 0$ and $\alpha \geq 0$, whereas in the present case of the bubble we have $\varphi \leq 0$, $\alpha > 0$ or $\varphi \geq 0$, $\alpha > 0$. This difference, however, is of no interest since we may change the sign of both φ and α at the same time.

An important difference between the case of the liquid drop and the case of the gas bubble is that (in physics) in the first case α is of the order of magni-

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tude of 10^{-5} , whereas in the second case α is only of the order of magnitude of $1/50$. Another significant difference is, of course, that $m \neq 0$, $n \neq 0$ and they are even not "very small" (see (1.12)).

We proceed to describe the main results of this paper. If $\varphi \leq 0$ then it is proved that there exists a unique solution and that $s(t)$ decreases monotonically to zero as $t \rightarrow t_0$ for some positive and finite number t_0 . More precisely if $\varphi(b) = 0$, then the above holds for any $\alpha > 0$, whereas if $\varphi(b) < 0$ (initial jump) it is proved to hold provided α satisfies (3.46). If $\varphi \geq 0$ then existence and uniqueness are proved for all $t > 0$, and $s(t)$ increases monotonically (like $t^{\frac{1}{2}}$) as t increases. Here α is assumed to satisfy the inequalities of (5.3). We also derive the approximation formulas

$$t_0 + m \int_0^{t_0} \frac{dt}{s(t)} \approx \frac{b^2}{2\alpha} + \frac{nb}{\alpha} \quad \text{where } b = s(0) \quad \text{in case } \varphi \geq 0,$$

$$s^2(t) \approx \frac{2\alpha m}{b} t \quad \text{in case } \varphi \geq 0.$$

These formulas improve others given in [1], [8]. In [7], [8], [9] there are given approximate formulas in the general situation that the temperature near the bubble's surface is not constant.

The methods of this paper in part are related to those of Part II and in part they are new. In §1 we formulate the main results in Theorem 8. The proof is given in §§2–6 and for details on its structure see the end of §1. In §7 we give a few approximation formulas, in addition to those given in §6.

1. Formulation of the Main Results.

The equations (1.1)–(1.4) below are the equations governing (under some simplifying assumptions) the behavior of one spherical gas bubble surrounded by liquid in which some of the same gas is dissolved. They are taken (with slight modification) from an unpublished manuscript by J. B. KELLER [5]. We first have to introduce some quantities:

- d = solubility coefficient;
- p_0 = pressure in the liquid at the bubble surface;
- σ = surface tension at the liquid-gas interface;
- ρ_0 = density of the gas at the pressure p_0 ;
- k = diffusibility of the gas in the liquid;
- p = pressure in the bubble;
- ρ = density of gas in the bubble;
- $\rho = \rho(p)$ is the equation of state; we shall assume that $\rho = ap$ (a constant);
in particular, $\rho_0 = ap_0$;
- $c_s = d\rho_0$ is the saturation density;
- $s(t)$ = radius of the bubble at time t ; $s(0) = b > 0$;
- x = distance from the bubble's center;
- $c(x, t)$ = density of the dissolved gas in the liquid; $c(x, 0) \equiv c_0(x)$.

The differential equation for $c(x, t)$ is

$$(1.1) \quad \frac{\partial c}{\partial t} = k \Delta c \quad \text{for } x > s(t), \quad t > 0,$$

where Δ is the Laplace operator in three dimensions. The initial condition is

$$(1.2) \quad c(x, 0) = c_0(x) \quad \text{for } x > b.$$

The equation expressing the fact that $c(x, t)$ is a continuous function on the bubble's surface is

$$(1.3) \quad c(s(t), t) = d\rho(p) \quad \text{and} \quad p = p_0 + \frac{2\sigma}{s(t)} \quad (t > 0).$$

In formulating the law of conservation of mass we assume that the change of mass of the bubble is proportional to the gradient of the density of the dissolved gas on the bubble's surface. The equation we then get is

$$(1.4) \quad \frac{d}{dt} \left(\frac{4\pi}{3} s^2(t) \rho \right) = 4\pi s^2(t) k c_x(s(t), t) \quad (t > 0).$$

In [5] it is assumed that on the right side of (1.4) there appears also a term $4\pi s^2(t) \dot{s}(t) c(s(t), t)$ due to convection. However, even if we make this assumption, all the results of this paper remain unchanged, provided in (1.6) we define n and α by

$$n = \frac{2 - 3d}{3 - 3d} m, \quad \alpha = \frac{d}{1 - d}.$$

Note (by (1.12)) that d is small.

Making the transformation $t \rightarrow t/k$ and defining

$$(1.5) \quad \tilde{c}_0(x) = \frac{c_0(x) - c^*}{d\rho_0}$$

where

$$c^* = d\rho \quad \text{and} \quad \rho = \rho \left(p_0 + \frac{2\sigma}{b} \right),$$

$$(1.6) \quad m = \frac{2\sigma}{p_0}, \quad n = \frac{2}{3} m, \quad \alpha = d,$$

$$(1.7) \quad \tilde{c}(x, t) = \frac{c(x, t) - c^*}{d\rho_0},$$

we obtain, for $\tilde{c}$, s , the following system:

$$(1.8) \quad \frac{\partial \tilde{c}}{\partial t} = \Delta \tilde{c} \quad \text{for } x > s(t), \quad t > 0,$$

$$(1.9) \quad \tilde{c}(x, 0) = \tilde{c}_0(x) \quad \text{for } x > b,$$

$$(1.10) \quad \tilde{c}(s(t), t) = \frac{m}{s(t)} - \frac{m}{b} \quad \text{for } t > 0,$$

$$(1.11) \quad \alpha \tilde{c}_x(s(t), t) = \left(1 + \frac{n}{s(t)}\right) \dot{s}(t) \quad \text{for } t > 0.$$

In the case of air-water under normal circumstances we roughly have, in CGS units,

$$(1.12) \quad p_0 = 10^3, \quad a = 10^{-6}, \quad d = \frac{1}{50}, \quad \sigma = 70, \quad m = \frac{3}{20},$$

$$n = \frac{1}{10}, \quad \alpha = \frac{1}{50}.$$

We assume that $\tilde{c}_0(x)$ (or $c_0(x)$) satisfies the following conditions:

$$(1.13) \quad \int_b^\infty x |\tilde{c}_0(x)| dx < \infty, \quad \text{l.u.b.}_{b \leq x < \infty} x |\tilde{c}_0(x)| < \infty, \quad \lim_{x \rightarrow \infty} \frac{d}{dx} [x \tilde{c}_0(x)] = 0.$$

We now put on the solution $c(x, t)$ the following condition as $x \rightarrow \infty$:

$$(1.14) \quad x \tilde{c}(x, t) \quad \text{and} \quad \frac{\partial}{\partial x} [x \tilde{c}(x, t)] \quad \text{remain bounded} \\ \text{as } x \rightarrow \infty, \text{ uniformly in } t \text{ in finite intervals.}$$

