Boxing Inequalities

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Introduction. In his measure theoretic investigations on surface area in n dimensional coordinate space W. H. FLEMING encountered, then posed, the following question: Is areal measure generated by polyhedral covers equivalent, in the sense of having the same null sets, to that generated by spherical covers? In order to attack this problem I shall first recast it in a combinatorially more amenable form.

Areal measure generated by spheres is clearly equivalent to that generated by coordinate cubes, that is, cubes all of whose faces lie in coordinate planes, and, indeed, to that by rational coordinate cubes, henceforth called merely cubes.

Moreover, it is not hard to see that areal measure generated by polyhedra is equivalent to that generated by coordinate polyhedra, that is, polyhedra all of whose faces lie in coordinate planes, and, indeed, to that by rational coordinate polyhedra, henceforth called polycubes.

Thus we are led to reformulate FLEMING's problem as follows: Are cubic and polycubic areal measures equivalent? I shall show that they are.

This result is proved by means of the following cubic combinatorial covering inequality—the boxing inequality: There is a finite constant b , depending only on the dimension n , such that any polycube P can be covered by finitely many nonoverlapping cubes Q whose total area $\sum \sigma Q$ is not too large compared with the area σP of P in the sense that

$$\sum \sigma Q \leq b \sigma P.$$

Cubes and polycubes. Let n dimensional real coordinate space R^n be endowed with the cubic metric whereby the distance between two points is their maximum absolute coordinate difference. Denote the diameter of a set X in this space by $|X|$ and the boundary of X by ∂X .

A (closed rational coordinate) cube Q is, in the cubic metric, a sphere with rational center and diameter. Its diameter $|Q|$ is its edge length, so its volume

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τQ is $|Q|^n$. Its boundary ∂Q consists of $2n$ square faces, that is, cubes of next lower dimension $n - 1$, each with edge length $|Q|$, so the (elementary) area σQ of Q is $2n |Q|^{n-1}$.

Since a polycube is a closed bounded polyhedron all of whose faces lie in rational coordinate planes, the union of finitely many polycubes is also a polycube. In particular, the union of finitely many cubes is a polycube. Conversely, as shown below, any polycube may be realized as a union of finitely many cubes.

To this end define for any positive rational number λ the λ grill $\mathfrak{G}_\lambda$ to be the class of all cubes of diameter λ , called λ blocks, cut out by the coordinate planes whose fixed coordinate is an integral multiple of λ . Any finite union of λ blocks we shall call a λ polycube, an integral polycube if $\lambda = 1$.

Consider now a polycube P . It has only finitely many faces, each on a rational coordinate plane, so all such rational coordinate values are commensurable, that is, are integral multiples of some positive rational number λ . Consequently every boundary point of P lies on some λ grill plane, so any λ block whose interior meets P lies wholly in P . Hence P , being, as polyhedron, the closure of its interior, is the union of all λ blocks it contains. Furthermore, being bounded, it contains only finitely many such blocks. Thus P is a λ polycube. Evidently its volume τP is λ^n times the number of its constituent λ blocks.

A face of a λ block we shall call a λ facet. A given λ facet is face of just two λ blocks; we call it an inside, bounding, or outside facet of a λ polycube P according as both, one, or none of these two blocks lie in P . The (elementary) area σP of P is thus λ^{n-1} times the number of λ facets bounding P .

We define, for two λ polycubes P and Q , the partial area $\sigma(P, Q)$ of P in Q as that amount of area of P inside Q ; it is thus λ^{n-1} times the number of bounding λ facets of P inside Q . Evidently for any finite set of *nonoverlapping* polycubes Q we have

$$\sum \sigma(P, Q) \leq \sigma P,$$

an inequality expressing the superadditivity of partial area with regard to nonoverlapping containers.

Areal measures. Let $\mathfrak{Q}$ be a countable class of cubes Q . We define the mesh of $\mathfrak{Q}$ as $\sup |Q|$ and the total area $\sigma \mathfrak{Q}$ of $\mathfrak{Q}$ as $\sum \sigma Q$; these values may be infinite.

For any positive δ ($0 < \delta \leq \infty$) we denote by $\mathfrak{Q}_\delta$ the (countable) class of all cubes with diameter $\leq \delta$. This class $\mathfrak{Q}_\delta$ generates an outer measure, δ mesh cubic areal measure q_δ , as follows. The measure value $q_\delta X$ on an arbitrary set X in the space is declared to be $\inf \sigma \mathfrak{Q}$, taken over all cubic covers of X by subclasses $\mathfrak{Q}$ of $\mathfrak{Q}_\delta$, by classes $\mathfrak{Q}$, that is, of mesh $\leq \delta$.

As δ decreases, $\mathfrak{Q}_\delta$ also decreases, so, for any given set X , the measure value $q_\delta X$ increases, though maybe not strictly. We define $q_0 X$ to be the limit of $q_\delta X$ as $\delta \rightarrow 0$; q_0 is an outer measure.

