*Partial Differential Equations with More Than Two Independent Variables in the Complex Domain**

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Introduction. In this paper we shall be primarily concerned with linear partial differential equations of the second order whose coefficients are analytic. We shall be interested in properties of the solutions of such equations which can be studied by means of an analytic continuation into the domain of complex values of the independent variables. The case of two independent variables has been thoroughly investigated from this point of view, and our contribution here will therefore be to problems involving three or more variables. The significance of such a study lies in the hope it offers of extending to several variables some of the theories for partial differential equations in two variables which depend on conformal mapping or on the introduction of characteristic coordinates.

The possibility of mapping an arbitrary two-dimensional surface conformally onto a plane permits us to reduce the general linear elliptic partial differential equation of the second order in two independent variables to a canonical form in which the highest derivatives appear as a Laplacian. This result is connected with the reduction to canonical form of a hyperbolic equation in two variables by means of the introduction of characteristic coordinates. The two processes are, indeed, equivalent for an equation with analytic coefficients, and the relation between them is derived through extension of the solution into the complex domain. For more independent variables a simple reduction to canonical form is not feasible, but it is not unnatural to expect that extension into the complex domain might nevertheless yield a successful analysis that could compensate in part for this deficiency. Such is the case, for example, with HADAMARD'S construction [7] of the fundamental solution of a second order equation in several variables, which he motivates by identifying the Riemann function as the coefficient of the logarithmic factor in the fundamental solution of an equation in two variables, and which he then performs in the complex domain.

One of the principal theorems about elliptic partial differential equations with analytic coefficients is that every solution must be analytic. For linear or non-

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linear elliptic equations in two independent variables, LEWY [9] has given a proof of this result based on extension into the complex domain by means of the solution of hyperbolic initial value problems. HADAMARD [7] uses his fundamental solution to prove the theorem in the case of linear equations with several independent variables, but for non-linear equations in space of more dimensions no proof through direct analytic extension into the complex domain is known. Similarly, in the plane LEWY [10] has developed an elaborate theory of the reflection of solutions of elliptic partial differential equations across curves upon which an analytic boundary condition has been imposed, but no satisfactory generalization of his results has heretofore been achieved in space. The interest in such a generalization stems partly from the usefulness of the method of images for the explicit solution of a wide class of boundary value problems.

The importance of investigating partial differential equations in the complex domain is made evident by these few remarks. However, we wish to emphasize that in the present work we are only able to make a beginning in this direction. Even in situations where our theory is complete, we shall prefer to present it through a study of special, but typical, examples rather than through a treatment of the most general cases. Our objective will be to simplify in so far as is possible the complicated geometry of the characteristics of a partial differential equation in the complex domain, and at the same time to retain the essential features and difficulties of each problem.

Our first topic will be the solution of Cauchy's problem for a second order partial differential equation in several independent variables. We shall develop a method of analytic continuation and contour integration in the complex domain which yields an especially simple derivation of the classical expression for the solution of Cauchy's problem for a linear equation in terms of the fundamental solution. Our procedure, which is related to the calculus of residues, seems much more natural and direct than HADAMARD's approach based on the concept of the finite part of an improper integral. It permits us, for example, to explain Huygens' principle in space of higher dimensions with a minimum of explicit calculation. We discuss the formula for the solution of Cauchy's problem also for elliptic partial differential equations. Use of complex variables is essential for this aspect of our work, which is motivated by applications which arise in connection with inverse methods for solving various free boundary problems of mathematical physics.

In addition to our study of Cauchy's problem, we sketch briefly a theory of mixed boundary value problems for the wave equation in several independent variables. This material is included because it plays an important role in our subsequent discussions and because the earlier treatment of the same subject by PROTTER [11, 12] is not sufficiently complete for our purposes.

One of our principal contributions is the introduction of what we shall describe as the reflected fundamental solution of a linear elliptic partial differential equation. If a fundamental solution is given on one side of an analytic section

of a boundary surface along which it vanishes, we call its analytic continuation into the region on the other side of the surface a reflected fundamental solution with respect to this surface. We give a construction of the reflected fundamental solution in terms of the solutions of certain mixed boundary value problems, and we investigate the locus of its singularities. Our approach has much in common with HADAMARD's representation of the fundamental solution and leans heavily on the theory of the characteristics of a partial differential equation in the complex domain.

Our technique of contour integration for the solution of Cauchy's problem in space can be applied to the reflected fundamental solution mentioned above in order to derive a reflection rule for the solutions of a linear elliptic partial differential equation which vanish on a given analytic surface. This rule expresses the solution at a point on one side of the analytic surface in terms of a rather complicated integral of it and its derivatives over a region on the other side, which is small when the point in question lies near the surface. Such a representation serves to establish that the analytic continuation across the given surface can be performed in a fixed neighborhood which does not depend on the particular solution involved, but only on the boundary condition that has been imposed. A result of this more general type can also be obtained for non-linear elliptic equations, under appropriate hypotheses.

The integral in our reflection rule can be simplified in special cases if we permit expressions to appear which only have significance in the large. We shall present an example of this phenomenon which can be based on an application of Huygens' principle. It is worth mentioning in this connection that the difference in the nature of the fundamental solution of a partial differential equation in space of an even number of dimensions and in space of an odd number of dimensions crops up throughout our analysis.

Cauchy's Problem for the Wave Equation. In the next two sections we wish to present a simple derivation, based on Green's theorem and contour integration in the complex domain, of the classical formula in terms of the fundamental solution for the calculation of solutions of Cauchy's problem for a linear hyperbolic partial differential equation in several independent variables. Our method depends on performing an analytic continuation into the domain of complex values of the independent variables and is therefore applicable only to equations with analytic coefficients. We begin by developing it for the wave equation with two space variables, a case in which the geometry involved can be easily visualized.

Our point of departure is the familiar formula

$$(1) \quad u(x, y, z) = \frac{1}{4\pi} \iint_s \left[\frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{1}{r} \right] d\sigma$$

for a harmonic function u of the three variables x, y, z in terms of data at points

ξ, η, ζ of a closed surface S containing the point x, y, z in its interior. Here

$$(2) \quad \frac{1}{r} = \frac{1}{[(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2]^{\frac{1}{2}}}$$

is the fundamental solution of Laplace's equation, $d\sigma$ is the element of area on S , and n is the outer normal of S . The integral on the right in (1) depends analytically on x, y and z and is defined for complex values of these three arguments in a domain restricted so that r does not vanish for any choice of the point ξ, η, ζ lying on the surface S . Suppose that the origin is located inside S and that x and y are real, but that

$$(3) \quad z = it$$

is pure imaginary. From the Cauchy-Riemann equations it is evident that

$$(4) \quad \frac{\partial^2 u}{\partial z^2} = -\frac{\partial^2 u}{\partial t^2},$$

and therefore u satisfies the wave equation

$$(5) \quad u_{xx} + u_{yy} - u_{tt} = 0$$

as a function of x, y and t . For a fixed choice of x, y and t , which we can at first assume to be small relative to S , our objective is to determine the locus of points ξ, η, ζ such that r vanishes. We shall then proceed to deform the surface S down around this locus and shall thus be led to a representation of the solution of the initial value problem for (5).

Since

$$(6) \quad r^2 = (x - \xi)^2 + (y - \eta)^2 + (it - \zeta)^2,$$

we see that r vanishes only on the circle

$$(7) \quad (\xi - x)^2 + (\eta - y)^2 = t^2, \quad \zeta = 0$$

of radius t about the point $x, y, 0$ in the (ξ, η) -plane. The integral on the right in (1) is independent of the surface of integration S , by Green's theorem, and therefore we can draw S down upon a pair of coincident discs of the form

$$(8) \quad (\xi - x)^2 + (\eta - y)^2 \leq (t - \epsilon)^2, \quad \zeta = 0$$

with their edges joined around the circle (7) by the torus

$$(9) \quad [(\xi - x)^2 + (\eta - y)^2 + \zeta^2 - t^2 - \epsilon^2]^2 = 4t^2(\epsilon^2 - \zeta^2),$$

which can be generated through rotation of a circle of radius ϵ (see Fig. 1). On the two discs r must have opposite signs, because it is defined by a square root for which the circle (7) is a branch line. On the other hand, the outer normal n has opposite directions on the two discs, and this change in sign cancels the previous one, so that the two contributions to the integral (1) over the discs are equal and combine to form one single integral there. We now let ϵ approach

FIGURE 1. Geometry of the surface of integration S for $m = 2$.

zero. Because r involves a square root, the integral (1) extended over the torus (9) consists only of terms in odd powers of $\epsilon^{\frac{1}{2}}$ and cannot yield any residue in the limit. We are therefore left with an integral over the full disc

$$(10) \quad (\xi - x)^2 + (\eta - y)^2 \leq t^2, \quad \zeta = 0$$

which represents the solution of Cauchy's problem for the wave equation (5) with data prescribed there.