We next define

$$(1.15) \quad u(x, t) = x \tilde{c}(x, t), \quad \varphi(x) = x \tilde{c}_0(x).$$

The system (1.8)–(1.11), (1.14) then takes the form

$$(1.16) \quad u_{xx} = u_t \quad \text{for } x > s(t), \quad t > 0,$$

$$(1.17) \quad u(x, 0) = \varphi(x) \quad \text{for } x > b,$$

$$(1.18) \quad u(s(t), t) = m - \frac{ms(t)}{b} \quad \text{for } t > 0,$$

$$(1.19) \quad \alpha u_x(s(t), t) = (n + s(t)) \dot{s}(t) + \alpha m \left(\frac{1}{s(t)} - \frac{1}{b} \right) \quad \text{for } t > 0,$$

$$(1.20) \quad u(x, t) \quad \text{and} \quad u_x(x, t) \quad \text{remain bounded} \\ \text{as } x \rightarrow \infty, \text{ uniformly in } t \text{ in finite intervals.}$$

The conditions (1.14) become:

$$(\Phi) \quad ||\varphi|| = \text{l.u.b.}_{b \leq x < \infty} |\varphi(x)| < \infty, \quad \dot{\varphi}(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty,$$

$$\text{and} \quad \int_b^\infty |\varphi(x)| dx < \infty.$$

Before we formulate the main theorem of this paper, let us note that in case $m = n = 0$ our problem is formally identical with that of the drop in Part II.

Indeed, if in Part II, §1 we define $u = x(z - 1)$ (instead of $u = xz$), then we obtain for u the system (1.16)–(1.20). Such a transformation would slightly simplify the calculations in Part II and would make the existence theorems of Part II valid for a somewhat wider range of α . Since in the present problem α is not “very small” (see (1.12)), the above simplifying definition of u (in (1.7), (1.15)) is of importance.

Definition. We say that u, s form a solution of the system (1.16)–(1.20) if they satisfy these equations and if, in addition, they satisfy the regularity assumptions in the definition of a solution in Part II, §1. Recall, in particular, that if $\varphi(b) = 0$ then $\dot{s}(t)$ is continuous for $t \geq 0$, whereas if $\varphi(b) \neq 0$ then $t^{\frac{1}{2}}\dot{s}(t)$ is continuous for $t \geq 0$.

Theorem 8. Assume that $\varphi(x)$ satisfies (Φ) and that $m > 0, n > 0, \alpha > 0$. (i) If $\varphi(x) \leq 0$ ($\varphi \equiv 0$) and $\varphi(b) = 0$, then the system (1.16)–(1.20) has a unique solution u, s for all $t < t_0$ ($t_0 > 0$). The curve $s(t)$ is strictly monotone decreasing in t and it intersects the t axis at $t = t_0$. (ii) If $\varphi \leq 0$ ($\varphi \equiv 0$) and $\varphi(b) < 0$, then the above assertions about existence and monotonicity remain true provided α satisfies (3.46). Uniqueness holds under the assumption (3.48). (iii) If $\varphi \geq 0$ ($\varphi \equiv 0$) then the system (1.16)–(1.20) has a solution u, s for all $t > 0$ provided α satisfies (5.3) and the solution is determined uniquely by any one of the assumptions (3.42), (3.48). Furthermore, $s(t)$ is strictly monotone increasing in t , and $s(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Physical interpretation. The assumption $\varphi \leq 0$ means that

$$(1.21) \quad c_0(x) \leq c^* \equiv d\rho_0 + \frac{2a}{b} \frac{d\sigma}{b} = c_s \left(1 + \frac{m}{b}\right).$$

Thus, even in case the initial gas density $c_0(x)$ in the liquid exceeds the saturation density c_s , it is possible to have $\varphi \leq 0$ (and hence, $s(t) \searrow 0$ as $t \nearrow t_0$) provided $c_0(x)$ satisfies (1.21). Moreover, for any given density $c_0(x)$, the inequality (1.21) is satisfied if b is sufficiently small. Hence not only undersaturated solutions but also supersaturated solutions are stable, in the sense that a bubble cannot start to develop if there is no external interference. This fact was first noticed by KELLER [5].

Consider next the case $\varphi \geq 0$ or, equivalently,

$$(1.22) \quad c_0(x) \geq c^* = c_s \left(1 + \frac{m}{b}\right).$$

This inequality holds if the initial gas density $c_0(x)$ exceeds the saturation density c_s , and if, in addition, the initial radius of the bubble is sufficiently large. c^* may be called the *equilibrium density* since if $c_0(x) \leq c^*$ the bubble decreases whereas if $c_0(x) \geq c^*$ the bubble grows. To examine more closely the condition (5.3), we notice that (1.22) implies

$$(1.23) \quad \frac{1}{b} \leq \frac{c_0(x) - c_s}{c_s m}.$$

Inserting the values of α , m from (1.12), we find that (5.3) is a consequence of the inequalities

$$(1.24) \quad \frac{||x[c_0(x) - c^*]||}{c_s} \frac{||c_0(x) - c_s||}{c_s} < \frac{3}{8}, \quad \frac{||c_0(x) - c_s||}{c_s} < \frac{5}{2}.$$

Structure of the proof. The proof of Theorem 8 is given in §§2–6. In §2 we reduce the problem of existence to that of solving a nonlinear integral equation. In §3 we solve the integral equation for small t . In §4 we prove that $s(t)$ is a strictly monotone function in t . In §5 we continue the solution for all $t > 0$ as long as $s(t)$ stays positive and finite. Finally, in §6 we prove that if $\varphi \geq 0$ then $s(t)$ remains bounded in any finite time and if $\varphi \leq 0$ then the bubble disappears in a finite time t_0 . The new ideas in the proof of Theorem 8 which do not appear in the previous Parts I, II are mostly contained in §§3, 4. The main part of §3 is devoted to the derivation of various inequalities on $s(t)$ in terms of $u_x(s(t), t)$, using equation (1.19). The important point is that $s(t)$ is not solved explicitly in terms of $u_x(s(t), t)$ (as in Parts I, II). Once these inequalities are derived, the existence of a solution for small t follows by the methods of Parts I, II. The novelty in §4 is that the boundary values of $\tilde{c}$ on $s(t)$ (see (1.10)) involve $s(t)$ which is *a priori* unknown. The main difficulty which thereby arises is in proving that $s(t)$ is monotone for t immediately above $t = 0$. This difficulty is overcome by an approximation method, and use is made of the existence results of §3.

