*Representations of Lie Algebras of Classical Type with Applications to Linear Groups**

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Introduction. A Lie algebra of classical type is a Lie algebra $\mathfrak{L}$ over an algebraically closed field Ω of characteristic $p \neq 2, 3, 5, 7$ with a commutative Cartan subalgebra $\mathfrak{H}$ such that $\mathfrak{L}$ and $\mathfrak{H}$ satisfy the axioms given by MILLS & SELIGMAN [10]. Any Lie algebra over Ω whose Killing form is non-degenerate is known to satisfy the axioms, so that the semi-simple Lie algebras over the complex field as well as the Lie algebras investigated in [5] and [6] all belong to the class of algebras to be studied in this paper.

In Chapter I it is shown that the methods of CARTAN, WEYL, and HARISH-CHANDRA in the representation theory of semi-simple Lie algebras over the complex field can be applied with only minor changes to obtain a complete classification of all finite dimensional, irreducible, restricted representations of an arbitrary Lie algebra $\mathfrak{L}$ of classical type.

The irreducible representations of the classical groups defined over fields of characteristic $p > 0$ have never been determined. Chapter II contains some contributions towards the solution of this problem. Let $\mathfrak{L}$ be a Lie algebra of classical type, and let G be the algebraic component of the identity in the algebraic group of all automorphisms of $\mathfrak{L}$. When $\mathfrak{L}$ is a simple Lie algebra of type A_n , B_n , C_n , or D_n , the group G can be identified with one of the classical groups by the method given by REE [11], and by Theorem 3 of this paper. It is proved that an irreducible projective representation of G can be constructed explicitly from each of the irreducible restricted representations of $\mathfrak{L}$, and that inequivalent representations of $\mathfrak{L}$ give rise to inequivalent projective representations of G .

If G is a connected Lie group, it is easily proved by the methods in [2] that every continuous finite dimensional projective representation of G can be lifted to an ordinary representation of the simply connected covering group of G . The problem arises whether it is possible to construct an algebraic covering group G^* of G to which all the irreducible projective representations of G can

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be lifted, when G is the algebraic component of the identity in the automorphism group of a Lie algebra of classical type. In §II.3, we construct a covering group G^* of G with the desired lifting property, but we do not prove that G^* is an algebraic group in all cases. A theorem is proved which enables us to identify G^* with a group of linear transformations on a vector space, provided that the Lie algebra $\mathfrak{L}$ has certain special properties. These conditions are satisfied in case $\mathfrak{L}$ is a simple Lie algebra of type A_n or C_n , and in §II.4 it is proved that the covering group G^* in these cases turns out to be the unimodular group $SL(n+1, \Omega)$ or the symplectic group $Sp(2n, \Omega)$ respectively.

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CHAPTER I

CLASSIFICATION OF THE IRREDUCIBLE MODULES

I.1. Preliminary discussion. In Chapter I, $\mathfrak{L}$ denotes a Lie algebra over an algebraically closed field Ω of characteristic $p \neq 2, 3, 5, 7$. We assume that $\mathfrak{L}$ contains a commutative Cartan subalgebra $\mathfrak{H}$ such that $\mathfrak{L}$ and $\mathfrak{H}$ satisfy the axioms (i)-(v) of MILLS & SELIGMAN [10]. The axioms are known to be satisfied, in particular, if the Killing form of $\mathfrak{L}$ is non-degenerate. Let $\alpha, \beta, \dots$ be the roots of $\mathfrak{L}$ with respect to $\mathfrak{H}$, with zero counted as a root. Then the root spaces

$$\mathfrak{L}_\alpha = \{x \mid x \in \mathfrak{L}, [xh] = \alpha(h)x \text{ for all } h \in \mathfrak{H}\}$$

corresponding to the non-zero roots are one dimensional, and we have

$$\mathfrak{L} = \mathfrak{H} + \sum_{\alpha \neq 0} \mathfrak{L}_\alpha \quad (\text{direct sum}).$$

For each root $\alpha \neq 0$, we choose a fixed element $e_\alpha \neq 0$ in $\mathfrak{L}_\alpha$. Then L has the

following multiplication table:

$$[hh'] = 0, \quad h, h' \in \mathfrak{H};$$

$$[e_\alpha h] = \alpha(h)e_\alpha, \quad h \in \mathfrak{H}, \quad \alpha \neq 0 \text{ a root};$$

$$[e_\alpha e_\beta] = \begin{cases} 0 & \text{if } \alpha + \beta \neq 0 \text{ and } \alpha + \beta \text{ is not a root,} \\ n_{\alpha\beta}e_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root } \neq 0, \text{ where } n_{\alpha\beta} \in \Omega, \\ h_\beta \in \mathfrak{H} & \text{if } \alpha + \beta = 0. \end{cases}$$

The following additional properties of the roots are essential for the rest of Chapter I: the results (i)-(xix) of [10] and Lemma 4 and Propositions 1 and 2 of [6]. The proofs given in [6] of these last three results are all valid as presented, even if the hypothesis is changed from the assumption that the Killing form of $\mathfrak{L}$ is nondegenerate to the assumption that $\mathfrak{L}$ satisfies the axioms (i)-(v) of [10], and consequently has the properties (vi)-(xix) as well. No other properties of the roots are needed in Chapter I; in particular, it is unnecessary to appeal to the classification of systems of roots given in [10], §§4-11.

Now we establish some notations. We may assume that all the elements e_α are chosen in such a way that $\alpha(h_\alpha) = 2$. Then $\beta(h_\alpha)$ belongs to the prime field Ω_0 in Ω for any non-zero roots α and β . By (viii) there exist roots $\epsilon_1, \dots, \epsilon_l$ such that $h_{\epsilon_1}, \dots, h_{\epsilon_l}$ form a basis of $\mathfrak{H}$ over Ω . With respect to this basis for $\mathfrak{H}$, we may introduce an order relation $\alpha < \beta$ among the roots; its properties are described in Lemma 4 of [6]. In particular, we may speak of the positive and negative roots of $\mathfrak{L}$ with respect to $\mathfrak{H}$. We shall denote by Δ the maximal simple system consisting of the simple roots determined according to [6], p. 96, by the order relation $\alpha < \beta$. Then the roots α_i in Δ are all positive, and every positive root α can be expressed in the form $\alpha = \sum_1^s \alpha_i$, $\alpha_i \in \Delta$, where all the partial sums $\sum_1^k \alpha_i$, $k \leq s$, are positive roots.

It will become necessary to introduce a p -power operation in $\mathfrak{L}$ so that $\mathfrak{L}$ becomes a restricted Lie algebra in the sense of JACOBSON [9], if the characteristic $p \neq 0$. For uniformity of style, we agree to call $\mathfrak{L}$ a restricted Lie algebra if $p = 0$, and set $x^p = x$ for all x in $\mathfrak{L}$ in this case. Now let $p \neq 0$, and consider the basis of $\mathfrak{L}$ consisting of the root elements e_α , $\alpha \neq 0$, and $h_{\epsilon_1}, \dots, h_{\epsilon_l}$ already defined in the preceding paragraphs. By (xviii) and (xv) of [10], we have $(\text{ad } e_\alpha)^p = 0$ for all non-zero roots α , and $(\text{ad } h_{\epsilon_i})^p = \text{ad } h_{\epsilon_i}$, since, for every root $\alpha \neq 0$, $\alpha(h_{\epsilon_i}) \in \Omega_0$. We then define $e_\alpha^p = 0$ for all non-zero roots α , and $h_{\epsilon_i}^p = h_{\epsilon_i}$, $1 \leq i \leq l$. By a result of JACOBSON ([9], pp. 17, 18) the p -power operation that we have defined on the basis elements can be extended uniquely to all elements of $\mathfrak{L}$ in such a way that $\mathfrak{L}$ becomes a restricted Lie algebra.

I.2. Classification theorems based on the theory of weights. A *restricted linearization* $(\psi, \mathfrak{A})$ of $\mathfrak{L}$ is a linear mapping from $\mathfrak{L}$ into an associative algebra

$\mathfrak{A}$ with an identity such that $\mathfrak{A}$ is generated by the identity element and the elements $\psi(s)$, $s \in \mathfrak{L}$, and

$$\psi([st]) = \psi(s)\psi(t) - \psi(t)\psi(s), \quad \psi(s^p) = \psi(s)^p, \quad s, t \in \mathfrak{L},$$

where the last p -power denotes $\psi(s)$ multiplied by itself p times in $\mathfrak{A}$ if $p > 0$. There exists a universal restricted linearization $(\psi^*, \mathfrak{U})$ with the property that if $(\psi, \mathfrak{A})$ is any restricted linearization of $\mathfrak{L}$, then there exists an associative homomorphism ϕ of $\mathfrak{U}$ onto $\mathfrak{A}$ such that $\psi = \psi^*\phi$ (because we write operators on the right, the right hand side of this equation means first ψ^* then ϕ). It is known ([1], [9]) that the mapping ψ^* is one to one, and therefore we may identify $\mathfrak{L}$ with $\psi^*(\mathfrak{L})$. Thus $\mathfrak{L}$, together with the identity element, forms a set of generators for the algebra $\mathfrak{U}$. We shall call $\mathfrak{U}$ the universal associative algebra of $\mathfrak{L}$, or briefly, the u -algebra of $\mathfrak{L}$. We shall classify the finite dimensional irreducible right $\mathfrak{U}$ -modules; we shall also speak of them as irreducible $\mathfrak{L}$ -modules. The problem is equivalent to the task of classifying all restricted linearizations $(\psi, \mathfrak{A})$ where $\mathfrak{A}$ is an irreducible algebra of linear transformations on a finite dimensional vector space over Ω .