Thus for all mesh or limit mesh values $0 \leq \delta \leq \infty$ we obtain reversely ordered

areal measures $q_0 \geq q_\delta \geq q_\infty$: called fine if $\delta = 0$, mediate if $0 < \delta < \infty$, coarse if $\delta = \infty$.

Just as cubes generate cubic areal measures q_δ so also do polycubes generate polycubic areal measures p_δ .

Comparison of measures. Two outer measures over the same space are called equivalent if they yield the same null sets (sets of measure 0), biequivalent if they yield the same null sets and also the same huge sets (sets of measure ∞), commensurate if each is bounded above by a finite multiple of the other. Clearly these comparison relations become stronger in the order named.

It is not difficult to prove that cubic areal measures of different mesh compare as follows: All, whether fine, mediate, or coarse, are equivalent; mediate and coarse are biequivalent; any two mediate measures are commensurate.

Consider now comparison of cubic and polycubic areal measures of the same mesh. Since every cube is a polycube, it is clear that $p_\delta \leq q_\delta$ for all $\delta > 0$, hence also for the limit value $\delta = 0$. I shall show that there is a finite constant b , depending only on the dimension n , such that $q_\delta \leq b p_\delta$ for all $\delta > 0$, hence also for $\delta = 0$. Therefore cubic and polycubic areal measures of the same mesh, whether it be fine, mediate, or coarse, are commensurate. Consequently polycubic measures of different mesh compare the same as do the corresponding cubic measures; whereupon all measures, cubic and polycubic, are equivalent, regardless of mesh.

Combinatorial boxing inequalities. The desired measure inequality $q_\delta \leq b p_\delta$ will surely hold if the following proposition does: Every polycube P of diameter $\leq \delta$ admits a cubic cover $\mathfrak{Q}$ with mesh $\leq \delta$ and total area $\sigma \mathfrak{Q} \leq b \sigma P$.

Let us simplify this proposition without logical loss. First, we may disregard the mesh requirement, hence omit all reference to δ . For if a polycube P of diameter $\leq \delta$ admits an unsatisfactory cubic cover $\mathfrak{Q}'$ of mesh $> \delta$ and total area $\sigma \mathfrak{Q}' \leq b \sigma P$, it also admits a satisfactory cubic cover, namely a single covering cube Q with the same diameter as P , so Q has diameter $\leq \delta$ and, since $\mathfrak{Q}'$ contains some cube of diameter $> \delta$, has area $\sigma Q < \sigma \mathfrak{Q}' \leq b \sigma P$. Second, by making a suitable similarity transformation $\lambda \rightarrow 1$, we may evidently suppose P an integral polycube composed of unit grill blocks.

The above covering proposition, thus simplified, we now strengthen by imposing the following additional requirements on the sought for cubic cover: There shall be only finitely many cubes in the cover; all cubes shall be integral; no two shall overlap. Such a cover I call an integral boxing.

We are thus led to formulate the following cubic combinatorial covering inequality.

Integral boxing inequality. *There is a finite constant b , depending only on the dimension n , such that any integral polycube P admits a finite nonoverlapping integral cubic cover $\mathfrak{Q}$ with total area*

$$\sigma\Omega \leq b \sigma P.$$

Of all such n dimensional boxing constants b there is a least, say b_n . It is not hard to see that $b_n \geq n$. For let P be an integral polycubic plate one unit high erected on an $n - 1$ dimensional square base P' of diameter δ . Thus P is composed of δ^{n-1} blocks and has area $\sigma P = 2 \delta^{n-1} \theta$ where $\theta \rightarrow 1$ as $\delta \rightarrow \infty$. The cubes Q of any cubic cover Ω of P meet the plane of the base P' in squares covering P' , so $\sum |Q|^{n-1} \geq \delta^{n-1}$, whence $\sigma\Omega \geq 2n\delta^{n-1}$. Therefore $\sigma\Omega/\sigma P \geq n/\theta \rightarrow n$ as $\delta \rightarrow \infty$; whereupon $b_n \geq n$.

It is obvious that $b_1 = 1$; that $b_2 = 2$ I shall show in the appendix; I conjecture generally that $b_n = n$ but am unable to prove it for $n > 2$.

Indeed, every general boxing constant which I have been able to determine greatly exceeds this conjectured minimum because my method of proof establishes the following stronger inequality, uniformly valid for each covering cube, wherefrom the boxing inequality results by summation.

Uniform boxing inequality. *There is a finite constant B , for example $B = 2n \cdot 2^n$, depending only on the dimension n , such that any integral polycube P admits a finite nonoverlapping integral cubic cover Ω , each cube Q of which has area*

$$\sigma Q \leq B \sigma(P, Q).$$

Summing this inequality over all covering cubes Q , we get for the total area of Ω

$$\sigma\Omega \leq B \sum \sigma(P, Q) \leq B \sigma P,$$

since partial area is superadditive with regard to nonoverlapping containers. Hence any uniform boxing constant B is also a boxing constant b .