2. The Integral Equation.

Denoting

$$K(x, t; \xi, \tau) = \frac{1}{2\pi^{\frac{1}{2}}(t - \tau)^{\frac{1}{2}}} \exp \left\{ -\frac{(x - \xi)^2}{4(t - \tau)} \right\}$$

and using Green's formula, we obtain the following representation for u :

$$(2.1) \quad \begin{aligned} u(x, t) = & \int_b^\infty K(x, t; \xi, 0) \varphi(\xi) d\xi - m \int_0^t \left(1 - \frac{s(\tau)}{b} \right) K(x, t; s(\tau), \tau) \dot{s}(\tau) d\tau \\ & - m \int_0^t K_x(x, t; s(\tau), \tau) \left(1 - \frac{s(\tau)}{b} \right) d\tau \\ & - \int_0^t K_x(x, t; s(\tau), \tau) u_x(s(\tau), \tau) d\tau. \end{aligned}$$

Define

$$(2.2) \quad v(t) = u_x(s(t), t).$$

If we differentiate the third term on the right side of (2.1) with respect to x , we get

$$-m \int_0^t K_{xx}(x, t; s(\tau), \tau) \left(1 - \frac{s(\tau)}{b} \right) d\tau$$

$$\begin{aligned}
(2.3) \quad &= m \int_0^t \left[\frac{d}{d\tau} K(x, t; s(\tau), \tau) \right] \left(1 - \frac{s(\tau)}{b} \right) d\tau \\
&\quad - m \int_0^t K_{\xi}(x, t; s(\tau), \tau) \left(1 - \frac{s(\tau)}{b} \right) \dot{s}(\tau) d\tau \\
&= \frac{m}{b} \int_0^t K(x, t; s(\tau), \tau) \dot{s}(\tau) d\tau \\
&\quad - m \int_0^t K_{\xi}(x, t; s(\tau), \tau) \left(1 - \frac{s(\tau)}{b} \right) \dot{s}(\tau) d\tau.
\end{aligned}$$

Note that the last integral cancels with the x -derivative of the second term on the right side of (2.1). Differentiating (2.1) with respect to x , using (1.19), (2.2), (2.3), then letting $x \rightarrow s(t) + 0$ and using Lemma 1 of Part I, we obtain

$$\begin{aligned}
(2.4) \quad v(t) &= 2 \int_b^{\infty} K_x(s(t), t; \xi, 0) \varphi(\xi) d\xi \\
&\quad + \frac{2m}{b} \int_0^t K(s(t), t; s(\tau), \tau) \left[\frac{\alpha v(\tau)}{n + s(\tau)} - \frac{\alpha m}{n + s(\tau)} \left(\frac{1}{s(\tau)} - \frac{1}{b} \right) \right] d\tau \\
&\quad - 2 \int_0^t K_x(s(t), t; s(\tau), \tau) v(\tau) d\tau,
\end{aligned}$$

and $s(t)$ is defined, in terms of $v(t)$, by

$$(2.5) \quad \alpha v(t) = (n + s(t))\dot{s}(t) + \alpha m \left(\frac{1}{s(t)} - \frac{1}{b} \right) \quad \text{for } t > 0; \quad s(0) = b.$$

Conversely one can show that if $s(t) > 0$ ($0 \leq t \leq \delta$) is defined by (2.5) and if $v(t)$ (or $t^{\frac{1}{2}} v(t)$) is a continuous solution of (2.4) for $0 \leq t < \delta$ where $\delta > 0$, then $u(x, t)$ defined by (2.1) with $u_x(s(t), t)$ replaced by $v(t)$ forms, together with $s(t)$, a solution of the system (1.16)–(1.20). It thus remains to consider the integral equation (2.4).

3. Existence of a Solution for Small Times.

Proceeding with the method of Part II, §3 we define

$$(3.1) \quad \bar{v}(t) = t^{\frac{1}{2}} v(t).$$

Denoting the right side of (2.4) by Tv and multiplying both sides of (2.4) by $t^{\frac{1}{2}}$ we obtain for $v(t)$ the nonlinear integral equation

$$(3.2) \quad \bar{v}(t) = t^{\frac{1}{2}} Tv \equiv \bar{T}\bar{v}.$$

We have to prove that $\bar{T}$ has a fixed point $\bar{v}$. We introduce the space C_{σ} of continuous functions $\bar{w}(t)$ with the norm $\|\bar{w}\| = \text{l.u.b.}_{0 \leq t \leq \sigma} |\bar{w}(t)|$, and denote by $C_{\sigma, M}$ the set $\|\bar{w}\| \leq M$. We shall prove that $\bar{T}$ has a fixed point in $C_{\sigma, M}$ for some positive σ, M .

To do that, we need to establish *a priori* inequalities analogous to (3.4),

(3.5), (3.16) and (3.17) of Part II. We state these inequalities in separate lemmas. In what follows $\bar{v}(t)$ is any function in $C_{\sigma, M}$ and $s(t)$ is defined by

$$(3.3) \quad \alpha t^{\frac{1}{2}} \bar{v}(t) = (n + s(t)) \dot{s}(t) + \alpha m \left(\frac{1}{s(t)} - \frac{1}{b} \right), \quad s(0) = b.$$

Lemma 4. *If σ satisfies the inequality*

$$(3.4) \quad \frac{4\alpha m}{3b^2} \sigma + \frac{8\alpha M}{3b} \sigma^{\frac{1}{2}} < \frac{b}{2},$$

then $s(t)$ exists (as a solution of (3.3)) for $0 \leq t \leq \sigma$, and

$$(3.5) \quad \frac{b}{2} < s(t) < \frac{3b}{2} \quad \text{for } 0 \leq t \leq \sigma.$$

Proof. Introducing the functions

$$(3.6) \quad z(t) = s(t) + n, \quad y(t) = [z(t)]^2,$$

(3.3) is reduced to

$$(3.7) \quad \dot{y}(t) + \frac{2\alpha m}{(y(t))^{\frac{1}{2}} - n} - \frac{2\alpha m}{b} = 2\alpha \frac{\bar{v}(t)}{t^{\frac{1}{2}}}, \quad y(0) = (b + n)^2.$$

A unique solution then exists for all $t \geq 0$ as long as $(y(t))^{\frac{1}{2}} - n$ stays positive. Thus, for $\epsilon > 0$ sufficiently small, a solution exists for $0 \leq t \leq \epsilon$ and it satisfies

$$(3.8) \quad \frac{b}{2} \leq (y(t))^{\frac{1}{2}} - n \leq \frac{3b}{2} \quad \text{if } 0 \leq t \leq \epsilon.$$