We write $(\alpha_1, \dots, \alpha_m)$ for the set of positive roots, ordered so that $\alpha_i < \alpha_{i+1}$ for all i . Then the positive roots α_i , together with their negatives, constitute the full set of non-zero roots of $\mathfrak{L}$ with respect to $\mathfrak{S}$. The elements

$$e_{\alpha_1}, \dots, e_{\alpha_m}, h_1, \dots, h_l, e_{-\alpha_1}, \dots, e_{-\alpha_m},$$

where $h_i = h_{\alpha_i}$, form a basis for $\mathfrak{L}$. It is known ([1], [9]) that $\mathfrak{U}$ has a basis over Ω consisting of the standard monomials

$$u(P, Q, R) = e_{\alpha_1}^{p_1} \dots e_{\alpha_m}^{p_m} h_1^{q_1} \dots h_l^{q_l} e_{-\alpha_1}^{r_1} \dots e_{-\alpha_m}^{r_m},$$

where $P = (p_1, \dots, p_m)$, $Q = (q_1, \dots, q_l)$, and $R = (r_1, \dots, r_m)$ are vectors with non-negative integer coefficients, subject to no other restrictions if the characteristic $p = 0$, but with all coefficients between 0 and $p - 1$, inclusive, if $p > 0$. We write $u^0 = 1$ for all u in $\mathfrak{U}$ and write 0 for the zero vector $(0, \dots, 0)$.

1.2.1. *The subspace $\mathfrak{U}_+$ generated by the elements $u(P, 0, 0)$, $P \neq 0$, forms a subalgebra of $\mathfrak{U}$. If M is a finite dimensional right $\mathfrak{U}$ -module with annihilating ideal $\mathfrak{B}$, then there exists a positive integer m such that $\mathfrak{U}_+^m \subset \mathfrak{B}$.*

Proof. We prove first a somewhat stronger result than the statement that $\mathfrak{U}_+$ forms a subalgebra. First we define the *extent* $E(e_\alpha)$ of a root element e_α , $\alpha > 0$, to be the uniquely determined length of a chain of elements from Δ which generates the root α (see Lemma 3 of [6]). We then define the extent of a standard monomial $u(P, 0, 0)$ by $\sum p_k E(\alpha_k)$. We shall prove that if $u = u(P, 0, 0)$ and $\alpha_i > 0$, then every standard monomial appearing in the expression for ue_{α_i} with a non-zero coefficient has the form $u(P', 0, 0)$ and has extent equal to the sum of the extents of u and e_{α_i} . The proof is by induction on the total extent of u . We assume as an induction hypothesis that the result

holds for any u and $\alpha_k > \alpha_i$, and for any u' of extent $< E(u)$ together with α_i . We see that the result holds for u of extent one or for $\alpha_i = \alpha_m$, so that the induction hypothesis is legitimate. Let $u = e_{\alpha_1}^{p_1} \cdots e_{\alpha_k}^{p_k}$, $p_k > 0$. We can suppose $\alpha_k > \alpha_i$, otherwise ue_{α_i} is a standard monomial as it stands. We have

$$ue_{\alpha_i} = e_{\alpha_1}^{p_1} \cdots e_{\alpha_k}^{p_k-1} e_{\alpha_i} e_{\alpha_k} + e_{\alpha_1}^{p_1} \cdots e_{\alpha_k}^{p_k-1} [e_{\alpha_k} e_{\alpha_i}].$$

In the first summand we can apply the induction hypothesis to $e_{\alpha_1}^{p_1} \cdots e_{\alpha_k}^{p_k-1} e_{\alpha_i}$, and then to all the resulting monomials multiplied on the right by e_{α_k} , since $\alpha_k > \alpha_i$, and obtain the required conclusion for all the standard monomials making a contribution to this summand. The second summand is either zero or a multiple of $e_{\alpha_1}^{p_1} \cdots e_{\alpha_k}^{p_k-1} e_{\alpha_k + \alpha_i}$ where $\alpha_k + \alpha_i$ is a root $> \alpha_i$. Again we can apply the induction hypothesis, obtaining the desired result for the standard monomials making a contribution to the second summand, and our assertion is proved.

The statement that $\mathfrak{U}_+$ is a subalgebra is now clear. Let M be a finite dimensional $\mathfrak{U}$ -module. If the characteristic $p = 0$, it follows from the fact that vectors of different weights are linearly independent that some power of every root element lies in $\mathfrak{B}$. This statement is implied by the fact that $e_\alpha^p = 0$ for all non-zero roots α in case $p > 0$. It follows that the extent of all standard monomials $u(P, 0, 0) \notin \mathfrak{B}$ is bounded for all P . An application of the result obtained in the first part of the proof then shows that for some integer $m > 0$, $\mathfrak{U}_+^m \subset \mathfrak{B}$. This completes the proof of I.2.1.

Of course, a similar result can be proved for the space $\mathfrak{U}_-$ generated by the standard monomials $u(0, 0, R)$, $R \neq 0$.

Now let M be an irreducible right $\mathfrak{U}$ -module. By I.2.1 and the argument given in the proof of Theorem 3 of [5] we have

$$M = \sum_{\lambda \in \mathfrak{S}^*} M_\lambda \quad (\text{direct sum})$$

where the subspaces $M_\lambda = \{x \mid xh = \lambda(h)x \text{ for all } h \text{ in } \mathfrak{S}\}$ are defined for each $\lambda \in \mathfrak{S}^*$, the dual space of $\mathfrak{S}$. Those subspaces $M_\lambda \neq 0$ are called the *weight spaces*, and the corresponding linear functions the *weights* of M . The number of distinct weights is finite. The space M is a completely reducible $\mathfrak{S}$ -module, and the irreducible $\mathfrak{S}$ -submodules are one-dimensional. Consequently, for any $\mathfrak{S}$ -submodule N of M we have $N = \sum_\lambda (N \cap M_\lambda)$. For any subset $\mathfrak{S}$ of $\mathfrak{U}$ we shall write $(0: \mathfrak{S})$ for the subspace of M consisting of all elements x such that $x\mathfrak{S} = 0$.

I.2.2. *Let M be an irreducible right $\mathfrak{U}$ -module. Then $(0: \mathfrak{U}_+) \neq 0$, and there exists a unique weight λ such that $(0: \mathfrak{U}_+) \subset M_\lambda$.*

Proof. We have $(0: \mathfrak{U}_+) \neq 0$ by I.2.1. A vector $x \in (0: \mathfrak{U}_+)$ if and only if $xe_\alpha = 0$ for all roots $\alpha > 0$. It follows that $(0: \mathfrak{U}_+)$ is an $\mathfrak{S}$ -module, and has a basis consisting of weight vectors. We have to show that any two of them have the same weight. Suppose that the result is false and that $(0: \mathfrak{U}_+)$ contains

non-zero vectors x and x' belonging to different weights λ and λ' . By Lemma 6 of [5], whose proof is valid in the present case because of the properties of the order relation on the roots described in Lemma 4 of [6], any non-zero weight vector $v \in (0: \mathfrak{U}_+)$ has the property that the vectors $vu(0, 0, R)$ generate the space M . Then we can write

$$x = \xi_0 x' + \sum_{R \neq 0} \xi_R x' u(0, 0, R), \quad x' = \eta_0 x + \sum_{S \neq 0} \eta_S x u(0, 0, S),$$

with the ξ 's and η 's in Ω . Since vectors of different weights are linearly independent, $\xi_0 = \eta_0 = 0$, and upon substituting one expression in the other and using I.2.1 we obtain $x' = xu$, $u \in \mathfrak{U}_-$. The second assertion of I.2.1 implies that $x' = 0$, contrary to our original assumption, and I.2.2 is proved.

Definition. The non-zero vectors belonging to $(0: \mathfrak{U}_+)$ are called the *maximal vectors* in M ; those in $(0: \mathfrak{U}_-)$ the *minimal vectors*. The uniquely determined weight λ of an irreducible right $\mathfrak{U}$ -module M such that $(0: \mathfrak{U}_+) \subset M_\lambda$ is called the *maximal weight* of M ; similarly the *minimal weight* of M is defined.