The simplest case of the formula which we have just described is that in which u vanishes identically for $t = 0$, or, in other words, for $z = 0$. Here letting $\epsilon \rightarrow 0$ immediately yields the result

$$(11) \quad v(x, y, t) = \frac{1}{2\pi} \iint_{(\xi-x)^2 + (\eta-y)^2 \leq t^2} \frac{v_r(\xi, \eta, 0) d\xi d\eta}{[t^2 - (\xi - x)^2 - (\eta - y)^2]^{\frac{1}{2}}},$$

where we have substituted $\zeta = i\tau$ and where

$$(12) \quad v(x, y, t) = u(x, y, it)$$

can be interpreted as a real solution of the wave equation (5). Notice that taking the partial derivative of (11) with respect to t provides an expression for the solution v_t of the wave equation which has the arbitrary initial values $v_t(x, y, 0)$ and a vanishing initial derivative

$$(13) \quad v_{tt}(x, y, 0) = v_{xx}(x, y, 0) + v_{yy}(x, y, 0) = 0$$

at $t = 0$. This remark establishes that the general representation for the solution

of Cauchy's problem for the wave equation in two space dimensions, without any assumption about the initial data, is

$$(14) \quad v(x, y, t) = \frac{1}{2\pi} \frac{\partial}{\partial t} \iint_{(\xi-x)^2 + (\eta-y)^2 \leq t^2} \frac{v(\xi, \eta, 0) d\xi d\eta}{[t^2 - (\xi-x)^2 - (\eta-y)^2]^{\frac{1}{2}}} \\ + \frac{1}{2\pi} \iint_{(\xi-x)^2 + (\eta-y)^2 \leq t^2} \frac{v_r(\xi, \eta, 0) d\xi d\eta}{[t^2 - (\xi-x)^2 - (\eta-y)^2]^{\frac{1}{2}}},$$

PARSEVAL's classical formula.

The initial value problem for the wave equation

$$(15) \quad v_{x_1 x_1} + \cdots + v_{x_m x_m} - v_{tt} = 0$$

in m space variables can be solved in a similar manner. Here we set

$$(16) \quad r^2 = (x_1 - \xi_1)^2 + \cdots + (x_m - \xi_m)^2 + (x_{m+1} - \xi_{m+1})^2,$$

and we denote by $\omega_m/(m-1)$ the m -dimensional area of the surface of the unit sphere in $(m+1)$ -dimensional space. We consider now the expression

$$(17) \quad u(x_1, \cdots, x_{m+1}) = \frac{1}{\omega_m} \int_S \left[\frac{1}{r^{m-1}} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{1}{r^{m-1}} \right] d\sigma$$

for a harmonic function u of $m+1$ variables $x_1, \cdots, x_{m+1}$ as an integral over a closed surface S of dimension m . Once more we make the complex substitution

$$(18) \quad x_{m+1} = it,$$

and for $x_1, \cdots, x_m$ real and x_{m+1} pure imaginary we deform the surface of integration S down upon the m -dimensional analogue of a pair of discs and a torus about the $(m-1)$ -dimensional sphere

$$(19) \quad (\xi_1 - x_1)^2 + \cdots + (\xi_m - x_m)^2 = t^2, \quad \xi_{m+1} = 0$$

where r vanishes. The generalized torus is generated by an $(m-1)$ -parameter family of circles of radius ϵ , and the generalized discs coincide with the solid sphere

$$(20) \quad (\xi_1 - x_1)^2 + \cdots + (\xi_m - x_m)^2 \leq (t - \epsilon)^2, \quad \xi_{m+1} = 0.$$

There are now two cases to be distinguished, according as m is even or odd. If m is even, we have a situation like that for $m=2$ which we described in detail above. Both the sign of r^{m-1} and the direction of the outer normal n are reversed as we proceed from one of the generalized discs of integration to the other in (17), so that these two contributions are equal and combine to form one integral over the solid sphere (20). As $\epsilon \rightarrow 0$, the integral over the generalized torus about the $(m-1)$ -dimensional sphere (19) cannot leave a residue, since it has an expansion in exclusively odd powers of $\epsilon^{\frac{1}{2}}$. Hence we obtain in the limit

a representation for the solution of Cauchy's problem as an m -dimensional integral over the solid sphere

$$(21) \quad (\xi_1 - x_1)^2 + \cdots + (\xi_m - x_m)^2 \leq t^2, \quad \xi_{m+1} = 0.$$

For odd values of m , matters turn out to be quite different. In this case, only even powers of r appear in our calculations, whence no square roots are involved. As we proceed from one of our generalized discs of integration to the other, r^2 does not change its sign, and hence the reversal in the direction of the normal serves to cancel out the two contributions to the integral (17) there altogether. Thus no integral over the solid sphere (20) remains at all. We are left with an integration over the generalized torus of radius ϵ around the $(m-1)$ -dimensional sphere (19). For $\epsilon \rightarrow 0$, this latter term has an expansion in integral powers of ϵ , and it can be computed by the calculus of residues. The representation which we obtain in this manner for the solution of Cauchy's problem with m odd is an integral over the $(m-1)$ -dimensional surface (19) alone. The observation that the answer does not depend on data over the interior of the m -dimensional solid sphere (21) is merely a restatement of Huygens' principle. Thus we are able to verify that Huygens' principle is valid for odd m , but not for even m , with a bare minimum of calculation, since our proof is based only on the fact that even powers of r are single-valued, but odd powers are not, in the complex domain.

The solution of the initial value problem for the wave equation is much easier to calculate by our method when m is odd than it is when m is even, since in the former case the limit process of letting ϵ approach zero can be facilitated by means of the calculus of residues. In higher dimensions, this brings higher derivatives of the initial data into the answer. For the simplest case $m = 3$ with u itself initially zero, the integral (17) clearly reduces, except for a numerical factor, to the average of the partial derivative u_t over the sphere (19). The numerical factor can be evaluated from the special case $u \equiv t$, and the result can be used in the same fashion that led us from (11) to (14) in order to yield the general representation

$$(22) \quad \begin{aligned} v(x_1, x_2, x_3, t) = & \frac{1}{4\pi} \frac{\partial}{\partial t} t \int_0^{2\pi} \int_0^\pi v(x_1 + t \sin \theta \cos \varphi, \\ & x_2 + t \sin \theta \sin \varphi, x_3 + t \cos \theta, 0) \sin \theta \, d\theta \, d\varphi \\ & + \frac{t}{4\pi} \int_0^{2\pi} \int_0^\pi v_t(x_1 + t \sin \theta \cos \varphi, \\ & x_2 + t \sin \theta \sin \varphi, x_3 + t \cos \theta, 0) \sin \theta \, d\theta \, d\varphi \end{aligned}$$

of the solution

$$(23) \quad v(x_1, x_2, x_3, t) = u(x_1, x_2, x_3, it)$$

of the initial value problem for the wave equation in three space dimensions with no restriction on the initial data. This is POISSON'S classical formula.

Cauchy's Problem for Hyperbolic and Elliptic Equations. Our method as described thus far applies only to the wave equation with initial data prescribed in a hyperplane. We shall show next how to generalize it not only to an arbitrary space-like initial surface, but also to an arbitrary partial differential equation of the hyperbolic type with analytic coefficients. The most important innovation required for this purpose is the replacement of analytic continuation under the sign of integration in expressions such as (17) by deformation of the surface S of integration through the complex domain, which is feasible by virtue of Cauchy's theorem. Such an artifice is needed because in more than two independent variables it is not possible to convert the general hyperbolic partial differential equation into an elliptic one by means of a simple complex substitution.