Defining

$$(3.9) \quad \lambda(t) = \frac{2\alpha m}{(y(t))^{\frac{1}{2}} - n}$$

we then have

$$(3.10) \quad \frac{4\alpha m}{3b} \leq \lambda(t) \leq 4 \frac{\alpha m}{b}.$$

Integrating both sides of (3.7) with respect to t and using (3.10), we get

$$(3.11) \quad |y(t) - (b + n)^2| \leq \frac{2\alpha m}{b} t + 4\alpha M t^{\frac{1}{2}}.$$

Using (3.8) we conclude that

$$(3.12) \quad |(y(t))^{\frac{1}{2}} - (n + b)| = \frac{|y(t) - (n + b)^2|}{|(y(t))^{\frac{1}{2}} + n + b|} \leq \frac{2}{3b} \left(\frac{2\alpha m}{b} t + 4\alpha M t^{\frac{1}{2}} \right) < \frac{b}{2}$$

provided ϵ satisfies

$$(3.13) \quad \frac{4\alpha m}{3b^2} \epsilon + \frac{8\alpha M}{3b} \epsilon^{\frac{1}{2}} < \frac{b}{2}.$$

We have thus proved that (3.8) implies

$$(3.14) \quad \frac{b}{2} < (y(t))^{\frac{1}{2}} - n < \frac{3b}{2} \quad \text{for } 0 \leq t \leq \epsilon$$

provided ϵ satisfies (3.13). A simple argument of continuation now shows that (3.14) remains valid for all $t \leq \sigma$ provided σ satisfies (3.4). Using the definition of $y(t)$ in (3.6), the proof of the lemma is completed.

Lemma 5. *If σ satisfies (3.4) and if*

$$(3.15) \quad 2m\sigma^{\frac{1}{2}} < bM,$$

then, for $0 \leq \tau < t \leq \sigma$,

$$(3.16) \quad |s(t) - s(\tau)| \leq \frac{5\alpha M}{bt^{\frac{1}{2}}} (t - \tau).$$

Proof. By (3.3) it follows that

$$s(t) - s(\tau) = \int_{\tau}^t \dot{s}(\lambda) d\lambda = \int_{\tau}^t \frac{\alpha \bar{v}(\lambda)}{(n + s(\lambda))\lambda^{\frac{1}{2}}} d\lambda - \int_{\tau}^t \frac{\alpha m}{n + s(\lambda)} \left(\frac{1}{s(\lambda)} - \frac{1}{b} \right) d\lambda.$$

Applying Lemma 4, noting that

$$\int_{\tau}^t \frac{d\lambda}{\lambda^{\frac{1}{2}}} \leq \frac{2}{t^{\frac{1}{2}}} (t - \tau)$$

and using (3.15), we get

$$|s(t) - s(\tau)| \leq \frac{4\alpha M(t - \tau)}{bt^{\frac{1}{2}}} + \frac{2\alpha m(t - \tau)}{b^2} \leq \frac{5\alpha M(t - \tau)}{bt^{\frac{1}{2}}},$$

and the proof is completed.

Lemma 6. *Assume that σ satisfies (3.4), (3.15) and that*

$$(3.17) \quad 8\alpha m\sigma < \frac{b^3}{3}, \quad 30 \frac{\alpha M^2}{b^2} \sigma^{\frac{1}{2}} \leq 1.$$

If $\bar{v}_0$ belongs to $C_{\sigma, M}$ and if s_0 corresponds to v_0 in the same way that s corresponds to v by (3.3), then

$$(3.18) \quad |s(t) - s_0(t)| \leq \frac{6\alpha\epsilon}{b} t^{\frac{1}{2}} \quad \text{for } 0 \leq t \leq \sigma$$

where

$$(3.19) \quad \epsilon = ||v - v_0||.$$

Proof. Using the notation (3.6) and defining

$$(3.20) \quad \eta(t) = y(t) - y_0(t)$$

where y_0, z_0 corresponds (via (3.6)) to s_0 , we obtain, by (3.7),

$$(3.21) \quad \dot{\eta}(t) + \mu(t)\eta(t) = \frac{2\alpha}{t^{\frac{1}{2}}} (\bar{v}(t) - \bar{v}_0(t)), \quad \eta(0) = 0,$$

where

$$(3.22) \quad \mu(t) = -2\alpha m \{ [(y(t))^{\frac{1}{2}} - n][(y_0(t))^{\frac{1}{2}} - n][(y_0(t))^{\frac{1}{2}} + (y(t))^{\frac{1}{2}}] \}^{-1}.$$

Solving (3.21) for η and using the inequalities (see (3.8))

$$(3.23) \quad 0 \leq -\mu(t) \leq \frac{8\alpha m}{b^{\frac{1}{2}}},$$

we get

$$(3.24) \quad |\eta(t)| \leq 4\alpha\epsilon t^{\frac{1}{2}} \exp \left\{ \frac{8\alpha m\sigma}{b^{\frac{1}{2}}} \right\} \leq 6\alpha\epsilon t^{\frac{1}{2}}$$

where we made use of (3.17). Using the definitions (3.6), (3.20), we obtain

$$|s(t) - s_0(t)| = |z(t) - z_0(t)| = \frac{|y(t) - y_0(t)|}{(y(t))^{\frac{1}{2}} + (y_0(t))^{\frac{1}{2}}} \leq \frac{6\alpha\epsilon}{b} t^{\frac{1}{2}},$$

and the proof of the lemma is completed.

Lemma 7. *Assume that all the assumptions of Lemma 6 are satisfied and that, in addition, σ satisfies*

$$(3.25) \quad 30\alpha M\sigma^{\frac{1}{2}} < b^2.$$

Then

$$(3.26) \quad |\dot{s}(t) - \dot{s}_0(t)| \leq 10 \frac{\alpha\epsilon}{b t^{\frac{1}{2}}} \quad \text{for } 0 < t \leq \sigma.$$

Proof. From the proof of Lemma 6 we get

$$|\dot{y}(t) - \dot{y}_0(t)| \leq |\mu(t)| |y(t) - y_0(t)| + \frac{2\alpha}{t^{\frac{1}{2}}} \epsilon \leq \frac{8\alpha m}{b^{\frac{1}{2}}} 6\alpha\epsilon t^{\frac{1}{2}} + \frac{2\alpha}{t^{\frac{1}{2}}} \epsilon \leq \frac{4\alpha\epsilon}{t^{\frac{1}{2}}}$$

where in the last inequality we made use of (3.17). Using (3.6) we conclude that

$$(3.27) \quad |(s(t) + n)\dot{s}(t) - (s_0(t) + n)\dot{s}_0(t)| \leq \frac{4\alpha\epsilon}{t^{\frac{1}{2}}}.$$

Next, letting $\tau \rightarrow t$ in (3.16) we get

$$(3.28) \quad |\dot{s}(t)| \leq \frac{5\alpha M}{b t^{\frac{1}{2}}}.$$

Using (3.28) and Lemma 6 we obtain, upon using (3.25),

$$|(s(t) + n)\dot{s}(t) - (s_0(t) + n)\dot{s}_0(t)| = |\dot{s}(t)| |s(t) - s_0(t)| \leq 30 \frac{\alpha^2 M}{b^2} \epsilon < \frac{\alpha\epsilon}{t^{\frac{1}{2}}}.$$

Combining this inequality with (3.27) and using Lemma 4, we get

$$|\dot{s}(t) - \dot{s}_0(t)| \leq \frac{5\alpha\epsilon}{(s_0(t) + n)t^{\frac{1}{2}}} \leq 10 \frac{\alpha\epsilon}{bt^{\frac{1}{2}}},$$

and the proof is completed.