Let M be irreducible, and let x be a maximal vector in M . Since $\mathfrak{U}_-$ is nilpotent modulo the annihilating ideal of M , $x \notin x\mathfrak{U}_-$. On the other hand, M has a basis consisting of elements in $x\mathfrak{U}_-$ together with x . Therefore $x\mathfrak{U}_-$ is a hyperplane in M , and we have

$$(2) \quad M = \Omega x \oplus x\mathfrak{U}_-.$$

I.2.3. Let x be a maximal vector in an irreducible right $\mathfrak{U}$ -module M , and let $y = xe_{\gamma_1} \cdots e_{\gamma_r}$ belong to M_λ , where the γ_i are arbitrary roots. Then there exists an element $\xi \in \Omega$ which depends only upon the sequence $(\gamma_1, \dots, \gamma_r)$ and λ , and not upon the action of $\mathfrak{U}$ upon M , such that

$$y \equiv \xi x \pmod{x\mathfrak{U}_-}.$$

Proof. It is clear by (2) that ξ exists and is uniquely determined for a particular module M . The problem is to show that ξ depends only upon λ and the sequence of roots $\gamma_1, \dots, \gamma_r$; and this is established by the now familiar argument of WEYL ([15], p. 282).

With I.2.3 in hand, the rest of WEYL's argument ([15], p. 283) can be applied almost verbatim to obtain the following result.

Theorem 1. Let M and M' be irreducible right $\mathfrak{U}$ -modules with maximal weights λ and λ' . Then M and M' are $\mathfrak{U}$ -isomorphic if and only if $\lambda = \lambda'$.

Corollary. Let M be an irreducible right $\mathfrak{U}$ -module. Then $(0: \mathfrak{U}_+)$ is one-dimensional over Ω .

Proof of corollary. By Schur's Lemma, every $\mathfrak{U}$ -endomorphism of M is a scalar multiplication by an element of Ω . Let x and x' be two non-zero maximal vectors in M . By WEYL's proof of Theorem 1, the mapping $x \rightarrow x'$ can be extended to a $\mathfrak{U}$ -endomorphism of M , and hence $x = \xi x'$ for some ξ in Ω .

For each root α , we selected h_α as the element in $[\mathfrak{L}_-\mathfrak{L}_\alpha]$ such that $\alpha(h_\alpha) = 2$. Then $h_\alpha^p = h_\alpha$ for all non-zero roots α , and, as we have shown, $\mathfrak{S}$ has a basis consisting of some of the elements h_α . Let M be a finite dimensional irreducible $\mathfrak{U}$ -module, and let λ be the maximal weight of M . Then $\lambda(h_\alpha)^p = \lambda(h_\alpha)$ for every $\alpha \neq 0$, and if the characteristic $p > 0$, then $\lambda(h_\alpha) \in \Omega_0$ for every $\alpha \neq 0$. If $p = 0$, then it is known that $\lambda(h_\alpha)$ is a non-negative rational integer ([1], p. 17-07). We define a linear function $\lambda \in \mathfrak{S}^*$ to be *rational* if $\lambda(h_\alpha) \in \Omega_0$ for all h_α , and to be *integral* if $\lambda(h_\alpha) \in \mathbb{Z} \cdot 1 \subset \Omega_0$ for all h_α . If $p > 0$, then of course there is no distinction between rational and integral linear functions. Our final objective in this section is to prove that every integral linear function, with the additional restriction that $\lambda(h_\alpha) \geq 0$ for all roots $\alpha > 0$ if the characteristic $p = 0$, is the maximal weight of some finite dimensional irreducible $\mathfrak{U}$ -module. This result is new only in case $p > 0$, and even then HARISH-CHANDRA's proof of Theorem 1 of [8] applies with only a few modifications. Because HARISH-CHANDRA actually proved a more general result, some important simplifications are possible in the present case, and it seems worthwhile to include the details.

Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{U}$ generated by 1 and the elements of $\mathfrak{S}$. Let $\mathfrak{R} = \Omega[X_1, \dots, X_l]$ be the polynomial ring in l variables X_i , and let $\mathfrak{R}$ be the ideal in $\mathfrak{R}$ generated by the elements $X_i^p - X_i$, $1 \leq i \leq l$. If $p = 0$, then $X_i^p = X_i$, and $\mathfrak{R}$ is the zero ideal. For any p , $\mathfrak{C}$ is isomorphic to $\mathfrak{R}/\mathfrak{R}$. We denote by $\mathfrak{P}$ the right ideal in $\mathfrak{U}$ generated by the e_α , $\alpha > 0$; then $\mathfrak{P} = \sum_{\alpha > 0} e_\alpha \mathfrak{U}$. We let $\mathfrak{S}$ be the finitely generated right $\mathfrak{C}$ -module consisting of all elements $\sum e_{\gamma_1} \dots e_{\gamma_r} c_{\gamma_1, \dots, \gamma_r}$, where the γ_i are negative roots such that $r > 0$ and $\gamma_1 + \dots + \gamma_r = 0$, and the $c_{\gamma_1, \dots, \gamma_r} \in \mathfrak{C}$. Of course, $\mathfrak{S} = 0$ if the characteristic is zero.

For each rational linear function λ , let $\mathfrak{D}_\lambda = \sum_{1 \leq i \leq l} (h_i - \lambda(h_i))\mathfrak{U}$ and $\mathfrak{C}_\lambda = \sum_{1 \leq i \leq l} (h_i - \lambda(h_i))\mathfrak{C}$; then $\mathfrak{D}_\lambda$ is a right ideal in $\mathfrak{U}$ and $\mathfrak{C}_\lambda$ is an ideal in the commutative algebra $\mathfrak{C}$.

1.2.4. Every rational linear function λ on $\mathfrak{S}$ can be extended to a linear function λ^* on $\mathfrak{U}$ such that $\lambda^*(1) = 1$, $\lambda^*|_{\mathfrak{C}^1}$ is a homomorphism, and $\lambda^*|\mathfrak{P} = \lambda^*|\mathfrak{S} = \lambda^*|\mathfrak{C}_\lambda = 0$.

Proof. We first extend λ to $\mathfrak{L}$ by setting $\lambda(e_\alpha) = 0$ for all roots $\alpha \neq 0$. We then define λ^* on a set of basis elements of $\mathfrak{U}$ by the formulas

$$\lambda^*(1) = 1,$$

$$\lambda^*(u(P, Q, R)) = \lambda(e_{\alpha_1})^{p_1} \dots \lambda(e_{\alpha_m})^{p_m} \lambda(h_1)^{q_1} \dots \lambda(h_l)^{q_l} \lambda(e_{-\alpha_1})^{r_1} \dots$$

where $P = (p_i)$, $Q = (q_i)$, $R = (r_i)$ and at least one P , Q , or R is not zero.

Any linear function on the space of homogeneous elements of degree one in the polynomial ring $\mathfrak{R} = \Omega[X_1, \dots, X_l]$ can be extended uniquely to a homomorphism of $\mathfrak{R}$ into Ω and will induce a homomorphism of $\mathfrak{R}/\mathfrak{R}$ into Ω , provided that it vanishes on the elements of $\mathfrak{R}$. Since λ is a rational linear function, this condition is satisfied for the linear function f such that $f(X_i) = \lambda(h_i)$, and we

¹ $\lambda^*|_{\mathfrak{C}}$ means the restriction of the mapping λ^* to $\mathfrak{C}$.

conclude that λ^* defines a homomorphism of $\mathfrak{C}$ into Ω . We have $\lambda^* | \mathfrak{C}_\lambda = 0$ since λ^* is a homomorphism.

An application of I.2.1 shows that $u \in \mathfrak{P}$ if and only if $u = \sum \xi_i u(P_i, Q_i, R_i)$, $\xi_i \in \Omega$, where, for each i , $P_i \neq 0$. Then $\lambda^* | \mathfrak{P} = 0$.

For any roots $\gamma_1, \dots, \gamma_r$ and h in $\mathfrak{S}$ we have

$$\left(\prod_1^r e_{\gamma_i} \right) h = h \left(\prod_1^r e_{\gamma_i} \right) + (\gamma_1 + \dots + \gamma_r)(h) \prod_1^r e_{\gamma_i}.$$

If $\sum \gamma_i = 0$, then $\prod_1^r e_{\gamma_i}$ commutes with every element of $\mathfrak{C}$, and for any basis element of $\mathfrak{C}$ of the form $u(0, Q, 0)$ we have by I.2.1

$$\left(\prod_1^r e_{\gamma_i} \right) u(0, Q, 0) = u(0, Q, 0) \left(\prod_1^r e_{\gamma_i} \right) = \sum_{R_i \neq 0} \xi_i u(0, Q, R_i),$$

and it follows that $\lambda^* | \mathfrak{C} = 0$. This completes the proof of I.2.4.

The next two results are essentially contained in [8].

I.2.5. $(\mathfrak{C} + \mathfrak{P}) \cap \mathfrak{C} = 0$.

Proof. Let $c = \sum \xi_i u(0, Q_i, 0) \in (\mathfrak{C} + \mathfrak{P}) \cap \mathfrak{C}$. For every rational linear function λ on $\mathfrak{S}$, we have

$$\lambda^*(c) = \sum \xi_i \lambda^*(u(0, Q_i, 0)) = 0,$$

and it is easily verified by induction on l that these equations imply that $c = 0$.