We introduce the most general linear partial differential equation

$$(24) \quad L[u] = \sum_{i,k=1}^{m+1} a_{i,k} u_{x_i x_k} + \sum_{k=1}^{m+1} b_k u_{x_k} + cu = 0$$

of the second order in $m + 1$ independent variables $x_1, \dots, x_{m+1}$ with analytic coefficients $a_{i,k}$, b_k , c , and we consider its adjoint

$$(25) \quad M[v] = \sum_{i,k=1}^{m+1} (a_{i,k} v)_{x_i x_k} - \sum_{k=1}^{m+1} (b_k v)_{x_k} + cv = 0.$$

We assume without loss of generality that

$$(26) \quad a_{i,k} = a_{k,i},$$

and we define

$$(27) \quad B[u, v] = \sum_{k=1}^{m+1} \left\{ \sum_{j=1}^{m+1} [v a_{j,k} u_{x_j} - u (a_{j,k} v)_{x_j}] + b_k u v \right\} \\ \cdot (-1)^{k+1} dx_1 \cdots dx_{k+1} dx_{k-1} \cdots dx_{m+1}.$$

Using the standard conventions of the calculus of exterior differential forms, we can derive from Green's theorem the familiar formula

$$(28) \quad \int_D \{vL[u] - uM[v]\} dx_1 \cdots dx_{m+1} = \int_S B[u, v],$$

where S is an m -dimensional surface bounding the $(m + 1)$ -dimensional domain D . Suppose now that u and v are analytic and that $x_1, \dots, x_{m+1}$ are complex with real and imaginary parts given by

$$(29) \quad x_k = s_k + it_k \quad \text{for } k = 1, \dots, m; \quad x_{m+1} = t_{m+1} + is_{m+1}.$$

By Stokes' theorem and the Cauchy-Riemann equations, formula (28) remains valid when D is an arbitrary $(m + 1)$ -dimensional manifold in the $(2m + 2)$ -dimensional complex space of the variables $x_1, \dots, x_{m+1}$ and when S is its m -dimensional boundary there. We shall have occasion to use this generalized result for analytic solutions u and v of (24) and (25), since it then shows that the integral on the right in (28) is independent of the surface of integration S .

We can define a fundamental solution $G(P, Q)$ of (24) and (25) in the usual manner [7], where P denotes a point with the coordinates $x_1, \dots, x_{m+1}$ and Q denotes a point with the coordinates $\xi_1, \dots, \xi_{m+1}$. Let Γ be a non-trivial solution of the first order partial differential equation

$$(30) \quad \sum_{j,k=1}^{m+1} a_{j,k} \Gamma_{x_j} \Gamma_{x_k} = 4\Gamma$$

which vanishes along the characteristic conoid for (25) that has its vertex at Q . Then $G(P, Q)$ is a solution of (25) as a function of P and a solution of (24) as a function of Q which has a representation of the form

$$(31) \quad G(P, Q) = \frac{U(P, Q)}{\Gamma^{\frac{1}{2}(m-1)}} + V(P, Q) \log \Gamma,$$

where U and V are regular and $U(Q, Q) \neq 0$. The coefficient V drops out when m is even, and when m is odd it satisfies (25) as a function of P and (24) as a function of Q .

If (24) is an equation of normal hyperbolic type, we can make a real affine transformation of coordinates so that at the point Q it has the canonical form

$$(32) \quad u_{x_1 x_1} + \dots + u_{x_m x_m} - u_{x_{m+1} x_{m+1}} + \dots = 0,$$

where the dots stand for terms involving only u and its first derivatives. However, in the general case this canonical form can only be achieved at the single point Q . Suppose without loss of generality that

$$(33) \quad \xi_1 = \dots = \xi_{m+1} = 0$$

and let S denote for the moment the m -dimensional sphere

$$(34) \quad s_1^2 + \dots + s_{m+1}^2 = \epsilon^2, \quad t_1 = t_2 = \dots = t_{m+1} = 0$$

of radius ϵ in the $(2m + 2)$ -dimensional space of complex values of $x_1, \dots, x_{m+1}$ as indicated in (29). We can normalize the fundamental solution G so that

$$(35) \quad u(Q) = \frac{1}{i} \int_S B[u(P), G(P, Q)]$$

for any regular solution u of (24). The derivation of (35) follows from the fact that the integral on the right is, as we pointed out earlier, independent of the surface S and therefore of the radius ϵ , and as $\epsilon \rightarrow 0$ the integrand is asymptotically equivalent to that in (17), by virtue of (32) and the hypothesis that

$$(36) \quad x_{m+1} = i s_{m+1}$$

is pure imaginary. We shall exploit our observation that (35) does not depend on the choice of the surface of integration S in a region where $\Gamma \neq 0$ and G is regular in order to shift S so that this formula reduces to a representation of the solution of Cauchy's problem.

Suppose that the Cauchy data are to be prescribed on a surface which is nowhere characteristic in the $(m + 1)$ -dimensional space with real coordinates $x_1, \dots, x_{m+1}$, and suppose that the characteristic conoid $\Gamma = 0$ cuts a section T out of this surface which is topologically equivalent to a solid m -dimensional sphere. Our purpose is to reduce the integral (35) over the spherical surface S to an integral over T . In order to understand how this should be done, we can make a preliminary transformation of coordinates such that the equation of the conoid $\Gamma = 0$ simplifies to become

$$(37) \quad x_1^2 + \dots + x_m^2 = x_{m+1}^2,$$

and such that the surface T lies in a hyperplane

$$(38) \quad x_{m+1} = \text{const.} > 0.$$

In the new coordinate system, the deformation of S through the complex domain which we shall use to draw it down upon T is the same for the general hyperbolic equation (24) as it would be for the wave equation (15). The appropriate deformation to use is suggested by our earlier treatment of the special case (15) through analytic continuation under the integral sign in (17).

We move S forward in an advancing hyperplane $t_{m+1} = \text{const.}$, and we augment the constant continuously until it attains the value occurring in (38). At the same time, we do not allow s_{m+1} to exceed the small positive amount ϵ , and this can all be achieved without letting S intersect the conoid $\Gamma = 0$. Of course, on S the sum of squares $s_1^2 + \dots + s_m^2$ must be permitted to assume large values to keep pace with the increasing size of t_{m+1}^2 . This geometry can best be understood and visualized in a cylindrical coordinate system about the x_{m+1} -axis, which serves to make it as simple as it is in the case $m = 1$. When we have reached the final hyperplane $t_{m+1} = \text{const.}$, we proceed to draw S down upon a coincident pair of solid spheres inside T , but slightly smaller than T , joined at their boundaries by a generalized torus around the intersection of T and the conoid $\Gamma = 0$, which is actually the boundary of T (see Fig. 2). The concluding step is to let the small radius of the generalized torus approach zero, so that by a limit process like that performed earlier, during our discussion of the wave equation, we obtain in the end an integral over T or its boundary alone. At all stages of the deformation of S that we have thus described, the integral (35) remains independent of the surface of integration S and continues to be equal to $u(Q)$, since we have kept S in the region $\Gamma \neq 0$ where u and G are, respectively, regular solutions of (24) and its adjoint (25). Therefore our final formula is HADAMARD's classical representation by means of the fundamental solution G of the solution of Cauchy's problem for the hyperbolic partial differential equation (24) with initial data given on the space-like section of m -dimensional surface T .

FIGURE 2. Deformation of S for $m = 1$.

Our procedure has the disadvantage that it applies only to the case of analytic coefficients $a_{i,k}$, b_k and c , since it involves integration in the complex domain. However, it does assure us in advance, by its very nature, that the final limit process of allowing the generalized torus to shrink into the boundary of T must, indeed, yield an explicit representation of the solution of Cauchy's problem. Also, several significant properties of this solution which are related to Huygens' principle can be read off by our method with no more detailed calculations of the actual result than those we have already presented. In particular, for even values of m the square root of the function Γ occurs throughout in (35), according to (31), and therefore, as in the special case of the wave equation, our final expression for the solution of Cauchy's problem here is an integral over the full solid sphere T . Thus Huygens' principle cannot hold for any partial differential equation in an odd number $m + 1$ of independent variables. On the other hand, when m is odd the only multiple-valued terms in (35) are those involving the logarithm of Γ . It follows from our earlier reasoning that in this case the solution of Cauchy's problem reduces to integrals over the boundary of T only, except for one term evaluated over the interior of T which is an integral there of $2\pi B[u, V]$. Thus Huygens' principle holds for a partial differential equation (24) with m odd if and only if the coefficient V of the logarithmic term in the definition (31) of the fundamental solution vanishes.

All the actual results that we have described thus far are well known, and the

only new issue we have raised is that of the reduction of their derivation to a bare minimum of effort. However, no strong distinction between the hyperbolic and elliptic cases appears in our procedure based on deformation of the surface of integration S through the complex domain, which we might view as a kind of generalized method of contour integration. Therefore we might well try to establish a representation of the form

$$(39) \quad u(Q) = \lim_{S \rightarrow T} \int_S B[u(P), G(P, Q)]$$

for the solution of Cauchy's problem for an equation (24) of the elliptic type with analytic Cauchy data, where the limit sign should indicate a suitable reduction of the closed surface of integration S to an appropriate section T in the complex domain of the manifold where the data are prescribed. This can be achieved for two independent variables by introducing complex characteristic coordinates, or by putting the equation (24) into canonical form and making a complex substitution like (18), which is really the same thing [1, 6]. For more than two variables a result of the type (39) can also be obtained, but the geometry involved is not so easy to visualize. We start with S defined to be a sphere about Q in the pure imaginary domain, and this must be deformed until it collapses on a portion of the manifold of Cauchy data in the complex domain. Meanwhile we must never allow S to cut the conoid $\Gamma = 0$ in the complex domain, which extends to infinity there. A principal difficulty is that the complex intersection of $\Gamma = 0$ with the manifold of initial data also extends to infinity, whereas we must draw S down upon a finite section T of this manifold of data bounded by a limited piece of the conoid.