As was previously mentioned, α (in physics) is not "too small" and it is therefore desirable to derive the inequalities of Lemmas 4–7 in the strongest form. This can be done by slightly modifying the previous proofs, and therefore we shall state here the final result without proof.

Lemma 8. *For any $\mu > 0$ there exists a positive number $\bar{\delta} < 1$, depending only on μ, α, m, n, M and b ($b \neq 0$), such that if*

$$(3.29) \quad 0 < \sigma \leq \bar{\delta},$$

then the following inequalities hold:

$$(3.30) \quad (1 - \mu)b < s(t) < (1 + \mu)b \quad \text{for } 0 \leq t \leq \sigma,$$

$$(3.31) \quad |s(t) - s(\tau)| \leq (2 + \mu) \frac{\alpha M}{bt^{\frac{1}{2}}} (t - \tau) \quad \text{for } 0 \leq \tau < t \leq \sigma,$$

$$(3.32) \quad |s(t) - s_0(t)| \leq (2 + \mu) \frac{\alpha\epsilon t^{\frac{1}{2}}}{b} \quad \text{for } 0 \leq t \leq \sigma,$$

$$(3.33) \quad |\dot{s}(t) - \dot{s}_0(t)| \leq (2 + \mu) \frac{\alpha\epsilon}{bt^{\frac{1}{2}}} \quad \text{for } 0 < t \leq \sigma.$$

Completion of the proof. Using Lemma 8 we find (compare Part II, §3) that if $\bar{w} \in C_{\sigma, M}$ then

$$(3.34) \quad \|T\bar{w}\| \leq 2 \|\varphi\| + \frac{5\alpha M^2}{4b} + A\sigma^{\frac{1}{2}}$$

where μ is a sufficiently small universal constant, and A depends only on d, α, M, b, m, n . We next find by the methods of Part II, §3 and Lemma 8 that

$$(3.35) \quad \begin{aligned} \|\bar{T}\bar{w} - \bar{T}\bar{w}_0\| &\leq 13 \frac{\alpha \|\varphi\| \epsilon}{b} + 3 \frac{\alpha^2 M \|\varphi\| \epsilon}{b^2} + 6 \frac{\alpha M \epsilon}{b} \\ &\quad + 22 \frac{\alpha^3 M^3 \epsilon}{b^3} + A_0 \sigma^{\frac{1}{2}} \epsilon \end{aligned}$$

where μ is a sufficiently small universal constant and A_0 is a constant depending only on α, M, b, m, n , provided α satisfies the inequality (compare Part II, (3.30))

$$(3.36) \quad 4\alpha M < b.$$

If we assume that

$$(3.37) \quad \frac{\alpha \|\varphi\|}{b} < \frac{1}{20},$$

and take

$$(3.38) \quad M = \frac{5}{2} \|\varphi\|,$$

then it easily follows that there exist positive constants δ' , ϑ ($\vartheta < 1$) depending only on $m, n, \alpha, b, \|\varphi\|$ such that if

$$(3.39) \quad \sigma \leq \delta',$$

then

$$(3.40) \quad \|\bar{T}\bar{w}\| \leq M, \quad \|\bar{T}\bar{w} - \bar{T}\bar{w}_0\| \leq \vartheta \|\bar{w} - \bar{w}_0\| \quad (\vartheta < 1).$$

Clearly (3.36) also follows from (3.37), (3.38).

From (3.40) it follows that $\bar{T}$ has a fixed point. Furthermore, the fixed point is uniquely determined by the condition

$$(3.41) \quad \lim_{t \rightarrow 0} |\bar{v}(t)| \leq M = \frac{5}{2} \|\varphi\|.$$

Using (3.3), this condition becomes

$$(3.42) \quad \lim_{t \rightarrow 0} |t^{\frac{1}{2}} \dot{s}(t)| \leq \frac{5\alpha \|\varphi\|}{2(b+n)}.$$

A second proof. We may replace the first term on the right side of (2.4) by

$$(3.43) \quad 2\varphi(b)K(s(t), t; b, 0) + 2 \int_b^\infty K(s(t), t; \xi, 0)\dot{\varphi}(\xi) d\xi$$

and then proceed by the previous method. By analogy with (3.34), (3.35) we obtain

$$(3.44) \quad \|\bar{T}\bar{w}\| \leq \frac{|\varphi(b)|}{\pi^{\frac{1}{2}}} + \frac{5\alpha M^2}{4b} + A\sigma^{\frac{1}{2}},$$

$$(3.45) \quad \|\bar{T}\bar{w} - \bar{T}\bar{w}_0\| \leq \frac{\alpha |\varphi(b)| \epsilon}{b} + 6 \frac{\alpha M \epsilon}{b} + 22 \frac{\alpha^3 M^3 \epsilon}{b^3} + A_0 \sigma^{\frac{1}{2}} \epsilon,$$

provided (3.36) is satisfied. Assuming that

$$(3.46) \quad \frac{\alpha |\varphi(b)|}{b} < \frac{1}{8}$$

and taking

$$(3.47) \quad M = |\varphi(b)|,$$

we easily find that (3.40) and (3.36) are satisfied. Hence there exists a solution $\bar{v}(t)$ ($0 \leq t \leq \sigma$) of the integral equation $\bar{v} = \bar{T}\bar{v}$. It is uniquely determined by the condition (compare (3.42))

$$(3.48) \quad \lim_{t \rightarrow \infty} |t^{\frac{1}{2}} \dot{s}(t)| \leq \frac{\alpha |\varphi(b)|}{b+n}.$$

The case $\varphi(b) = 0$. Replacing the first term on the right side of (2.4) by (3.43) we see that Tv consists only of terms which behave regularly at $t = 0$. Hence it is not necessary to introduce $\bar{v} = t^{\frac{1}{2}}v(t)$ and to consider the transformation $\bar{T}$. As in Part II it can be shown without difficulty that T has a unique fixed point in $C_{\sigma, M}$ for any value of $\alpha > 0$. Here we may take

$$(3.49) \quad M = 2 \|\dot{\varphi}\| + 1.$$

4. Monotonicity of $s(t)$.

In this section we shall prove that if $\varphi(x) \leq 0$ for all $b \leq x \leq \infty$, then $s(t)$ is strictly monotone decreasing in t . In a similar way it can also be proved that if $\varphi(x) \geq 0$ for all $b \leq x \leq \infty$, then $s(t)$ is strictly monotone increasing in t . The proof is given in three steps. In each step we suppose that the solution u, s is defined for $0 \leq t \leq \gamma$ for some $\gamma > 0$. We recall that $\varphi(x) = x\tilde{c}_0(x)$.