For s in $\mathfrak{S}$, the mapping

$$s \rightarrow A(s): u \rightarrow [us] = us - su, \quad u \in \mathfrak{U},$$

is a linear representation of $\mathfrak{S}$ by linear transformations acting in the vector space $\mathfrak{U}$. Following HARISH-CHANDRA [8], we define for each $P = (p_i)$ a linear function $\text{rank}(P)$ on $\mathfrak{S}$, namely $\text{rank}(P) = \sum_i p_i \alpha_i$. Because $A(s)$ is a derivation of the associative algebra $\mathfrak{U}$, it follows that each standard monomial $u(P, Q, R)$ is a weight vector for the representation A of weight $\text{rank}(P) - \text{rank}(R)$, and $\mathfrak{U}$ has a basis consisting of weight vectors. If the elements u and u' in $\mathfrak{U}$ have weights μ and μ' respectively, then uu' has weight $\mu + \mu'$. Finally we introduce the notation $|P| = \sum_i p_i$ for $P = (p_i)$.

I.2.6. Let λ be a rational linear function on $\mathfrak{S}$. Then $\mathfrak{P} + \mathfrak{D}_\lambda \neq \mathfrak{U}$.

Proof. Suppose, to the contrary, that $1 \in \mathfrak{P} + \mathfrak{D}_\lambda$. Then we must have a relation of the form

$$(3) \quad 1 = \sum_{\alpha > 0} e_\alpha u_\alpha + \sum_{1 \leq i \leq l} (h_i - \lambda(h_i)) v_i,$$

where the elements u_α and v_i belong to $\mathfrak{U}$. Since elements belonging to different weights are linearly independent, we may assume that each u_α has weight $-\alpha$ while each v_i has weight zero. We can write $v_i = \sum_j \beta_{ij} u(P_j, Q_j, R_j)$, where $\beta_{ij} \in \Omega$ and $\text{rank}(P_j) - \text{rank}(R_j) = 0$. If $|P_j| > 0$, then $u(P_j, Q_j, R_j) \in \mathfrak{P}$,

and the corresponding term in (3) is in $\mathfrak{P}$ since the elements of $\mathfrak{S}$ commute in $\mathfrak{U}$ with elements of weight zero. If $|P_i| = 0$, then either $|R_i| = 0$ and $u(P_i, Q_i, R_i) \in \mathfrak{C}$ or $|R_i| > 0$ and we have $\text{rank}(R_i) = 0$. In the latter case we have $u(P_i, Q_i, R_i) \in \mathfrak{S}$. Thus all terms on the right side of (3) are in $\mathfrak{P} + \mathfrak{S} + \mathfrak{C}_\lambda$, and there exist $u \in \mathfrak{P}$, $s \in \mathfrak{S}$, and $c \in \mathfrak{C}_\lambda$ such that $1 = u + s + c$. Then $1 - c \in (\mathfrak{S} + \mathfrak{P}) \cap \mathfrak{C} = 0$ by I.2.5. Therefore $1 = c \in \mathfrak{C}_\lambda$. Let λ^* be the linear function extending λ defined in the proof of I.2.4. Then we obtain $1 = \lambda^*(1) = \lambda^*(c) = 0$, and we have a contradiction. Therefore $\mathfrak{P} + \mathfrak{D}_\lambda \neq \mathfrak{U}$ as required.

Theorem 2. *Let λ be an arbitrary integral linear function on $\mathfrak{S}$, with the additional restriction that $\lambda(h_\alpha) \geq 0$ when $\alpha > 0$ if the characteristic of Ω is zero. Then there exists a finite dimensional irreducible right $\mathfrak{U}$ -module whose maximal weight is λ .*

Proof. By I.2.6, $\mathfrak{P} + \mathfrak{D}_\lambda \neq \mathfrak{U}$. Let $\mathfrak{I}$ be a maximal right ideal containing $\mathfrak{P} + \mathfrak{D}_\lambda$, and let $\mathfrak{U}^* = \mathfrak{U}/\mathfrak{I}$. Then $\mathfrak{U}^*$ is an irreducible right $\mathfrak{U}$ -module. We shall write u^* for the coset in $\mathfrak{U}^*$ containing an element u in $\mathfrak{U}$. Then $1 \notin \mathfrak{I}$, and $1^* \neq 0$ in $\mathfrak{U}^*$. Since $\mathfrak{P} \subset \mathfrak{I}$, $1^* \in (0: \mathfrak{U}_+)$. Since $\mathfrak{D}_\lambda \subset \mathfrak{I}$, we have, for each i , $1 \cdot (h_i - \lambda(h_i)) \in \mathfrak{D} \subset \mathfrak{I}$, and hence $1^* h_i = \lambda(h_i) 1^*$. We have proved that 1^* is a maximal vector of weight λ in $\mathfrak{U}^*$. If the characteristic $p > 0$, then $\mathfrak{U}^*$ is finite dimensional and Theorem 2 is proved. If the characteristic of Ω is zero, then for the first time we use the fact that λ is an integral function, rather than an arbitrary rational function, in order to prove that $\mathfrak{U}^*$ is finite dimensional. For a proof of this last fact we refer to [8], p. 42, or to [1], p. 17-07.

Corollary. *There is a one-to-one correspondence between the set of integral linear functions λ on $\mathfrak{S}$ (with the restriction that $\lambda(h_\alpha) \geq 0$ for $\alpha > 0$ if the characteristic of Ω is zero) and the set of all equivalence classes of isomorphic irreducible finite dimensional right $\mathfrak{U}$ -modules. In particular, if $p > 0$, then the number of non-isomorphic irreducible right $\mathfrak{U}$ -modules is exactly p^l , where l is the rank of $\mathfrak{S}$.*

CHAPTER II

APPLICATIONS TO GROUP THEORY

In Chapter II, $\mathfrak{L}$ denotes a Lie algebra over an algebraically closed field Ω of characteristic $p \neq 2, 3, 5, 7$ for which the assumptions and notations described in §I.1 and §I.2 are in force.

II.1. The group of invariant automorphisms of $\mathfrak{L}$. For each root element e_α , $(\text{ad } e_\alpha)^4 = 0$. Consequently $\exp(\text{ad } \xi e_\alpha) = \sum_{k=0}^{\infty} (k!)^{-1} (\text{ad } \xi e_\alpha)^k$, $\xi \in \Omega$, is defined and is an automorphism of $\mathfrak{L}$ (see [13], §3). Let G be the subgroup of the group $A(\mathfrak{L})$ of all automorphisms of $\mathfrak{L}$ generated by the automorphisms

$$\sigma = \sigma(\alpha, \xi) = \exp(\text{ad } \xi e_\alpha).$$

We emphasize that the generators of G are associated with the root elements belonging to a fixed Cartan decomposition of $\mathfrak{L}$. Therefore G is a subgroup of the group I of invariant automorphisms defined by SELIGMAN ([13], [14]). The group I is generated by the automorphisms $\exp(\operatorname{ad} \xi e_\alpha)$ where the e_α run through all root elements corresponding to all Cartan subalgebras satisfying the axioms of [10].

The next theorem has been established, in a slightly different context, by ONO (*Jour. Math. Soc. Japan*, **10** (1958), pp. 307–313).

Theorem 3. *The group G is an irreducible algebraic subgroup of the group $A(\mathfrak{L})$ of all automorphisms of $\mathfrak{L}$. If the Killing form of $\mathfrak{L}$ is non-degenerate, then G is the algebraic component of the identity in $A(\mathfrak{L})$, and the Lie algebra of G is isomorphic to $\mathfrak{L}$.*

Proof. The terminology and results concerning algebraic groups are all taken from CHEVALLEY'S book [3]. For any root $\alpha \neq 0$, let G_α be the set of automorphisms $\exp(\operatorname{ad} \xi e_\alpha)$, $\xi \in \Omega$. The argument on p. 159 of [3] shows, even in the case of characteristic $p > 0$, that G_α is an irreducible algebraic group. Since G is generated by the groups G_α and since Ω is assumed to be algebraically closed, it follows from Corollary 3, p. 123 of [3] that G is an irreducible algebraic group.

Now we assume that the Killing form of $\mathfrak{L}$ is non-degenerate. Because of this assumption, the Lie algebra $\mathfrak{D}$ of all derivations of $\mathfrak{L}$ is identical with the Lie algebra $\operatorname{ad}(\mathfrak{L})$ consisting of the inner derivations $\operatorname{ad} s$, $s \in \mathfrak{L}$. Now let $\mathfrak{g}$ be the Lie algebra of the algebraic group G . Let X be a transcendental element over Ω ; then $\exp(\operatorname{ad} X e_\alpha)$ is an automorphism of $\mathfrak{L}^{\Omega(X)}$ and is a generalized point of G . Let D be the derivation with respect to X in $\Omega(X)$. Then D can be extended to a derivation D' of the algebra $\mathfrak{G}(\mathfrak{L})^{\Omega(X)}$, and in this algebra we have $(X \operatorname{ad} e_\alpha)D' = \operatorname{ad} e_\alpha$. Therefore $\exp(X \operatorname{ad} e_\alpha) \operatorname{ad} e_\alpha = \exp(X \operatorname{ad} e_\alpha)D'$. By Proposition 4, p. 132 of [3], $\operatorname{ad} e_\alpha$ belongs to $\mathfrak{g}^{\Omega(X)}$. Since $\operatorname{ad} e_\alpha$ belongs to the set of linear transformations in $\mathfrak{L}$ over Ω , we have $\operatorname{ad} e_\alpha \in \mathfrak{g}$. The elements $\operatorname{ad} e_\alpha$ generate $\operatorname{ad}(\mathfrak{L})$, and we have $\operatorname{ad}(\mathfrak{L}) \subset \mathfrak{g}$. By Example II, p. 143 of [3] $\mathfrak{g} \subset \mathfrak{D} = \operatorname{ad}(\mathfrak{L})$, and we have $\mathfrak{g} = \operatorname{ad}(\mathfrak{L})$. Because the center of $\mathfrak{L}$ is zero, we obtain finally $\mathfrak{g} \cong \operatorname{ad}(\mathfrak{L}) \cong \mathfrak{L}$.