We can always introduce new coordinates so that the manifold where the Cauchy data are prescribed reduces to the hyperplane

$$(40) \quad x_{m+1} = \text{const.},$$

and then the issue becomes whether or not there exists within this hyperplane in the complex domain a finite section of the conoid $\Gamma = 0$ which is topologically equivalent to the $(m - 1)$ -dimensional surface of a sphere. Since we persist in our assumption (33), Γ has a convergent Taylor's series expansion of the form

$$(41) \quad \Gamma = \sum_{j,k=1}^{m+1} A_{j,k} x_j x_k + \dots$$

in the neighborhood of the origin, where the matrix $A_{j,k}$ is the inverse of the matrix $a_{j,k}$ evaluated at Q , and where the dots denote terms of degree higher than two. That such a regular expansion exists can be verified by direct substitution of (41) into the partial differential equation (30) and examination of the resulting recursion formulas. A linear transformation of the variables $x_1, \dots, x_{m+1}$ can be made which diagonalizes the quadratic form on the right in (41), but which at most alters x_{m+1} by a numerical factor. Hence there is no loss of generality if we assume in the first place that

$$(42) \quad A_{j,k} = 0 \quad \text{when } j \neq k, \quad A_{kk} = 1, \quad \text{for } j, k = 1, \dots, m+1.$$

Let us denote by ϵ the constant on the right in (40), and suppose that we restrict $x_1, \dots, x_m$ to have the order of magnitude ϵ . Then the equations for the intersection of the conoid $\Gamma = 0$ with the hyperplane (40) take the form

$$(43) \quad \sum_{k=1}^m x_k^2 + \epsilon^2 = f(x_1, \dots, x_m, \epsilon), \quad x_{m+1} = \epsilon,$$

where $f(x_1, \dots, x_m, \epsilon)$ stands for terms which are of the order ϵ^3 or higher. If these terms drop out, the desired subset of the locus (43) which is topologically equivalent to the $(m-1)$ -dimensional surface of a sphere can be found by requiring

$$(44) \quad x_k = \text{pure imaginary}$$

for $k = 1, \dots, m$. This is essentially the case of a partial differential equation (24) with constant coefficients $a_{j,k}$.

If ϵ is small enough, we can treat the general case by rewriting the first relation (43) in the form

$$(45) \quad \sum_{k=1}^m [x_k(1 - \omega_k f/x_k^2)^{\frac{1}{2}}]^2 + \epsilon^2 = 0,$$

where $\omega_1, \dots, \omega_m$ are non-negative, infinitely differentiable functions of the complex variables $x_1, \dots, x_m$ such that

$$(46) \quad \sum_{k=1}^m \omega_k = 1$$

and such that ω_k vanishes identically when $|x_k|$ becomes smaller than an appropriate positive numerical factor times ϵ . With this rearrangement of terms, we can find a subset of the locus (43) topologically equivalent to the $(m-1)$ -dimensional surface of a sphere by setting

$$(47) \quad x_k(1 - \omega_k f/x_k^2)^{\frac{1}{2}} = \text{pure imaginary}$$

for $k = 1, \dots, m$. This means that if Q is sufficiently near to the manifold where the initial data are prescribed, we can determine a finite m -dimensional section T of this manifold bounded by a finite $(m-1)$ -dimensional portion of its intersection (43) with the conoid $\Gamma = 0$. We can draw the surface of integration S in (39) down upon this locus T just as we did in the hyperbolic case and we can thus derive the final desired representation of the solution of Cauchy's problem for an elliptic partial differential equation (24). Since T itself lies in the complex domain here, our use of analytic continuation is essential for a result of this nature.

We shall illustrate the usefulness of formula (39) by applying it to construct in space steady, irrotational flows of an incompressible fluid with a prescribed constant pressure free surface. We must assume that this surface is analytic,

and we shall denote it by R . The flow is governed by a velocity potential u which is harmonic in the region occupied by the fluid. Along R we impose the two boundary conditions

$$(48) \quad \frac{\partial u}{\partial n} = 0, \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 1,$$

which express the requirements that there be no flow through R and that the pressure be constant on R . It is to be noted that the relations (48) only determine u on R to within an arbitrary function of one parameter, since it merely is required to satisfy a first order partial differential equation there. Let us suppose, however, that this arbitrary function has been fixed in an analytic manner, so that u possesses analytic Cauchy data on R . Then our inverse free surface flow problem reduces to an analytic Cauchy problem for Laplace's equation in three independent variables x, y, z . As such it can be solved by means of the representation formula (39), where T must be chosen as an appropriate section of the extension of the surface R into the complex domain. Heretofore results of this type had only been stated for plane or axially symmetric flows [3].

With these few remarks we close our discussion of the Cauchy problem. We have only to add that our approach is not unrelated to recent work by LERAY [8].

Mixed Boundary Value Problems for the Wave Equation. For the most part, we confine our discussion of mixed boundary value problems in this section to the wave equation, although corresponding results are valid for the general case of any partial differential equation of the hyperbolic type. Our purpose is to indicate briefly the combinations of boundary conditions which lead to correctly posed problems, since this question will arise in what follows. Our attitude is that this can best be understood by examination of the simplest examples.

We consider the wave equation

$$(49) \quad u_{xx} + u_{yy} + u_{zz} - u_{tt} = 0$$

in three space variables x, y, z . We suppose that u alone is prescribed on the solid sphere

$$(50) \quad x^2 + y^2 + z^2 \leq t_0^2, \quad t = 0,$$

and on the mantle of the cone

$$(51) \quad x^2 + y^2 + z^2 = (t - t_0)^2, \quad t \geq 0$$

situated below the cone

$$(52) \quad (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = t^2, \quad t \geq 0,$$

where

$$(53) \quad x_0^2 + y_0^2 + z_0^2 < t_0^2, \quad t_0 > 0$$

FIGURE 3. Schematic indication of the geometry of the mixed problem.

(see Fig. 3). We shall prove that this mixed data determines u in a unique way in the region lying between the sphere (50), the mantle (51) and the cone (52). We shall give an explicit solution of the problem based on Huygens' principle.

There is no loss of generality if we assume that u vanishes on the sphere (50), since when this is not already the case we can subtract off a suitable solution (22) of Cauchy's problem to achieve such a situation. With u known on the characteristic (51), the wave equation (49) reduces there to an ordinary differential equation for the normal derivative $\partial u / \partial n$. Since $\partial u / \partial n$ can be calculated directly at the boundary of the sphere (50), it can be found along the mantle (51) by integration, and thus we are in possession of the Cauchy data there. By Huygens' principle, this data suffices to determine u at all points of the cone

$$(54) \quad x^2 + y^2 + z^2 = (t + t_0)^2, \quad -t_0 \leq t \leq 0,$$

such that

$$(55) \quad t^2 > (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2.$$

But because u is assumed to vanish at $t = 0$, it is an odd function of t , and it can therefore be found at the reflected image of the point set (54), (55) in the hyperplane $t = 0$. This image just serves to extend the mantle (51) to a complete cone, and our construction now defines u there. It is well known that the values of u on a complete section of characteristic cone reaching to the vertex determine the solution u of (49) explicitly in a region cut off by an opposed cone, and

therefore the mixed problem we have introduced can be solved in closed form as indicated.

The mixed boundary value problem for which we have just presented an exact solution can also be attacked by means of an appropriate integral equation. The latter approach is useful and will be described here because it can be generalized to cover partial differential equations other than the wave equation (49). In our development of this new method we shall allow the cone (51) to be replaced at the outset by any characteristic

$$(56) \quad \varphi(x, y, z, t) = 0$$

of the wave equation (49) which is very nearly conical. Otherwise the geometry of the problem should remain precisely as it was before.