Step 1: $\tilde{c}_0(b) = 0, d\tilde{c}_0(b)/dx < 0$. Using the transformation

$$(4.1) \quad y = x - s(t), \quad w(y, t) = \tilde{c}(x, t)$$

and using (1.8)–(1.11), (1.14), we find for w, s the following system:

$$(4.2) \quad \Delta w - w_t + \dot{s}(t)w_y = 0 \quad \text{for } y > 0, \quad 0 < t < \gamma,$$

$$(4.3) \quad w(y, 0) = \tilde{c}_0(y + b) \leq 0 \quad \text{for } y > 0,$$

$$(4.4) \quad w(0, t) = \frac{m}{s(t)} - \frac{m}{b} \quad \text{for } 0 < t < \gamma,$$

$$(4.5) \quad \alpha w_y(0, t) = \left(1 + \frac{n}{s(t)}\right)\dot{s}(t) \quad \text{for } 0 < t < \gamma,$$

$$(4.6) \quad w(y, t) \rightarrow 0 \quad \text{as } y \rightarrow 0, \quad \text{uniformly in } t, \quad 0 \leq t \leq \gamma.$$

Now $w_y(0, 0) = d\tilde{c}_0(b)/dx < 0$ and hence, by (4.5), $\dot{s}(0) < 0$. Thus $s(t)$ is strictly monotone decreasing in the interval $0 \leq t \leq \epsilon$ for some $\epsilon > 0$.

Assume that $s(t)$ is strictly monotone decreasing in an interval $0 \leq t \leq \lambda$ where $\lambda < \gamma$. If we prove that there exists $\epsilon > 0$ such that $s(t)$ is strictly monotone decreasing in the interval $0 \leq t \leq \lambda + \epsilon$, then the proof is completed.

We now remark that [4; Theorem 2] remains valid (with slight modifications in the proof) in the case that $P^0 = (x^0, t^0)$ lies on the intersection of the lateral boundary and the upper base of the domain D . Applied to our present case it says that if w attains its maximum in $0 \leq y < \infty, 0 \leq t \leq \lambda$ at the point $(0, \lambda)$, and if $w(y, t) < w(0, \lambda)$ for $0 < y \leq \eta, \lambda - \eta \leq t \leq \lambda$, for some $0 < \eta < \lambda$, then $w_y(0, \lambda) < 0$. (Note that the theorem in [4] is proved under some differentiability assumptions on the boundary; to satisfy these requirements we made the transformation (4.1).) Now, by assumption, $s(\lambda) \leq s(t) \leq b$ for $0 \leq t \leq \lambda$, hence, by (4.4), $w(0, \lambda) \geq w(0, t) \geq w(0, 0) = 0 \geq w(y, 0)$ and $w(y, 0) \equiv 0$. Using the strong maximum principle [6] we can conclude that

$$w(y, t) < w(0, \lambda) \quad \text{for } 0 < y < \infty, \quad 0 \leq t \leq \lambda.$$

We thus can use the above italicized statement and conclude that $w_b(0, \lambda) < 0$. By (4.5) it then follows that $\dot{s}(\lambda) < 0$ and hence $s(t)$ is strictly decreasing in an interval $0 \leq t \leq \lambda + \epsilon$ for some positive ϵ . We remark that reasoning similar to the above was given independently by J. B. KELLER in his unpublished manuscript [5].

Step 2: $\tilde{v}_0(b) = 0$. Let $\{\tilde{v}_j(x)\}$ be a sequence of functions which satisfy the properties

$$(4.7) \quad \begin{aligned} \tilde{v}_j(x) &= \tilde{v}_0(x) \quad \text{for } b+1 < x < \infty, \quad \tilde{v}_j(b) = 0, \quad \frac{d\tilde{v}_j(b)}{dx} < 0, \\ \text{l.u.b.}_{b \leq x \leq b+1} \left[|\tilde{v}_j(x) - \tilde{v}_0(x)| + \left| \frac{d}{dx} \tilde{v}_j(x) - \frac{d}{dx} \tilde{v}_0(x) \right| \right] &\rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Such a sequence always exists since $d\tilde{v}_0(b)/dx \leq 0$ (which follows from $\tilde{v}_0(b) = 0$, $\tilde{v}_0(x) \leq 0$). Using the results stated at the end of §3 we conclude that (for any $\alpha > 0$) there exists $\delta > 0$ such that for each j there exists for $0 \leq t \leq \delta$ a solution u_j , s_j (or $\tilde{v}_j$, s_j) of the system (1.16)–(1.20) (or (1.8)–(1.11), (1.14)). Denoting by v_j the corresponding solution of the integral equation we shall prove below that

$$(4.8) \quad \text{l.u.b.}_{0 \leq t \leq \delta} |v_j(t) - v(t)| \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

From (4.8), (2.5) it then follows that

$$(4.9) \quad \text{l.u.b.}_{0 \leq t \leq \delta} |s_j(t) - s(t)| \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

To each $s_j(t)$ we apply the proof given in Step 1 and thus conclude that each $s_j(t)$ is monotone decreasing in t . Using (4.9) it follows that $s(t)$ is monotone decreasing in t for $0 \leq t \leq \delta$. We can now repeat the argument given in Step 1 and conclude that $s(t)$ is *strictly* decreasing in t in the interval $\delta < t \leq \gamma$ (since $\dot{s}(t)$ is proved to be negative). Since δ can be taken as small as we please, it follows that $s(t)$ is *strictly* decreasing for $0 \leq t \leq \gamma$. To complete the proof of monotonicity it remains to establish (4.8).

Proof of (4.8). Defining

$$\epsilon = \text{l.u.b.}_{0 < t < \delta} |v_j(t) - v(t)|$$

and using the proof of Lemma 6, we get

$$|s_j(t) - s(t)| \leq \frac{6\alpha\epsilon}{b} t.$$

The method of proof that T is a contraction can now be used to derive the inequality

$$(4.10) \quad \|v_j - v\| \leq A_1 \delta^{\frac{1}{2}} \epsilon + A_2 \|\dot{\varphi} - \dot{\varphi}_j\| \quad (\varphi_j(x) = x\tilde{v}_j(x))$$

where A_1 depends only on $\|\dot{\varphi}\|$, α , m , n , b and A_2 is a universal constant. Taking δ so as to satisfy $2A_1\delta^{\frac{1}{2}} < 1$, we obtain

$$\limsup_{i \rightarrow \infty} \|v_i - v\| \leq 2A_2 \lim_{i \rightarrow \infty} \|\dot{\varphi} - \dot{\varphi}_i\| = 0$$

where we made use of (4.7).