Because G and $A(\mathfrak{L})$ have the same Lie algebra and G is irreducible, we deduce that G is the algebraic component of the identity in $A(\mathfrak{L})$ by the corollary on p. 133 of [3] and Theorem 2, p. 86 of [3].

II.2. Some irreducible projective representations of the group G . Let V be a vector space over Ω , and let $GL(V)$ be the group of all invertible linear transformations of V .

Definition. A projective representation of a group G is a mapping $\sigma \rightarrow F(\sigma)$ of G into some $GL(V)$ together with a mapping $(\sigma, \tau) \rightarrow a_{\sigma\tau}$ of $G \times G$ into the multiplicative group Ω^* of Ω such that $F(\sigma)F(\tau) = F(\sigma\tau) a_{\sigma\tau}$ for all σ, τ in G ,

and such that $F(1) = 1$. The function $(\sigma, \tau) \rightarrow a_{\sigma\tau}$ is called the *factor set* of the projective representation. The projective representation F is said to be *irreducible* provided that the vector space V has no proper subspaces which are sent into themselves by all the linear transformations $F(\sigma)$, $\sigma \in G$. Two projective representations F and F' are said to be *strictly equivalent* if there exists a one-to-one linear transformation S of V onto V' and a function $\sigma \rightarrow \xi_\sigma$ of G into Ω^* such that

$$(4) \quad SF'(\sigma) = F(\sigma)S\xi_\sigma$$

for all $\sigma \in G$.

The notion of strict equivalence is an appropriate definition of equivalence if we regard projective representations as defining homomorphisms from a group into the group of collineations of a projective space (see JACOBSON, *The Theory of Rings*, Math. Surveys No. II, New York, 1943; pp. 79 ff.).

Let G be the group defined in §II.I, and let B be the set of generators $\sigma = \sigma(\alpha, \xi)$ of G . Although we may have $\sigma(\alpha, \xi) = \sigma(\alpha', \xi')$ for different roots α and α' and elements $\xi, \xi' \in \Omega$, it is impossible to have a relation $\sigma(\alpha, \xi) = \sigma(\alpha', \xi')$ where $\alpha > 0$ and $\alpha' < 0$ unless $\sigma(\alpha, \xi) = \sigma(\alpha', \xi') = 1$. This fact is proved by letting $\sigma(\alpha, \xi)$ and $\sigma(\alpha', \xi')$ act on elements of H , and by recalling that $(\sum_{\alpha > 0} \mathfrak{L}_\alpha) \cap (\sum_{\alpha' < 0} \mathfrak{L}_{\alpha'}) = 0$. We may partition the elements of B into two classes B_+ and B_- , where $\sigma(\alpha, \xi) \in B_+$ if $\alpha > 0$ and $\sigma(\alpha, \xi) \in B_-$ if $\alpha < 0$. We have shown that the assignment of a generator to B_+ or B_- is independent of the way the generator is expressed in the form $\sigma(\alpha, \xi)$, and we have $B_+ \cap B_- = \{1\}$. Because $\sigma \in B$ implies $\sigma^{-1} \in B$, each automorphism $\tau \in G$ can be expressed as a product of the generators from B :

$$\tau = \sigma_1 \sigma_2 \cdots \sigma_t, \quad \sigma_i \in B.$$

For each $\tau \in G$, we select a fixed expression for τ with the minimal number of factors σ , from B . Although, in general, these minimal factorizations are not uniquely determined, the elements of B obviously do have a unique minimal expression as a product of elements of B .

Because of the definition of the p -power operation in $\mathfrak{L}$, it follows that all automorphisms τ of $\mathfrak{L}$ are restricted in the sense that they commute with the p -power operation. Therefore each automorphism τ of $\mathfrak{L}$ can be extended uniquely to an automorphism of the u -algebra $\mathfrak{U}$ of $\mathfrak{L}$, and we shall denote this extended automorphism also by τ .

Now let M be a finite dimensional irreducible right $\mathfrak{U}$ -module and let λ be the maximal weight of M . We shall define an irreducible projective representation of G by transformations acting in the space M . We select a fixed maximal vector x_+ and a fixed minimal vector x_- in M .

Let $\sigma \in G$; then we define a right $\mathfrak{U}$ -module $M^{(\sigma)}$ whose underlying vector space is M and whose composition is given by

$$x * u = xu^\sigma, \quad x \in M, \quad u \in \mathfrak{U}.$$

If $\sigma = \sigma(\alpha, \xi) \in B_+$, then, for any positive root β , we have

$$x_+ * e_\beta = x_+ e_\beta^\sigma = x_+ \left(\sum_k (k!)^{-1} \xi^k [\cdots \underbrace{[e_\beta e_\alpha] \cdots e_\alpha}_k] \right) = 0$$

since all the multiple commutators $[\cdots [e_\beta e_\alpha] \cdots]$ are either zero or are multiples of root elements belonging to positive roots. For $h \in \mathfrak{H}$ we have

$$x_+ * h = x_+ h = x_+ \left(\sum_k (k!)^{-1} \xi^k [\cdots \underbrace{[h e_\alpha] \cdots e_\alpha}_k] \right) = x_+ h = \lambda(h) x_+.$$

We have proved that x_+ is a maximal vector of $M^{(\sigma)}$ of weight λ . By Theorem 1, $M \cong M^{(\sigma)}$ and the mapping

$$F(\sigma): x_+ e_{\gamma_1} \cdots e_{\gamma_r} \rightarrow x_+ e_{\gamma_1}^\sigma \cdots e_{\gamma_r}^\sigma$$

defines an invertible linear transformation of M such that

$$(5) \quad (xu)F(\sigma) = xF(\sigma) * u = xF(\sigma)u^\sigma$$

for all $x \in M$, $u \in \mathfrak{U}$. In the characteristic zero case it can be shown that the mapping $F(\sigma) = \exp(\xi T(e_\alpha))$ where T is the representation of $\mathfrak{g}$ defined by the $\mathfrak{U}$ -module M ; however, this is not necessarily the case if Ω has characteristic $p > 0$.

If $\sigma \in B_-$, then the mapping

$$F(\sigma): x_- e_{\gamma_1} \cdots e_{\gamma_r} \rightarrow x_- e_{\gamma_1}^\sigma \cdots e_{\gamma_r}^\sigma$$

defines an invertible linear transformation of M such that (5) holds, by exactly the same reasoning. We set $F(1) = 1$; we have now defined in a unique way, given the choices of x_+ and x_- , the transformations $F(\sigma)$ for all elements $\sigma \in B$.

Now let $\tau \in G$, and let $\tau = \sigma_1 \sigma_2 \cdots \sigma_r$ be a minimal factorization of τ in terms of the generators σ_i in B . We define

$$F(\tau) = F(\sigma_1)F(\sigma_2) \cdots F(\sigma_r)$$

and verify at once that $F(\tau)$ satisfies the relation (5) for all $u \in \mathfrak{U}$. From (5) it follows that, for σ, τ in G , the transformation $F(\sigma)F(\tau)F(\sigma\tau)^{-1}$ belongs to the centralizer of the module M . Because M is absolutely irreducible, Schur's Lemma implies that there exists a function $(\sigma, \tau) \rightarrow a_{\sigma\tau}$ on $G \times G$ to Ω^* such that $F(\sigma\tau) = F(\sigma)F(\tau)a_{\sigma\tau}^{-1}$, and we have proved that F is a projective representation of G .

Because the expressions $\tau = \sigma_1 \cdots \sigma_r$, $\sigma_i \in B$, are not necessarily unique for an element τ not in B , there is some ambiguity in our definition of $F(\tau)$ for elements $\tau \notin B$; by Schur's Lemma, however, it follows that any two determinations of $F(\tau)$ for $\tau \notin B$ differ by a factor from Ω . Therefore the two projective representations of G which are obtained are strictly equivalent, and the multiplier function ξ_σ which appears in (4) has the additional property that $\xi_\sigma = 1$ for all $\sigma \in B$. The next theorem shows that there is no further ambiguity in the definition of F ; in other words, we cannot get strictly equivalent projective representations from non-isomorphic irreducible $\mathfrak{U}$ -modules.