To derive an integral equation, we note that prescribing u on the mantle of the characteristic (56) between the hyperplane $t = 0$ and the cone (52) is equivalent by (22) to giving the averages of $u_i(x, y, z, 0)$ over all spheres inside the closed surface

$$(57) \quad \varphi(x, y, z, 0) = 0$$

in three-dimensional space which touch this surface, but do not enclose the point x_0, y_0, z_0 . For a fixed point ξ, η, ζ inside (57), we consider all those amongst the above spheres which pass through ξ, η, ζ , but do not cut the segment from ξ, η, ζ to x_0, y_0, z_0 . We compute the integrals of $u_i(x, y, z, 0)$ over these spheres with respect to the element of surface area, and then we average the results with respect to the unit normals of the spheres at the point ξ, η, ζ . This process must yield a triple integral of $u_i(x, y, z, 0)$ multiplied by a certain non-negative kernel

$$(58) \quad K = K(x, y, z; \xi, \eta, \zeta).$$

The kernel K has an infinity at ξ, η, ζ whose principal part is identical with that which would be obtained if we were to integrate over planes through ξ, η, ζ rather than over spheres. This new process is immediately seen to lead to the kernel

$$(59) \quad K_1 = \frac{\pi}{[(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2]^{\frac{3}{2}}} = \frac{\pi}{r},$$

and thus K has a representation

$$(60) \quad K = K_1 + K_2,$$

where K_2 remains finite at ξ, η, ζ .

It is a familiar fact that

$$(61) \quad \Delta \iiint u_i(x, y, z, 0) K_1(x, y, z; \xi, \eta, \zeta) dx dy dz = -4\pi^2 u_i(\xi, \eta, \zeta, 0),$$

where

$$(62) \quad \Delta = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2}.$$

Therefore we can derive from the above considerations the Fredholm integral equation

$$(63) \quad 4\pi^2 u_i(\xi, \eta, \zeta, 0) - \iiint u_i(x, y, z, 0) \Delta K_2(x, y, z; \xi, \eta, \zeta) dx dy dz = g(\xi, \eta, \zeta)$$

for the initial derivative u_i , where g is a known function obtained from the boundary data. The kernel ΔK_2 of this integral equation grows at most like r^{-2} in the neighborhood of the point ξ, η, ζ , and hence the Fredholm alternative applies. Thus we have only to show that the corresponding homogeneous equation has no non-trivial solutions in order to prove that our mixed boundary value problem can always be solved and is properly set, since after $u_i(x, y, z, 0)$ is calculated $u(x, y, z, t)$ can be found from a Cauchy problem.

Actually, we can do somewhat better than to resort to the Fredholm alternative in order to solve (63). We have already established that for the case of data given on the conical mantle (51) our mixed problem has a solution that can be evaluated in closed form. Hence in that case the resolvent kernel for the integral equation (63) exists. We can now apply this resolvent kernel to both sides of (63) in the general case to obtain a new Fredholm equation whose Neumann series converges when the characteristic (56) is close enough to the cone (51). Operating thus with the known resolvent kernel even suffices to establish that the mixed problem is correctly posed for nearly flat initial surfaces and for arbitrary linear partial differential equations whose coefficients are near those of the wave equation. This result, which will be useful to us later on, follows because the general problem can always be reformulated as a Fredholm integral equation like (63). The reader can verify that neither the occurrence of a logarithmic term in the representation (31) of the fundamental solution nor a change in the number of independent variables allowed alters the validity of this statement. However, in spaces of higher dimension the Laplace operator must be applied more often to arrive at an appropriate analogue of (63), and if the number of independent variables is odd, the resolvent kernel is only known explicitly for axially symmetric geometries.

These few remarks about mixed boundary value problems will be adequate for our needs in the sections to follow.

The Reflected Fundamental Solution. For a partial differential equation (24) of the elliptic type with analytic coefficients $a_{i,k}$, b_k and c , consider a fundamental solution $G(P, Q)$ which vanishes when P lies on an analytic section R of boundary surface. For instance, G might be the Green's function of some domain. We are interested in studying the behavior of G on the opposite side of R from the parameter point Q . In particular, we ask for the locus there along which G might become infinite. In the two-dimensional case this is only a point, but for larger numbers of independent variables it will turn out in general to be a manifold of dimension two less than that of the space.

The detailed part of our discussion of this topic will be carried out only for Laplace's equation in four variables, although the results are easily extended to an arbitrary elliptic equation in any number of independent variables, with analytic coefficients. Since this generalization involves nothing essentially new, it seems preferable to explain the theory on the basis of the simplest and most easily understood special case. Our method of attack is through analytic continuation into the complex domain and analysis there of the singularities of G .

Since we confine our attention to Laplace's equation in four-dimensional space, G has the representation

$$(64) \quad G(P, Q) = \frac{1}{r^2} + H(P, Q),$$

where

$$(65) \quad r^2 = \sum_{k=1}^4 (x_k - \xi_k)^2$$

is the square of the distance from P to Q and where H is a regular function of P near Q . In the domain of complex values of $x_1, \dots, x_4$, the distance r vanishes on a cone along which G must be singular. Let C denote the intersection of this cone with the analytic extension into the complex domain of the surface R on which G vanishes. Through C we can pass two opposite complex characteristics of Laplace's equation. They can be defined as the loci

$$(66) \quad \varphi(x_1, \dots, x_4) = 0,$$

where φ is either one of the two solutions of the first order, second degree partial differential equation

$$(67) \quad \varphi_{x_1}^2 + \varphi_{x_2}^2 + \varphi_{x_3}^2 + \varphi_{x_4}^2 = 4\varphi$$

which reduce to r^2 on the surface R :

$$(68) \quad \varphi = r^2 \quad \text{on } R.$$

Clearly, $\varphi \equiv r^2$ is one of these two solutions and the original cone $r = 0$ is the corresponding characteristic of Laplace's equation. The other solution φ of (67) and (68) is in general non-trivial, and we shall call the corresponding locus (66) the reflected characteristic for Q with respect to R .

Since G is singular along the cone $r = 0$, but vanishes on R , it must have a singularity along the reflected characteristic (66) which just cancels on R the one along the cone. This follows because all singularities of any solution of a partial differential equation must occur along characteristics. We conclude that G can become infinite on the opposite side of R from Q at points satisfying the equation (66). Since φ is in general complex-valued even for real choices of $x_1, \dots, x_4$, this actually imposes two real conditions on the locus of reflected singularities of G , whence this manifold is two-dimensional in the real four-dimensional space, as we indicated earlier it would be. Additional singularities

of G can appear along the intersections with the real domain of characteristics emanating from further places where the locus (66) cuts R , and so on.

Examples are easy to give. If R is the parabolic cylinder

$$(69) \quad 4x_4 = x_1^2 + x_2^2$$

and Q is at the focus

$$(70) \quad \xi_1 = \xi_2 = \xi_3 = 0, \quad \xi_4 = 1,$$

then

$$(71) \quad r^2 = x_1^2 + x_2^2 + x_3^2 + (x_4 - 1)^2 = x_3^2 + (x_4 + 1)^2$$

on R , so that

$$(72) \quad \varphi = x_3^2 + (x_4 + 1)^2$$

is the desired solution of (67). Hence G becomes infinite on the directrix plane

$$(73) \quad x_3 = 0, \quad x_4 = -1$$

where φ vanishes in the real domain.

The next question which comes up is to find the precise nature of the singularity of G along the reflected characteristic (66). The answer is suggested by HADAMARD's construction [7] of the fundamental solution (31) of a linear partial differential equation of the second order. We seek a solution of Laplace's equation of the form

$$(74) \quad F = \frac{E}{\varphi} + W \log \varphi$$

which should have precisely the boundary values r^{-2} on R , where E and W are supposed to be regular functions in a neighborhood of the intersection C of the reflecting surface R with the characteristic cone $r = 0$ whose vertex is at Q . We shall call this type of solution a reflected fundamental solution with respect to R . The difference $r^{-2} - F$ need not be the same as G itself, since singularities of G not accounted for by F can appear along characteristics coming from the further intersections of (66) with R . However, for Q sufficiently close to R the term F serves to describe the most significant singularities of G on the opposite side of R . One can imagine here that we are trying to locate charges on this far side of R so that they compensate for a unit charge at Q in such a way as to establish an equilibrium potential along R , and F will determine the portion of these charges nearest to R .

In our proof of the existence of the reflected fundamental solution F , we take up at first a symmetric case where the geometry of the problem will be easy to visualize. We suppose that R has an equation

$$(75) \quad x_4 = p(x_1^2, x_2^2, x_3^2)$$

with an even right-hand side, and we suppose that Q lies on the x_4 -axis,

$$(76) \quad \xi_1 = \xi_2 = \xi_3 = 0.$$

In this situation we choose x_1 , x_2 and x_3 to be pure imaginary, but x_4 to be real, say

$$(77) \quad x_1 = is_1, \quad x_2 = is_2, \quad x_3 = is_3, \quad x_4 = t_4.$$

After this substitution, F becomes a solution of the wave equation

$$(78) \quad F_{s_1 s_1} + F_{s_2 s_2} + F_{s_3 s_3} - F_{t_4 t_4} = 0$$

with a prescribed singularity on the characteristic (66). The advantage of our simplifying assumptions (75) and (76) of symmetry is that both R and (66) acquire realizations as genuine three-dimensional surfaces in the four-dimensional space with coordinates s_1 , s_2 , s_3 and t_4 .