Step 3: $\bar{v}_0(b) < 0$. In this case $\varphi(b) < 0$. Using Lemma 8 and replacing the first two terms on the right side of (2.4) by (3.43), we get (compare with (3.44))

$$(4.11) \quad |\bar{v}(t) - 2\varphi(b)t^{\frac{1}{2}}K(s(t), t; b, 0)| \leq \text{const. } t^{\frac{1}{2}} + \frac{5\alpha M^2}{4b}.$$

Since we can take $M = |\varphi(b)|$ and since by Lemma 8 and (3.46)

$$|2t^{\frac{1}{2}}K(s(t), t; b, 0) - \pi^{-\frac{1}{2}}| \leq \frac{1}{64}e^{1/64},$$

we easily conclude from (4.11) that if

$$(4.12) \quad \frac{5\alpha |\varphi(b)|}{4b} < \frac{1}{2},$$

then the sign of $\bar{v}(t)$, for all sufficiently small t , is the same as the sign of $\varphi(b)$, that is, $\bar{v}(t) < 0$. Clearly, (4.12) follows from (3.46).

Having proved that $\bar{v}(t) < 0$ it follows, in particular, that $\bar{v}(0) < 0$. Hence, by (2.5), $\lim t^{\frac{1}{2}}s(t) < 0$ as $t \rightarrow 0$. It follows that $\dot{s}(t) < 0$ for all $t > 0$ sufficiently small. We can now complete the proof of monotonicity by using the argument of Step 1.

We sum up the results of this section. *If $\varphi(0) = 0$, $\varphi \leq 0$, then $s(t) \searrow$ as $t \nearrow$, for any $\alpha > 0$. If $\varphi(0) < 0$, $\varphi \leq 0$, then $s(t) \searrow$ as $t \nearrow$, provided α satisfies (3.46). Similarly, if $\varphi(0) = 0$, $\varphi \geq 0$, then $s(t) \nearrow$ as $t \nearrow$, for any $\alpha > 0$, and if $\varphi(0) > 0$, $\varphi \geq 0$, then $s(t) \nearrow$ as $t \nearrow$, provided α satisfies (3.46).*

5. Existence and Uniqueness in the Large.

In this section we shall prove that the solution u , s exists as long as $s(t)$ remains finite and positive. We shall consider separately the three cases: (i) $\varphi \geq 0$, (ii) $\varphi(b) = 0$, $\varphi \leq 0$, (iii) $\varphi(b) < 0$, $\varphi \leq 0$.

Case (i): $\varphi \geq 0$. Existence for small times was established in §3 provided α satisfies (3.37). Since (3.37) implies (3.46), it follows that if (3.37) holds, then $s(t)$ is strictly monotone increasing in t . In the next chapter we shall prove that $s(t)$ remains finite in finite times. Hence we can use the method of continuation of Part II, §4. It follows (see (3.37)) that a solution can be continued for all $t > 0$ provided α satisfies

$$(5.1) \quad \limsup_{t \rightarrow \rho} \text{l.u.b.}_x \frac{\alpha |u(x, t)|}{R} < \frac{1}{20}$$

for any $0 < \rho < \infty$, where $R = \lim_{t \rightarrow \rho} s(t)$. Now, by the maximum principle [6] and the monotonicity of $s(t)$,

$$(5.2) \quad |u(x, t)| \leq \max \left\{ \|\varphi\|, m \left(\frac{s(t)}{b} - 1 \right) \right\}.$$

Hence (5.1) holds if

$$(5.3) \quad \frac{\alpha \|\varphi\|}{b} < \frac{1}{20}, \quad \frac{\alpha m}{b} < \frac{1}{20}.$$

We have thus proved the existence of u, s for all $t > 0$ provided (5.3) holds. We recall that uniqueness holds under any one of the assumptions (3.42), (3.48).

Case (ii): $\varphi(b) = 0, \varphi < 0$. Following the method Part II, §6 (and using the monotonicity of $s(t)$), we can prove, for any $\alpha > 0$, the existence and uniqueness for all $t > 0$ as long as $s(t)$ remains positive. In the next section we shall prove that there exists $t_0 < \infty$ such that $s(t) \rightarrow 0$ as $t \rightarrow t_0$.

Case (iii): $\varphi(b) < 0, \varphi \leq 0$. In §3 we proved the existence of u, s in a small interval $0 \leq t \leq \sigma$ ($\sigma > 0$) provided α satisfies (3.46). Uniqueness was proved under the assumption (3.48). $s(t)$ was also proved (in §4) to be monotone decreasing in t . We can now proceed to prove the existence and uniqueness for $\sigma \leq t < t_0$ by the method of Case (ii).

6. Bounds on $s(t)$ (when $\varphi \geq 0$) and on t_0 (when $\varphi \leq 0$).

6.1. Bound on $s(t)$ (when $\varphi \geq 0$). Integrating (1.16) in the domain $s(\tau) < x < q, 0 < \tau < t$, we find that

$$(6.1) \quad \int_0^t u_x(q, \tau) d\tau - \int_0^t u_x(s(\tau), \tau) d\tau \\ = \int_{s(t)}^\infty u(x, t) dx - \int_b^\infty \varphi(x) dx + \int_0^t u(s(\tau), \tau) \dot{s}(\tau) d\tau.$$

Using (2.1) and the assumption (Φ) we find that

$$(6.2) \quad \lim_{x \rightarrow \infty} \text{g.l.b.}_{0 \leq \tau \leq t} |u_x(x, \tau)| = 0.$$

Combining (6.1), (6.2) and using (1.18), (1.19) we get, upon taking $q \rightarrow \infty$,

$$(6.3) \quad \frac{1}{2} \left(\frac{1}{\alpha} - \frac{m}{b} \right) (s^2(t) - b^2) + \left(\frac{n}{\alpha} + m \right) (s(t) - b) + m \int_0^t \frac{d\tau}{s(\tau)} - \frac{m}{b} t \\ = \int_b^\infty \varphi(x) dx - \int_{s(t)}^\infty u(x, t) dx.$$