Theorem 4. *The projective representation F is irreducible. If two projective representations F and F' of G belonging to irreducible $\mathfrak{U}$ -modules M and N' are strictly equivalent and if the function ξ_σ in (4) takes the value 1 for all $\sigma \in B$, then M and M' are $\mathfrak{U}$ -isomorphic.*

Proof. The proofs of both statements are based on the following result.

II.2.1. *Let $w \neq 0$ be an arbitrary element of the irreducible $\mathfrak{U}$ -module M , and let α be a non-zero root. Then there exist vectors $w_2, w_3, \dots$ in M , depending on w , such that for all $\xi \in \Omega$ we have*

$$(6) \quad wF(\sigma(\alpha, \xi)) = w + \xi we_\alpha + \xi^2 w_2 + \xi^3 w_3 + \dots$$

Proof of II.2.1. It is sufficient to consider the case $\alpha > 0$. The vector w is a linear combination of vectors of the form

$$v = x_+ e_{\gamma_1} \cdots e_{\gamma_r},$$

where the γ_i are all negative roots. As a second reduction, it is sufficient to prove the result for such a vector v . By the definition of $F(\sigma(\alpha, \xi))$ we have, for all $\xi \in \Omega$,

$$\begin{aligned} vF(\sigma(\alpha, \xi)) &= x_+ \left\{ \sum_k \frac{\xi^k}{k!} e_{\gamma_1} (\text{ad } e_\alpha)^k \right\} \cdots \left\{ \sum_m \frac{\xi^m}{m!} e_{\gamma_r} (\text{ad } e_\alpha)^m \right\} \\ &= y_0 + \xi y_1 + \xi^2 y_2 + \cdots, \end{aligned}$$

where the y_i 's are computed as though ξ is an indeterminate and at most a finite number of them are different from zero. We shall compute y_0 and y_1 . We have by inspecting the sum

$$y_0 = x_+ e_{\gamma_1} \cdots e_{\gamma_r} = v;$$

and

$$\begin{aligned} y_1 &= \sum_{i=1}^r \xi x_+ e_{\gamma_1} \cdots e_{\gamma_{i-1}} [e_{\gamma_i} e_\alpha] e_{\gamma_{i+1}} \cdots e_{\gamma_r} = \xi x_+ [e_{\gamma_1} \cdots e_{\gamma_r}, e_\alpha] \\ &= \xi x_+ e_{\gamma_1} \cdots e_{\gamma_r} e_\alpha = \xi v e_\alpha \end{aligned}$$

because $x_+ e_\alpha = 0$ and $\text{ad } e_\alpha$ is a derivation in the associative algebra $\mathfrak{U}$. This completes the proof of II.2.1.

Now we are ready to prove the irreducibility of F . Let $N \neq 0$ be a subspace of M such that $NF(\sigma) \subset N$ for all $\sigma \in G$, and let $w \neq 0$ be an arbitrary element of N . For any root α , choose $w_2, w_3, \dots$ according to II.2.1 such that (6) holds for all $\xi \in \Omega$. Then for all $\xi \neq 0$ we obtain

$$we_\alpha + \xi w_2 + \cdots \in N.$$

Because Ω is an infinite field, we can use a Vandermonde determinant argument to conclude that $we_\alpha \in N$. Because the e_α are generators of the Lie algebra $\mathfrak{g}$,

we have $N\mathfrak{U} \subset N$, and $N = M$ since M is an irreducible $\mathfrak{U}$ -module. Therefore F is an irreducible projective representation.

Finally let F and F' be strictly equivalent, and let S and ξ_σ be given as in (4) such that $\xi_\sigma = 1$ for all $\sigma \in B$. Then for all $\sigma(\alpha, \xi)$ in B and all $w \in M$ we have $wSF'(\sigma(\alpha, \xi)) = wF(\sigma(\alpha, \xi))S$. Apply II.2.1 to both w and wS , obtaining elements $w_2, w_3, \dots$ in M and $w'_2, w'_3, \dots$ in M' such that, for all ξ in Ω ,

$$wF(\sigma(\alpha, \xi)) = w + \xi we_\alpha + \xi^2 w_2 + \dots$$

and

$$wSF'(\sigma(\alpha, \xi)) = wS + \xi wSe_\alpha + \xi^2 w'_2 + \dots.$$

Applying (5), subtracting, and dividing by ξ we have, for all $\xi \neq 0$,

$$[(we_\alpha)S - (wS)e_\alpha] + \xi[w_2S - w'_2] + \xi^2[w_3S - w'_3] + \dots = 0.$$

By a Vandermonde determinant argument again we have $(we_\alpha)S = (wS)e_\alpha$ for all roots $\alpha \neq 0$. Because the e_α generate $\mathfrak{L}$, these relations imply that M and M' are $\mathfrak{U}$ -isomorphic, and Theorem 4 is proved.

Example. If we do not insist that M be a restricted $\mathfrak{L}$ -module, then the construction of a projective representation of G by the method we have outlined is impossible. For example, let $\mathfrak{L}$ be the 3-dimensional Lie algebra over a field Ω of characteristic $p > 2$ with basis elements $e_\alpha, e_{-\alpha}, h_\alpha$ such that $[e_\alpha h_\alpha] = 2e_\alpha, [e_{-\alpha} h_\alpha] = -2e_{-\alpha}$, and $[e_{-\alpha} e_\alpha] = h_\alpha$. Let $\tau = \exp(\text{ad } e_{-\alpha}) \exp(\text{ad } -e_\alpha)$; then $\tau \in G$ and $e_\alpha^\tau = -e_{-\alpha}$. By the methods of [7] it can be shown that there exists a finite dimensional irreducible representation T of $\mathfrak{L}$ such that $T(e_\alpha)^p = 1, T(e_{-\alpha})^p = 0$. Then the construction of this section, if applicable, would require that T be equivalent to the representation U such that $U(s) = T(s^\tau)$ for all s in $\mathfrak{L}$. But $U(e_\alpha) = -T(e_{-\alpha})$ and $U(e_\alpha)^p = (-1)^p T(e_{-\alpha})^p = 0$, so that U and T cannot be equivalent representations of $\mathfrak{L}$.

II.3. A covering group of G . Let $M_1, M_2, \dots$ be a complete set of non-isomorphic irreducible right $\mathfrak{U}$ -modules. By Theorems 1 and 2, there will be a countably infinite number of the M_i if the characteristic of Ω is zero and $p^i, i = \text{rank } \mathfrak{L}$, modules M_i in case the characteristic of Ω is $p > 0$. For each i , let F_i be the irreducible projective representation of G by linear transformations in M_i defined in the preceding section, and let G_i be the group consisting of all linear transformations $F_i(\sigma_1) \cdots F_i(\sigma_r)$, where $r \geq 1$ and the σ_i are arbitrary generators of G belonging to B .

Now consider the direct product $\prod_i G_i$ of all the groups G_i . For a fixed ordered set of generators $(\sigma_1, \dots, \sigma_r)$ of G , let $g = g(\sigma_1, \dots, \sigma_r)$ be the element of $\prod_i G_i$ whose i^{th} component g_i is given by $g_i = F_i(\sigma_1) \cdots F_i(\sigma_r)$. The set of all such elements g in $\prod_i G_i$ forms a subgroup G^* of $\prod_i G_i$, which is a subdirect product in the sense that all the projections $\pi_i: G^* \rightarrow G_i$ are homomorphisms of G^* onto G_i . The mapping $\omega: g(\sigma_1, \dots, \sigma_r) \rightarrow \sigma_1 \cdots \sigma_r$ of G^* onto G is single-valued because $g(\sigma_1, \dots, \sigma_r) = 1$ implies by (5) that, for all $s \in \mathfrak{L}$,

$$s^{\sigma_1 \cdots \sigma_r} - s \in \bigcap_{i=1}^{\infty} [(0: M_i) \cap \mathfrak{L}] = 0,$$

since $\mathfrak{L}$ has a faithful completely reducible restricted module, and we have $\sigma_1 \cdots \sigma_r = 1$. Then ω is a homomorphism of G^* onto G , and by (5) and Schur's Lemma we see that the kernel of ω is contained in the center of G^* . In this sense G^* is a covering group of G . For each i the projection π_i defines an irreducible representation of G^* by linear transformations acting in the space M_i , because, as we have shown, $G^*\pi_i = G_i$ and M_i is irreducible relative to G_i by Theorem 4.

Let M be an arbitrary finite dimensional irreducible right $\mathfrak{U}$ -module. For each $s \in \mathfrak{L}$, the transformation $x \rightarrow xs$, x in M , can be extended uniquely to a derivation $D(s)$ of the tensor algebra² $T(M)$ defined over the vector space M . The mapping $s \rightarrow D(s)$ is a restricted representation of $\mathfrak{L}$ by linear transformations acting in $T(M)$, and consequently the algebra $T(M)$ becomes a right $\mathfrak{U}$ -module. The finite dimensional submodules of $T(M)$ and their homomorphic images are called the $\mathfrak{U}$ -components of $T(M)$.