When we put (74) into (78), it becomes clear that the coefficient W of the logarithmic term must itself be a solution of (78), that is, W must be a harmonic function of $x_1, \dots, x_4$. Because of (67), the remaining contributions yield

$$(79) \quad \frac{2 \partial E / \partial \nu + E \square \varphi + 8E}{\varphi^2} = \frac{\square E + 2 \partial W / \partial \nu + W \square \varphi + 4W}{\varphi},$$

where

$$(80) \quad \frac{\partial}{\partial \nu} = \varphi_{s_1} \frac{\partial}{\partial s_1} + \varphi_{s_2} \frac{\partial}{\partial s_2} + \varphi_{s_3} \frac{\partial}{\partial s_3} - \varphi_{t_4} \frac{\partial}{\partial t_4}$$

and

$$(81) \quad \square = \frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2} + \frac{\partial^2}{\partial s_3^2} - \frac{\partial^2}{\partial t_4^2}.$$

The numerator on the left in (79) must vanish along the characteristic $\varphi = 0$, and this provides an ordinary differential equation there for E , since the differential operator $\partial/\partial \nu$ is in the tangential direction of the bicharacteristics. A similar ordinary differential equation for W is derived along the characteristic $\varphi = 0$ from the fact that not only the coefficient of φ^{-2} in (79) must vanish there, but also the coefficient of φ^{-1} . Therefore the values of both E and W on the surface $\varphi = 0$ are determined by their initial values at the intersection C with R . On R itself we must, of course, prescribe

$$(82) \quad E = 1, \quad W = 0.$$

It is easy to see that these requirements allow us a wide latitude in the choice of E so that (79) will be fulfilled. However, we find that W must satisfy, as a solution of (78), a mixed boundary value problem of the type studied in the previous section.

From our discussion of mixed boundary value problems, it is apparent that we cannot prescribe the values of W on R and also on the entire characteristic $\varphi = 0$. Indeed, the mantle of this characteristic where we can elect to give W

FIGURE 4. Schematic diagram of the construction of F .

lies precisely between the reflecting surface R and a central characteristic cone

$$(83) \quad s_1^2 + s_2^2 + s_3^2 - [t_4 - p(0, 0, 0)]^2 = 0$$

with its vertex located at the point where the t_4 -axis cuts R (see Fig. 4). The intersection of the pair of characteristics (66) and (83) is a branch line around which W may be multiple-valued, since beyond this line along (66) it has two determinations, one from the ordinary differential equation mentioned above and the other from the mixed boundary value problem with data prescribed along R and (66). It now follows from our treatment of mixed boundary value problems that W exists when Q is close enough to R so that the section of R where (82) must be imposed is sufficiently flat, but the single-valuedness of the result is only assured as far as the intersection of (66) with (83). It is important to notice that φ , as defined by the first order partial differential equation (67) and the initial condition (68), will be single-valued in a similar region, but that further on the characteristic $\varphi = 0$ can degenerate to a focal curve.

In this manner we are able to establish the existence of the reflected fundamental solution (74) when Q is sufficiently close to the analytic reflecting surface R . In the proof, our symmetry assumption (75) has helped us to understand the geometry of the problem, but it is actually not required for the construction, which is valid more generally in the complex domain, subject only to the restriction that Q be near to R . The generalization here is like our extension of the

representation (39) of the solution of Cauchy's problem from the hyperbolic to the elliptic case, where genuinely complex data are needed. Also, it is not too difficult to see how the procedure extends to arbitrary linear elliptic equations with analytic coefficients in any even number of independent variables. However, a more significant modification is required for an odd number of independent variables, since W then drops out of the picture and it is E that must satisfy a complicated mixed boundary value problem. Suffice it to mention that for $m + 1$ independent variables the reflected fundamental solution should have a representation

$$(84) \quad F = \frac{E}{\varphi^{\frac{1}{2}(m-1)}} + W \log \varphi$$

analogous to (31), with $W = 0$ for even choices of m .

The Reflection Principle in Space. Our purpose in this section is to develop a rule for extending, or reflecting, a solution u of an analytic elliptic partial differential equation (24) across an analytic surface R on which it vanishes. We are able to obtain an explicit rule of this kind in terms of integrals of u and its derivatives, of the fundamental solution G , and of the reflected fundamental solution F . The results show that for a fixed linear equation the analytic extension exists in a region depending only on the analytic reflecting surface R and not on the particular solution involved. Although the theorem holds in general, we simplify the presentation by confining our attention to Laplace's equation in four-dimensional space, just as we did in the previous section.

We begin the discussion by recalling an elegant remark due to LEWY about reflection of harmonic functions in the plane. Let $u(x, y)$ be a solution of Laplace's equation

$$(85) \quad u_{xx} + u_{yy} = 0$$

in the (x, y) -plane which vanishes for $x = 0$. If we consider u as a function of the complex variable $y = y_1 + iy_2$, it turns out to satisfy the wave equation

$$(86) \quad u_{xx} - u_{y_2 y_2} = 0$$

in any (x, y_2) -plane. Let u_1, u_2, u_3 and u_4 be the values of u at the consecutive corners of any square in such a plane whose edges are characteristics of (86). Then it is well known that the difference equation

$$(87) \quad u_1 - u_2 + u_3 - u_4 = 0$$

is satisfied. Now if the corners corresponding to the indices 2 and 4 lie in the plane $x = 0$, we conclude from the elementary rules of analytic continuation that $u_2 = u_4 = 0$. It follows that

$$(88) \quad u_1 = -u_3,$$

FIGURE 5. The Schwarz reflection principle.

but this is precisely a restatement of the Schwarz reflection principle

$$(89) \quad u(x, y) = -u(-x, y)$$

to the effect that u is an odd function of x (see Fig. 5).

Our reflection problem in four-dimensional space is to find for the value at a point Q of a harmonic function u vanishing on a nearby analytic surface R an expression which involves u and its derivatives exclusively on the far side of R from Q . Suppose, once more, that R has the symmetric equation (75) and that Q has coordinates of the form (76). As a first step in the direction of deriving a reflection rule for $u(Q)$, we can represent this value by the formula

$$(90) \quad u(Q) = \lim_{s \rightarrow R} \frac{1}{\omega_3} \int_s \left[\frac{1}{r^2} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{1}{r^2} \right] d\sigma$$

as the solution of a Cauchy problem with data prescribed on the reflecting surface R . The relation (90) is merely a special case of our general result (39). Since u satisfies the wave equation (78) as a function of the variables (77), Huygens' principle actually applies here, and hence the final locus of integration in (90) reduces to the intersection C of R with the characteristic cone $r = 0$ in the domain of real s_1, s_2, s_3 and t_4 .

Since u vanishes on R , the representation (90) depends only on the values of $\partial u / \partial n$ there. This would appear at first sight to mean that the term involving

the normal derivative of the fundamental solution r^{-2} drops out, so that we could replace r^{-2} in the integral (90) by the reflected fundamental solution F . As we have seen before in our discussion of Cauchy's problem, however, the integral over the major portion of R in (90) cancels out because r^{-2} is single-valued in the complex domain, and actually the whole contribution comes from a residual integration over C . We shall establish that, surprisingly enough, r^2 should be replaced by $-F$ in (90) to leave this integral unchanged, and a closer examination of the details will thus yield the less obvious result

$$(91) \quad u(Q) = -\lim_{S \rightarrow R} \frac{1}{\omega_3} \int_S \left[F \frac{\partial u}{\partial n} - u \frac{\partial F}{\partial n} \right] d\sigma.$$

We recall that for the limit process in (90), S is chosen to consist of two coincident cells of R , topologically equivalent to solid three-dimensional spheres, whose boundaries are joined by a generalized torus about C . In both (90) and (91), the integrations over the coincident cells cancel out. For the terms involving r^{-2} and E/φ this is an evident consequence of single-valuedness, whereas for the contributions from the term $W \log \varphi$ it follows only because both u and W vanish on R . Here it is worth while to remark in passing that for a general elliptic partial differential equation (24) the increments in $\log \varphi$ and $\log \Gamma$ over the torus about C would have opposite signs, so that the choice of a minus sign in (91) would just serve in that case to render the contributions from V and W to (90) and (91) equal. Finally all terms composed of integrals of $W \log \varphi$ or its derivatives over the generalized torus about C tend to zero as the small radius of the torus diminishes.