Using (2.1) we can estimate the last integral on the right side of (6.3) as in Part II, §5. We find, in case $\varphi \geq 0$ (then $s(t) \nearrow$ as $t \nearrow$), that

$$(6.4) \quad \left| \int_{s(t)}^{\infty} u(x, t) dx \right| \leq A_0 + A_1 t^{\frac{1}{2}} + A_2 s(t) + C_1 t + C_2 s^2(t)$$

where C_1, C_2 can be taken to be arbitrarily small and A_i are appropriate constants. We easily conclude (upon using (6.3), (6.4)) that $s(t)$ cannot remain bounded as $t \rightarrow \infty$. On the other hand, since, by (5.3), the coefficient of $s^2(t) - b^2$ in (6.3) is positive, we conclude (upon using (6.4)) that $s(t)$ is bounded in finite times. More precisely,

$$s(t) \leq B_1 + B_2 t^{\frac{1}{2}} \quad (B_1, B_2 \text{ constants}).$$

Using (5.3) and (1.6) we also derive the approximate formula

$$(6.5) \quad s^2(t) \approx \frac{2\alpha m}{b} t = \frac{4 d\sigma}{bp_0} t \quad \text{as } t \rightarrow \infty.$$

(Compare (6.5) with the formula $s(t) = \text{const. } t^{\frac{1}{2}} + \text{const.}$ given in [8].) Writing (6.5) for any two large values $t = t', t = t''$ we obtain the approximate formula

$$(6.6) \quad \frac{s(t')}{s(t'')} \approx \left(\frac{t'}{t''} \right)^{\frac{1}{2}}.$$

Notice that if $\alpha = 0$ in (1.16)–(1.20) then $s(t) \equiv b$. If $m = 0$ then for a solution to exist we must have $\varphi \equiv 0$. In that case, $s(t) \equiv b$ and $u \equiv 0$.

6.2. Bound on t_0 (when $\varphi \leq 0$). If $s(t)$ is monotone decreasing, then it can be proved that

$$(6.7) \quad \left| \int_{s(t)}^{\infty} u(x, t) dx \right| \leq A_0 + A_1 t^{\frac{1}{2}} + C_1 t$$

where C_1 again can be taken arbitrarily small. Using (6.7), (6.3) we easily conclude that the solution cannot exist for all t ; it must disappear in a finite time t_0 . We have thus completed the proof of Theorem 8.

In the next section we shall find bounds on t_0 using formulas more suitable than (6.3).

7. Numerical Results for the Decreasing Bubble.

To derive bounds on t_0 we find it more suitable to define $\tilde{c}(x, t)$, $\tilde{c}_0(x)$ in a different manner than in §1, namely,

$$(7.1) \quad \tilde{c}(x, t) = \frac{c(x, t)}{d\rho_0}, \quad \tilde{c}_0(x) = \frac{c_0(x)}{d\rho_0}.$$

We then define u, φ by (1.15) and obtain for u, s the equations (1.16), (1.17), (1.20) and

$$(7.2) \quad u(s(t), t) = s(t) + m \quad \text{for } t > 0,$$

$$(7.3) \quad \alpha u_x(s(t), t) = (n + s(t))\dot{s}(t) + \alpha \left(1 + \frac{m}{s(t)} \right) \quad \text{for } t > 0,$$

where m, n, α are defined, as before, in (1.6). Note that the assumption $\varphi \leq 0$ made in Theorem 8 now (with the new definition of φ) takes the form

$$\varphi(x) = x\tilde{c}_0(x) \leq x \frac{c^*}{d\rho_0} \leq x \left(1 + \frac{m}{b}\right).$$

We assume that the present function $\varphi(x)$ satisfies the assumption (Φ) in §1. The assertions of Theorem 8 can be proved for the present φ , provided α is sufficiently small. The analogue of (6.3) now takes the form

$$(7.4) \quad \frac{1}{2} \left(\frac{1}{\alpha} + 1 \right) (b^2 - s^2(t)) + \left(\frac{n}{\alpha} + m \right) (b - s(t)) - t - m \int_0^t \frac{d\tau}{s(\tau)} \\ + \int_b^\infty \varphi(x) dx = \int_{s(t)}^\infty u(x, t) dx.$$

We can prove that (6.7) holds. Hence, taking $t \rightarrow t_0$ and assuming that

$$(7.5) \quad K \equiv \frac{b^2}{2\alpha} + \left(\frac{n}{\alpha} + m \right) b$$

is sufficiently large, we obtain the approximate formula

$$(7.6) \quad t_0 + m \int_0^{t_0} \frac{d\tau}{s(\tau)} = K.$$

Furthermore, since $s(t)$ is monotone decreasing, we obtain

$$(7.7) \quad m \frac{t_0 - t}{s(t)} < K.$$

If surface tension is ignored, that is, if we put $m = n = 0$, then we get the approximate formula

$$(7.8) \quad t_0 = \frac{b^2}{2\alpha} = \frac{b^2}{2d}$$

which is similar to an approximation formula of [1].

We conclude this section by deriving an upper bound on t_0 by a different method. By the maximum principle [6] we find that $u(x, t) \geq 0$. Using (7.4) we get

$$(7.9) \quad \int_b^\infty \varphi(x) dx + \frac{1}{2} (1 + \alpha) b^2 + \left(\frac{n}{\alpha} + m \right) b \geq t_0 + m \int_0^{t_0} \frac{d\tau}{s(\tau)}.$$

This formula is weaker than (7.6) in the sense that we have here inequality instead of equality. However, it has the advantage that it holds for all $\alpha > 0$.

REFERENCES

- [1] P. S. EPSTEIN & M. S. PLESSET, On the stability of gas bubble in liquid-gas solution, *J. of Chemical Physics*, **18** (1950), pp. 1505-1509.
- [2] A. FRIEDMAN, Free boundary problems for parabolic equations I. Melting of solids, *J. Math. and Mech.*, **8** (1959), pp. 499-517.

- [3] A. FRIEDMAN, Free boundary problems for parabolic equations II. Condensation or evaporation of a liquid drop, *J. Math. and Mech.*, **9** (1960), pp. 19–66.
- [4] A. FRIEDMAN, Remarks on the maximum principle for parabolic equations and its applications, *Pacific J. Math.*, **8** (1958), pp. 201–211.
- [5] J. B. KELLER, Gas bubbles in liquid-gas solutions, unpublished manuscript, accepted and may appear in *J. Chemical Physics*.
- [6] L. NIRENBERG, A strong maximum principle for parabolic equations, *Comm. Pure and Appl. Math.*, **6** (1953), pp. 167–177.
- [7] M. S. PLESSETT & S. A. ZWICK, A nonsteady heat diffusion with spherical symmetry, *J. of Applied Physics*, **23** (1952), pp. 95–98.
- [8] M. S. PLESSETT & S. A. ZWICK, The growth of vapor bubbles in supersaturated liquids, *J. of Applied Physics*, **25** (1954), pp. 493–500.
- [9] S. A. ZWICK, The growth and collapse of vapor bubble, *Calif. Inst. Tech. Hydro. Lab.*, Report No. 21–19, Dec. 1954.

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