Theorem 5. *Let M_1 be an irreducible right $\mathfrak{U}$ -module such that every irreducible right $\mathfrak{U}$ -module is isomorphic to a $\mathfrak{U}$ -component of $T(M_1)$. Then the mapping π_1 of G^* onto G_1 is an isomorphism.*

Proof. For each $\sigma \in G$, let $A(\sigma)$ be the unique automorphism of $T(M_1)$ which which extends the invertible linear transformation $F_1(\sigma)$ acting in M_1 . Let $G(T)$ be the group of linear transformations in $T(M_1)$ generated by all the $A(\sigma)$, $\sigma \in G$. Then the mapping $F_1(\sigma) \rightarrow A(\sigma)$ can be extended to an isomorphism of G_1 onto $G(T)$. Now let V be a finite dimensional $G(T)$ -submodule of $T(M_1)$ and W be a maximal $G(T)$ -submodule of V . We shall prove that V/W is an irreducible $\mathfrak{U}$ -component of $T(M_1)$, and that if $V/W \cong M_i$, then there is a natural homomorphism of G_1 onto G_i .

We begin by proving the formula

$$(7) \quad t D(s) A(\sigma) = t A(\sigma) D(s'),$$

for all $\sigma \in G$, $t \in T(M_1)$, and $s \in \mathfrak{L}$. The formula holds for $t \in M_1$ by (5). If (7) holds for t_1 and t_2 , then a calculation shows that (7) holds for $t_1 t_2$. Moreover, it is sufficient to verify (7) for homogeneous elements of $T(M_1)$. From these remarks (7) follows at once by induction on the degree of t .

Next we prove that if $t \in T(M_1)$ and if α is a non-zero root, then there exists a finite set of elements $t_2, t_3, \dots$ in $T(M_1)$ such that, for all $\xi \in \Omega$, we have

$$(8) \quad t A(\sigma(\alpha, \xi)) = t + \xi t D(e_\alpha) + \xi^2 t_2 + \xi^3 t_3 + \dots$$

This result is valid for t in M_1 by II.2.1, and is easily proved in the general case by induction on the degree of t .

² See [3], Chapter I, §1. As in [3], we write tt' instead of $t \otimes t'$ to denote the product of the elements t and t' in $T(M_1)$.

By (8) it follows by the same Vandermonde determinant argument used in the proof of Theorem 4 that every $G(T)$ -submodule of $T(M_1)$ is also a $\mathfrak{U}$ -submodule.

In particular, V and W are $\mathfrak{U}$ -submodules of $T(M_1)$, and we can find a $\mathfrak{U}$ -submodule V_1 such that $W \subset V_1 \subset V$ and such that V_1/W is an irreducible $\mathfrak{U}$ -module. Let $v_1 \in V_1$ be a vector such that the coset $v_1 + W$ is the distinguished maximal vector of V_1/W selected as in §II.2. Let α and β be positive roots, and let $\sigma = \sigma(\alpha, \xi)$ for $\xi \in \Omega$. Then $v_1 A(\sigma) \notin W$ because W is a $G(T)$ -module, and we have by (7) the fact that $v_1 A(\sigma) D(e_\beta) = v_1 D(e_\beta^{\sigma^{-1}}) A(\sigma) \in W$. Therefore $v_1 A(\sigma) + W$ is also a non-zero maximal vector in V_1/W . By the Corollary to Theorem 1, the space of maximal vectors in V_1/W is one dimensional, and we have

$$(9) \quad v_1 A(\sigma) = v_1 A(\sigma(\alpha, \xi)) \equiv a(\xi) v_1 \pmod{W},$$

where $\xi \rightarrow a(\xi)$ is a mapping from Ω into Ω^* . Applying (7) we obtain

$$(10) \quad v_1 D(e_{\gamma_1}) \cdots D(e_{\gamma_r}) A(\sigma) \equiv a(\xi) v_1 D(e_{\gamma_1}^{\sigma}) \cdots D(e_{\gamma_r}^{\sigma}) \pmod{W}$$

for any roots γ_i . Because the cosets $v_1 D(s) + W$, $s \in \mathfrak{S}$, generate the irreducible $\mathfrak{U}$ -module V_1/W , we infer from (10) that V_1 is invariant relative to $A(\sigma)$. It follows from the definition of $F_1(\sigma)$ that $1 - F_1(\sigma)$ is nilpotent on M_1 . Then $1 - A(\sigma)$ is nilpotent on V_1 , and hence on V_1/W . Therefore the linear transformation induced by $A(\sigma)$ on V_1/W has only the eigenvalue 1, and we have $a(\xi) = 1$ for all $\xi \in \Omega$. The formula (10) becomes

$$(10) \quad v_1 D(e_{\gamma_1}) \cdots D(e_{\gamma_r}) A(\sigma) \equiv v_1 D(e_{\gamma_1}^{\sigma}) \cdots D(e_{\gamma_r}^{\sigma}) \pmod{W}$$

for all roots γ_i . Consequently the mapping induced by $A(\sigma)$ in V_1/W is identical with the mapping $F(\sigma)$ associated with the space $M = V_1/W$ according to §II.2. We can draw all the same conclusions for the mappings $A(\sigma)$ where $\sigma = \sigma(\alpha, \xi)$, $\alpha < 0$, and v_1 is replaced by a coset representative of the distinguished minimal vector in V_1/W . It follows that V_1 is invariant relative to all the transformations $A(\sigma)$, $\sigma \in G$, and we have $V_1 = V$. Moreover, we have from (10) and its counterpart for the coset representative of the minimal vector in V/W that the mapping

$$(11) \quad F_1(\sigma_1) \cdots F_1(\sigma_i) \rightarrow F_i(\sigma_1) \cdots F_i(\sigma_i)$$

is a homomorphism of G_1 onto G_i , where G_i is the group associated with the irreducible module $V/W \cong M_i$.

It is not necessarily true that every $\mathfrak{U}$ -submodule of $T(M_1)$ is also a $G(T)$ -submodule; nevertheless the hypothesis of the theorem does imply that every irreducible $\mathfrak{U}$ -module M_i is isomorphic to V/W , where V/W is an irreducible $G(T)$ -component of $T(M_1)$. Therefore the mapping (11) is a homomorphism for every i . Now let $g \in G^*$, and let $g\pi_1 = 1$. Then by (11) $g\pi_i = 1$ for every i , and $g = 1$. Therefore π_1 is an isomorphism, and Theorem 5 is proved.

Corollary. *Let M_1 be a module which satisfies the hypotheses of Theorem 5, and let M_i be any irreducible $\mathfrak{U}$ -module. Then the mapping*

$$\theta_i : F_1(\sigma_1) \cdots F_1(\sigma_r) \rightarrow F_i(\sigma_1) \cdots F_i(\sigma_r)$$

is an irreducible representation of G_1 by linear transformations acting in M_i . Moreover, two such representations θ_i and θ_j of G_1 are equivalent if and only if $i = j$.

II.4. Determination of the covering group in some special cases. The description of the algebras of type A_n and C_n is taken, with some minor changes, from [12], §XVII, the facts concerning their representations from [15].

(a) **Type A_n . The unimodular Lie algebra $\mathfrak{L}$.** Let V be a vector space of dimension $n + 1$ over Ω such that $p \nmid n + 1$, and let $v_1, \dots, v_{n+1}$ be a basis of V over Ω . Then $\mathfrak{L}$ is the Lie algebra of all linear transformations A of trace zero, while $\mathfrak{S}$ is the subalgebra of all linear transformation $H = \text{diag} \{ \eta_1, \dots, \eta_{n+1} \}$, where $\sum \eta_i = 0$. The mappings $\lambda_i : \lambda_i(H) = \eta_i$ are linear functions on $\mathfrak{S}$ whose sum is the zero linear function. The root elements in $\mathfrak{L}$ with respect to $\mathfrak{S}$ are the transformations $E_\alpha = E_{ij}$, $1 \leq i, j \leq n + 1$, $i \neq j$, where E_{ij} is the linear transformation such that $v_i E_{ij} = v_j$, $v_k E_{ij} = 0$, $k \neq i$. The root α belonging to E_{ij} is $\lambda_j - \lambda_i$. A maximal simple system of roots is given by

$$\alpha_1 = \lambda_1 - \lambda_{n+1}, \quad \alpha_2 = \lambda_2 - \lambda_1, \quad \dots, \quad \alpha_n = \lambda_n - \lambda_{n-1},$$

and we have

$$E_{\alpha_1} = E_{n+1,1}, E_{\alpha_2} = E_{12}, \dots, E_{\alpha_n} = E_{n-1,n};$$

$$H_{\alpha_1} = E_{11} - E_{n+1,n+1}, \dots, H_{\alpha_n} = E_{nn} - E_{n-1,n-1};$$

$$E_{-\alpha_1} = E_{1,n+1}, \dots, E_{-\alpha_n} = E_{n,n-1}.$$

Because the characteristic p of Ω does not divide $n + 1$, the Killing form of $\mathfrak{L}$ is non-degenerate, and L satisfies the axioms of [10].