The analysis of the integrals of r^{-2} and E/φ over the generalized torus is more difficult. In order to prove (91), it would now suffice to show that

$$(92) \quad \lim_{S \rightarrow C} \int_S \left[(r^{-2} + \varphi^{-1}) \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} (r^{-2} + \varphi^{-1}) \right] d\sigma = 0,$$

since $E = 1$ and $u = 0$ on C . Here the notation $S \rightarrow C$ is intended to indicate that we integrate over only the generalized torus about C and that we allow its radius to approach zero. We can establish (92) by conceiving of the integral involved as an iterated integral and by showing that each contribution evaluated over a small circle around the torus approaches zero separately as the radius diminishes. Because u vanishes on C , this is evident for those terms of the integrand which represent derivatives tangential to the torus, but perpendicular to such a circle. The remaining terms can best be examined in a new coordinate system x, y, α, β such that the circle in question lies in the (x, y) -plane with its center at the origin, such that the Laplace operator retains its canonical form at the new origin as far as the second order derivatives are concerned, and such that

$$(93) \quad r^2 \equiv x + iy, \quad \varphi \equiv x - iy.$$

It is an elementary matter to verify that such a transformation of variables

can be achieved locally. The significant terms in (92) which do not drop out in an obvious fashion in the limit are only those of the highest order involving differentiations and integrations with respect to x and y . Hence (92) reduces asymptotically to a statement of the form

$$(94) \quad \lim_{\epsilon \rightarrow 0} \int_0^{2\pi} \left[\frac{x}{x^2 + y^2} \frac{\partial u}{\partial \rho} - u \frac{\partial}{\partial \rho} \frac{x}{x^2 + y^2} \right] \rho d\theta = 0,$$

where ρ and θ are polar coordinates in the (x, y) -plane, where the integral is extended over the circle $\rho = \epsilon$ of radius ϵ , and where u vanishes for $y = 0$, that is, for $r^2 = \varphi$. The result now follows upon expansion of u in a Taylor's series.

A detailed calculation has thus led us to a complete proof of (91). We now point out that the integral on the right in (91) does not depend on the surface of integration S , so that the more general formula

$$(95) \quad u(Q) = -\frac{1}{\omega_3} \int_S \left[F \frac{\partial u}{\partial n} - u \frac{\partial F}{\partial n} \right] d\sigma$$

is valid whenever S can be deformed, without cutting the characteristic $\varphi = 0$, into the original pair of coincident cells, joined by a torus, with which we started. We can proceed to deform S back towards the domain of real values of the coordinates $x_1, \dots, x_4$ in a manner analogous to that which served to establish the representation (39) of the solution of Cauchy's problem. If, in particular, the surface R is a hyperplane, then

$$(96) \quad F = \frac{1}{r^{*2}},$$

where r^* is the distance to the reflected image Q^* of Q in R , and (95) reduces immediately to the classical Schwarz reflection principle

$$(97) \quad u(Q) = -u(Q^*).$$

When R is not flat, however, we are only assured of the existence of the reflected fundamental solution F in a region extending to the intersection of the reflected characteristic $\varphi = 0$ with the central characteristic cone (83). Therefore we draw S down upon the mantle N of this cone lying between its vertex and the characteristic $\varphi = 0$ to obtain the relation

$$(98) \quad u(Q) = -\lim_{S \rightarrow N} \frac{1}{\omega_3} \int_S \left[F \frac{\partial u}{\partial n} - u \frac{\partial F}{\partial n} \right] d\sigma,$$

where the limit sign is to indicate that, as in similar situations earlier, a residue calculation is needed at the boundary of N in the characteristic $\varphi = 0$. As usual, S is required to form a generalized torus around this boundary which collapses upon it in the limit.

The usefulness of formula (98) for the problem of reflection lies in the fact that it serves to represent $u(Q)$ exclusively in terms of Cauchy data along the

FIGURE 6. Geometry of the domain of dependence q .

mantle N of the central cone (83). This data can be expressed in turn as the result of Cauchy problems with data prescribed in the domain of real $x_1, \dots, x_4$ on the opposite side of R from Q . For this purpose the representation (22) can be utilized. To be specific, u and its derivatives at a point of N with the coordinates s_1, s_2, s_3 and t_4 depend on data over the sphere

$$(99) \quad x_1^2 + x_2^2 + x_3^2 + (x_4 - t_4)^2 = [p(0, 0, 0) - t_4]^2, \quad s_1 x_1 + s_2 x_2 + s_3 x_3 = 0$$

in the real domain. In order to insure that all such spheres lie in the region on the opposite side of R from Q , it is only necessary to take Q near enough to R in the first place. As the point s_1, s_2, s_3, t_4 sweeps over N , the spheres (99) generate a four-dimensional domain q on the far side of R (see Fig. 6) where u and its derivatives must be known in order for us to compute $u(Q)$ according to the reflection rule defined by (98). We can now write this reflection rule explicitly in terms of an integral over q of the form

$$(100) \quad u(Q) = -\lim \frac{1}{\omega_3^2} \int_{S_1} \int_{S_2} \left[F \frac{\partial r^{-2}}{\partial n_1} \frac{\partial u}{\partial n_2} - \frac{\partial F}{\partial n_1} r^{-2} \frac{\partial u}{\partial n_2} \right. \\ \left. - F \frac{\partial^2 r^{-2}}{\partial n_1 \partial n_2} u + \frac{\partial F}{\partial n_1} \frac{\partial r^{-2}}{\partial n_2} u \right] d\sigma_2 d\sigma_1,$$

where the limit sign is supposed to mean that we have collapsed the closed manifolds S_1 and S_2 in such a fashion that the final result is a singular integration of u and $\partial u / \partial n$ over q only. Just how singular this integral can actually become is shown by the outcome (97) in the special case of a flat reflecting surface R .

The reflection principle embodied in formula (100) is the main result of this paper. It is valid with almost the identical derivation for an arbitrary linear elliptic partial differential equation (24) with analytic coefficients and in any

number of independent variables. In the general case it takes the form

$$(101) \quad u(Q) = -\lim \int_{s_1} \int_{s_2} B\{B[u, G], F\} d\sigma_2 d\sigma_1 ,$$

where G is the fundamental solution and F is an appropriate reflected fundamental solution of (24). For more than four independent variables care must be taken to select the coefficient E in the representation (84) of F so that the residues on the intersection C of the reflecting surface R and the reflected characteristic $\varphi = 0$ come out properly. When the number of independent variables is odd instead of even, the contour integrations implicit in (101) are modified in a manner analogous to the difference between the Parseval and Poisson solutions (14) and (22) of the wave equation. The correctness of our proof of the theorem (101) depends on the hypothesis that Q lies sufficiently near to R . However, the values of u and its derivatives which are required for evaluation of the integral (101) occur in a small region q which depends only on the reflecting surface R and the coefficients $a_{i,k}$, b_k and c of the differential equation (24), and not on the particular solution u involved. Hence (101) shows that the analytic extension of u across the reflecting surface R exists in a neighborhood of R which does not depend on u itself, a result that we announced in our introduction.

It should be emphasized that the symmetry assumptions (75) and (76) are not at all essential to our derivation of (100), which is valid for any analytic reflecting surface R . These assumptions were only introduced in order to simplify what is really a quite complicated procedure in the complex domain, and to bring out in this way the main ideas of the argument while suppressing the less important details. We can also comment that the basic method of reflection that we have employed here should apply to more general boundary conditions than

$$(102) \quad u = 0 \quad \text{on} \quad R.$$

Finally, the standard process of successive approximations might allow one to extend some aspects of our theory to non-linear partial differential equations.

It should be pointed out that our construction of the reflected fundamental solution F and our explicit representation (101) of the reflection principle are new even for two independent variables and permit us to express in closed form LEWY's reflection laws for a linear elliptic equation in the plane. This is not a difficult matter, and we shall refrain from going into further details about it here. However, it is interesting that for two variables the data needed on the right in (101) occur on a one-dimensional curve q extending to a specific image point Q^* , whereas even for three variables we require q to be a solid. For example, for Laplace's equation

$$(103) \quad u_{xx} + u_{yy} + u_{zz} = 0$$

this solid is obtained by rotating a circular disc of variable diameter. This phenomenon can be simplified in certain special cases, and such reductions will be the topic of our next section.

Reflection Rule for a Cylinder. Up to this point we have presented a theory which is valid for general elliptic partial differential equations, and we have only utilized special cases to facilitate the exposition. We have gone out of our way to avoid procedures and statements which cannot be generalized. However, from now on we shall take the opposite tack and consider particular examples for their own sake. Our purpose will be to simplify the reflection rule (100) for harmonic functions in four-dimensional space when the reflecting surface R is a cylinder. To be specific, we shall take R to be the spherical cylinder

$$(104) \quad x_1^2 + x_2^2 + x_3^2 = 1.$$

In this case we can give a reflection rule requiring data in the large which is based on a somewhat different geometrical configuration than (100) and is in a certain sense more elementary.