The space V is an irreducible $\mathfrak{L}$ -module with maximal vector v_n and maximal weight λ_n . We shall prove that V satisfies the hypothesis of Theorem 5. Let $T(V)$ be the tensor algebra defined over V , and for each k , $1 \leq k \leq n$, let S_k be the submodule of $T(V)$ consisting of the skew symmetric tensors of degree k . The space S_k has a basis consisting of the tensors

$$[v_{i_1}, \dots, v_{i_k}] = \sum \pm v_{i_1} v_{i_2} \cdots v_{i_k},$$

where the sum is over all permutations $(1', 2', \dots, k')$ of $(1, 2, \dots, k)$ and the sign is $+$ or $-$ according as the permutation is even or odd. For each k , S_k is an irreducible $\mathfrak{L}$ -module with maximal weight $\Lambda_k = \lambda_{n-k+1} + \dots + \lambda_n$, $1 \leq k \leq n$, and maximal vector $v^{(k)} = [v_{n-k}, \dots, v_n]$. The maximal weights Λ_k have the property that $\Lambda_k(H_{\alpha_{n-k+1}}) = 1$, $\Lambda_k(H_{\alpha_i}) = 0$, $i \neq n - k + 1$. Now let M be any irreducible $\mathfrak{L}$ -module whose maximal weight is $\lambda = a_1^* \Lambda_1 + \dots + a_n^* \Lambda_n$, where the a_i^* belong to the prime field in Ω and are the images of non-negative rational integers a_i . Then $v = (v^{(1)})^{a_1} \cdots (v^{(n)})^{a_n}$ is a maximal vector in $T(V)$ of weight λ . Consider the cyclic module $v\mathfrak{L} \subset T(V)$, and let W be a submodule

of $v\mathfrak{U}$ such that $v \notin W$ and any submodule properly containing W contains v . Then W is a maximal submodule of $v\mathfrak{U}$, and $v\mathfrak{U}/W$ is an irreducible $\mathfrak{U}$ -module whose maximal weight is λ . By Theorem 1, $v\mathfrak{U}/W \cong M$, and we have proved that every irreducible $\mathfrak{U}$ -module is isomorphic to a $\mathfrak{U}$ -component of $T(V)$.

By Theorem 5, the projection π_1 of G^* upon G_1 , the group associated with the module V , is an isomorphism. We proceed to identify the group G_1 . We can write the basis elements of V in the form $v_n, v_n E_{nj}, 1 \leq j \leq n-1, v_n E_{n, n+1}$. Thus all the basis elements except for v_n have the form $v_n E_{-\alpha}$, where α is a positive root. For any generator $\sigma = \sigma(\alpha, \xi), \alpha > 0$, of G , using the particular form of the basis elements and the fact that $E_\alpha^2 = 0$, we verify that

$$F(\sigma) = \exp \xi E_\alpha = 1 + \xi E_\alpha.$$

Similarly we see that for $\sigma = \sigma(\alpha, \xi), \alpha < 0$, we have $F(\sigma) = 1 + \xi E_\alpha$. Therefore G_1 is generated by the transformations $1 + \xi E_{ij}, i \neq j$, and it follows that $G_1 \cong SL(n+1, \Omega)$ (see [11], p. 396). It is also easily verified that G is isomorphic to the quotient of G_1 by its center, and hence $G \cong PSL(n+1, \Omega)$.

(b) **Class C_n . The symplectic Lie algebra $\mathfrak{L}$.** Let V be a $2n$ -dimensional space over Ω , such that $p \nmid n+1$, with basis $v_1, \dots, v_{2n}$. The Lie algebra $\mathfrak{L}$ in this case is the set of all linear transformations A in V whose matrices (α) with respect to the given basis satisfy the equation

$$(\alpha) = -S^{-1}(\alpha)'S$$

where

$$S = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix},$$

with 1_n the n by n identity matrix, and $(\alpha)'$ is the transpose of (α) . The Lie algebra $\mathfrak{L}$ has a Cartan subalgebra $\mathfrak{H}$ consisting of the transformations $H = \text{diag} \{ \eta_1, \dots, \eta_n, -\eta_1, \dots, -\eta_n \}$. Let $\lambda_i, 1 \leq i \leq n$, be the linear functions on $\mathfrak{H}$ such that $\lambda_i(H) = \eta_i$. The algebra $\mathfrak{L}$ has a maximal simple system of roots

$$\alpha_i = \lambda_i - \lambda_{i+1}, \quad 1 \leq i \leq n-1; \quad \alpha_n = 2\lambda_n.$$

Upon defining E_{ij} as in (a) we have

$$\begin{aligned} E_{\alpha_i} &= E_{i+1, i} - E_{i+n, i+n+1}, & 1 \leq i \leq n-1; & \quad E_{\alpha_n} = E_{2n, n}; \\ H_{\alpha_i} &= E_{ii} - E_{i+1, i+1} - E_{i+n, i+n} + E_{i+n+1, i+n+1}, & 1 \leq i \leq n-1; \\ H_{\alpha_n} &= E_{nn} - E_{2n, 2n}; \\ E_{-\alpha_i} &= E_{i, i+1} - E_{i+n+1, i+n}, & 1 \leq i \leq n-1; & \quad E_{-\alpha_n} = E_{n, 2n}. \end{aligned}$$

Because $p \nmid n+1$, the algebra $\mathfrak{L}$ has a non-degenerate Killing form and satisfies the axioms of [10].

The space V is an irreducible $\mathfrak{u}$ -module with maximal weight λ_1 and maximal vector v_1 . As in (a) let S_k be the space of skew symmetric tensors of degree k defined over V , $1 \leq k \leq n$. Then S_k has a maximal vector $[v_1, \dots, v_k]$ of weight $\Lambda_k = \lambda_1 + \dots + \lambda_k$. The weights Λ_k satisfy the conditions $\Lambda_k(H_{\alpha_k}) = 1$, $\Lambda_k(H_{\alpha_i}) = 0$, $i \neq k$. Exactly as in (a) we see that every irreducible $\mathfrak{u}$ -module is isomorphic to a $\mathfrak{u}$ -component of $T(V)$.

By Theorem 5, it is important again to identify the group G_1 associated with the module V . By induction on i it follows that among the negative root elements the following transformations appear:

$$\begin{aligned} E_{i,n+i}, \quad 1 \leq i \leq n; \\ E_{1i} - E_{n+i,n+1}, \quad 2 \leq i \leq n; \\ E_{1,n+i} + E_{i,n+1} = [E_{1i} - E_{n+i,n+1}, E_{i,n+i}], \quad 2 \leq i \leq n. \end{aligned}$$

Therefore V has a basis consisting of the maximal vector v_1 together with vectors of the form $v_1 E_{-\alpha}$, $\alpha > 0$. Then as in case (a) we obtain for $\sigma = \sigma(\alpha, \xi)$, $\alpha > 0$,

$$F(\sigma) = \exp \xi E_{\alpha} = 1 + \xi E_{\alpha}.$$

The same result holds for $\sigma = \sigma(\alpha, \xi)$, $\alpha < 0$. Therefore G_1 is generated by the transformations $1 + \xi E_{\alpha}$ as α ranges over the set of all roots of $\mathfrak{g}$. Thus we obtain among the generators of G_1 all the generators of the group $Sp(2n, \Omega)$ (see [11], p. 396 and p. 394), and it follows that G_1 contains the symplectic group $Sp(2n, \Omega)$. Since all the transformations $1 + \xi E_{\alpha}$ belong to $Sp(2n, \Omega)$, we have $G_1 = Sp(2n, \Omega)$. In this case, G is isomorphic to the quotient of $Sp(2n, \Omega)$ by its center.

We may summarize the results of this section in the following theorem.

Theorem 6. *Let $\mathfrak{g}$ be a simple Lie algebra of type A_n , $n \geq 1$, or C_n , $n \geq 2$. Then there exists an irreducible $\mathfrak{u}$ -module M_1 which satisfies the hypothesis of Theorem 6. The group G_1 associated with M_1 is isomorphic to $SL(n+1, \Omega)$ in the first case and to $Sp(2n, \Omega)$ in the second. In both cases, G is isomorphic to G_1/Z_1 , where Z_1 is the center of G_1 .*

II.5. Concluding remarks. We list three problems which remain unsettled and which we plan to take up in subsequent paper.

(i) Is it possible to give an explicit formula, similar to WEYL's formula in the case of semi-simple Lie algebras of characteristic zero, for the dimensions of the irreducible $\mathfrak{u}$ -modules discussed in Chapter I?

(ii) In all cases it should be possible to identify the covering group G^* of G with an algebraic group in such a way that the representations π_i are rational representations of G^* . The question arises whether all rational irreducible representations of G^* can be obtained by some direct construction from the representations π_i .

(iii) If the parameters ξ which define the "one parameter subgroups" G_{α} are

restricted to a subfield Ω' of Ω , a subgroup of G is obtained. For example, when Ω' is a finite field, finite subgroups of G arise. It seems to be worthwhile to identify these groups, and to determine whether the irreducible projective representations of G give rise to irreducible projective representations of these subgroups.

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