We suppose that Q has coordinates satisfying

$$(105) \quad 0 \leq \xi_1 < 1, \quad \xi_2 = \xi_3 = \xi_4 = 0,$$

so that

$$(106) \quad r^2 = (x_1 - \xi_1)^2 + x_2^2 + x_3^2 + x_4^2.$$

We can define C as the intersection of (104) with either the characteristic cone $r = 0$ or the parabolic cylinder

$$(107) \quad 1 + \xi_1^2 - 2\xi_1 x_1 + x_4^2 = 0.$$

We shall keep x_1 , x_2 and x_3 real, but let

$$(108) \quad x_4 = i\tau_4$$

be pure imaginary, whence u satisfies the wave equation

$$(109) \quad u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3} - u_{\tau_4 \tau_4} = 0.$$

It can be verified that the reflected characteristic $\varphi = 0$ has the parametric equations

$$(110) \quad x_1 = (1 + \nu) \cos \alpha - \xi_1 \nu \cos 2\alpha,$$

$$(111) \quad x_2 = (1 + \nu - 2\xi_1 \nu \cos \alpha) \sin \alpha \cos \beta,$$

$$(112) \quad x_3 = (1 + \nu - 2\xi_1 \nu \cos \alpha) \sin \alpha \sin \beta,$$

$$(113) \quad \tau_4 = (1 - \nu)(1 - 2\xi_1 \cos \alpha + \xi_1^2)^{\frac{1}{2}}.$$

In the present context we choose to replace (67) in our analysis by the first order partial differential equation

$$(114) \quad \varphi_{x_1}^2 + \varphi_{x_2}^2 + \varphi_{x_3}^2 = 1$$

of the characteristics of the wave equation (109) and to define φ as the solution of (114) found by eliminating α , β and ν from the four relations (110), (111), (112), and

$$(115) \quad \varphi = \tau_4 - (1 - \nu)(1 - 2\xi_1 \cos \alpha + \xi_1^2)^{\frac{1}{2}}.$$

The Green's function of the interior of the cylinder (104) with pole at Q clearly becomes infinite outside the cylinder along the closed algebraic surface $\varphi = 0$ in the hyperplane $x_4 = 0$. To find the reflected fundamental solution F here, we could solve (110), (111), (112) and (115) for φ , ν , α and β as functions of x_1 , x_2 , x_3 and τ_4 and introduce them as new independent variables. However, we shall be content to point out that F should have the form

$$(116) \quad F = \frac{E}{\varphi} + W \log \varphi,$$

with E and W regular, and should reduce to r^{-2} on the cylinder (104). The coefficient W satisfies a mixed boundary value problem for the wave equation (109), and in the present geometry it is single-valued in the portion of the half-space $\tau_4 \geq 0$ lying outside (104) and underneath the reflected characteristic $\varphi = 0$. This is a region which does not shrink to a point as Q approaches the reflecting surface (104). We shall make use of these remarks to derive a reflection rule according to which $u(Q)$ depends on data in the shell between the sphere (104) and the closed algebraic surface $\varphi = 0$ of the three-dimensional hyperplane $x_4 = 0$. The result will have the disadvantage that it is not of local character, but involves a construction in the large. However, it sheds some light on the interesting fact that in space we can have more than one reflection principle. According to Huygens' principle, we can express $u(Q)$ as a residue integral of the type (90) over the intersection C of (104) with (107), which can be found by putting $\nu = 0$ in (110), (111), (112) and (113). We want to replace r^{-2} by $-F$ in this integral to obtain an analogue of (91). However, because of the peculiarity of the region in which we have found F to be single-valued in the present geometry, we must choose S here to be an open manifold topologically equivalent to a three-dimensional shell, with its two boundaries coinciding in the sphere (104) of the hyperplane $\tau_4 = 0$. At first, S should climb up the cylinder (104), then surround C as a generalized torus in the complex domain, and finally return to this sphere. Afterwards we draw S down to the real domain without cutting the reflected characteristic $\varphi = 0$. Eventually it is required to collapse upon the shell q which lies between the sphere (104) and the closed algebraic surface $\varphi = 0$ in the hyperplane $x_4 = 0$ (see Fig. 7). Of course, it must approach the latter surface through a family of generalized tori, and hence we must perform the usual residue calculation there. At all events, we derive in the end the reflection rule

$$(117) \quad u(Q) = -\lim_{S \rightarrow q} \frac{1}{\omega_3} \int_S \left[F \frac{\partial u}{\partial n} - u \frac{\partial F}{\partial n} \right] d\sigma,$$

which is in some ways simpler than (100).

The result (117) depends strongly on the use we have made of Huygens' principle in order to avoid Cauchy data inside the cylinder (104). Also, once we have attained the real domain we can see that the integral on the right in (117)

FIGURE 7. Reflection in a cylinder.

is independent of the particular shell by which we join the closed surface $\varphi = 0$ to the cylinder (104), and this could therefore be taken to be a curved manifold. Furthermore, when Q lies on the central axis of the cylinder, the coefficient W vanishes and the integral over the shell drops out altogether, so that we are left with merely the residual integration over the edge of q . Indeed, in this latter case we have the explicit expression

$$(118) \quad F = \sum_{k=1}^{\infty} \left[\frac{1}{(\tau_4 - 2k)^2 - x_1^2 - x_2^2 - x_3^2} + \frac{1}{(\tau_4 + 2k)^2 - x_1^2 - x_2^2 - x_3^2} \right]$$

for the reflected fundamental solution F , and our special reflection principle (117) reduces to the elementary formula

$$(119) \quad u(0, 0, 0, 0) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \frac{\partial}{\partial \nu} \nu \cdot u(\nu \cos \alpha, \nu \sin \alpha \cos \beta, \nu \sin \alpha \sin \beta, 0) |_{\nu=2} \sin \alpha \, d\alpha \, d\beta.$$

This rule is easily understood when we notice that the significant term in the expansion (118) of F is $-1/r^{*2}$, where

$$(120) \quad r^{*2} = x_1^2 + x_2^2 + x_3^2 + (x_4 - 2i)^2,$$

so that (117) can be replaced by

$$(121) \quad u(0, 0, 0, 0) = \lim_{s \rightarrow c} \frac{1}{\omega_3} \int_s \left[\frac{1}{r^{*2}} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{1}{r^{*2}} \right] d\sigma.$$

Comparison of (121) with (90) yields

$$(122) \quad u(0, 0, 0, 0) = u(0, 0, 0, 2i),$$

and it is not hard to deduce from (22) that this is equivalent to (119).

REFERENCES

- [1] R. COURANT & D. HILBERT, *Methoden der mathematischen Physik*, vol. 2, Berlin, 1937.
- [2] P. FRANK & R. VON MISES, *Die Differential- und Integralgleichungen der Mechanik und Physik*, vol. 2, Braunschweig, 1935.
- [3] P. R. GARABEDIAN, An example of axially symmetric flow with a free surface, *Studies in mathematics and mechanics presented to Richard von Mises*, pp. 149–159, New York, 1954.
- [4] P. R. GARABEDIAN, Applications of analytic continuation to the solution of boundary value problems, *J. Ratt. Mech. Anal.*, **3** (1954), pp. 383–393.
- [5] P. R. GARABEDIAN, H. LEWY & M. SCHIFFER, Axially symmetric cavitation flow, *Ann. Math.*, **56** (1952), pp. 560–602.
- [6] P. R. GARABEDIAN & H. M. LIEBERSTEIN, On the numerical calculation of detached bow shock waves in hypersonic flow, *J. Aero. Sci.*, **25** (1958), pp. 109–118.
- [7] J. HADAMARD, *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques*, Paris, 1932.
- [8] J. LERAY, Le problème de Cauchy pour une équation linéaire à coefficients polynomiaux, *C.R. Acad. Sci. Paris*, **242** (1956), pp. 953–959.
- [9] H. LEWY, Neuer Beweis des analytischen Charakters der Lösungen elliptischer Differentialgleichungen, *Math. Ann.*, **101** (1929), pp. 609–619.
- [10] H. LEWY, *A theory of terminals and the reflection laws of partial differential equations*, Technical Report No. 4, Stanford University, 1952.
- [11] M. H. PROTTER, A boundary value problem for the wave equation and mean value theorems, *Contributions to the theory of partial differential equations*, pp. 249–257, Princeton, 1954.
- [12] M. H. PROTTER, New boundary value problems for the wave equation and equations of mixed type, *J. Ratt. Mech. Anal.*, **3** (1954), pp. 435–446.

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