Let B_n be the minimal uniform boxing constant for dimension n . Then $B_n \geq b_n \geq n$. To find a better lower bound for B_n consider boxing a single block P . Let Q be that cube containing P in a boxing of P ; we seek to minimize the areal ratio $\sigma Q/\sigma(P, Q)$. Now Q cannot coincide with P , else $\sigma(P, Q) = 0$; so there remain only two minimizing possibilities: that Q have diameter 2 and P be one of its corners, or that Q have diameter 3 and P be its center; in which cases the areal ratio has values 2^n or 3^{n-1} respectively. Therefore $B_n \geq \min(2^n, 3^{n-1})$. I conjecture this lower bound to be the actual value of B_n , so that $B_1 = 1$ as is obvious, $B_2 = 3$, and $B_n = 2^n$ for $n > 2$. At any rate we see that the uniform boxing constant here obtained, namely $B = 2n \cdot 2^n$, is not excessively inflated.

Method of proof. The uniform boxing inequality with given constant would fail for a cube Q partially covering a polycube P if the area of P inside Q were too small compared with the area of Q . This could happen if hardly any or almost all blocks of Q were in P , that is, if the partial density

$$\rho(P, Q) = \frac{\tau PQ}{\tau Q}$$

of that part* PQ of P in Q were too small (near 0) or too large (near 1). On the other hand, if P had moderate density in Q , it seems plausible that the area of P inside Q could not be too small compared with the area of Q . Thus we are led to seek a boxing of P by cubes Q each containing a moderate amount of P .

We now formulate these suggestions precisely.

Boxing lemma. *For any number r between 0 and 1 and any integral polycube P there exists a finite nonoverlapping integral cubic cover $\mathfrak{Q}$ of P by cubes Q in each of which P has moderate partial density in the sense that*

$$2^{-n}r \leq \rho(P, Q) < r.$$

Area lemma.** *If the partial density of an integral polycube P in an integral cube Q is moderate in the sense that*

$$\theta r \leq \rho(P, Q) \leq r,$$

where θ and r are numbers between 0 and 1, then

$$\sigma Q \leq B \sigma(P, Q),$$

where $B = n/2\theta r (1 - r)$.

From these lemmas the uniform boxing inequality easily follows. For P can be covered, according to the boxing lemma, by nonoverlapping integral cubes Q in each of which P has moderate density in the sense that

$$2^{-n}r \leq \rho(P, Q) < r,$$

where r is left to be determined. Thus, taking $\theta = 2^{-n}$ in the area lemma, we see that

$$\sigma Q \leq B \sigma(P, Q)$$

for each cube Q , where B , as function of r , assumes its minimal value $2n \cdot 2^n$ when $r = \frac{1}{2}$.

Proof of boxing lemma. Let $\mathfrak{G}^k$ denote the dyadic grill $\mathfrak{G}_{2^k}$ ($k = 0, 1, 2, \dots$)

*The polycube PQ is the combinatorial, not the set, intersection of P and Q ; it is the union of all blocks contained both in P and in Q , hence is also the closure of the set intersection of the interiors of P and of Q .

**For fixed θ the value of B as given by the area lemma is minimized when $r = \frac{1}{2}$ with minimum value $B = 2n/\theta$. A slight improvement in the proof of this lemma affords the following slightly better value B' as a function of θ and r :

$$B' = n/2 \min [\theta r(1 - \theta r), r(1 - r)],$$

which, for fixed θ , is minimized when $r = (1 + \theta)^{-1}$ with minimum value $B' = n(1 + \theta)^2/2\theta$. The ratio of these two minimal values $B/B' = 4/(1 + \theta)^2$ is nearly 4 for small θ , hence, when $\theta = 2^{-n}$, for large n .

whose blocks are nonoverlapping integral cubes of diameter 2^k . Each block of $\mathfrak{G}^{k+1}$ is composed of 2^n blocks of $\mathfrak{G}^k$. Moreover, since each dyadic block is contained in just one block of any coarser dyadic grill, we see that any two dyadic blocks, whether in the same grill or not, are either comparable, that is, one contains the other, or else nonoverlapping.

Let there now be given a number r between 0 and 1 and an integral polycube P . We shall construct a boxing $\mathfrak{Q}$ of P all of whose cubes Q are dyadic grill blocks. To this end call a block Q of $\mathfrak{G}^{k+1}$ a *critical cube* if $\rho(P, Q) < r$ and if, furthermore, $\rho(P, Q') \geq r$ for at least one of the 2^n blocks Q' of $\mathfrak{G}^k$ which compose Q . Thus the partial density of P in a critical cube satisfies the inequality

$$\rho(P, Q) = \frac{\tau PQ}{\tau Q} \geq \frac{\tau PQ'}{2^n \tau Q'} = 2^{-n} \rho(P, Q') \geq 2^{-n} r,$$

so is moderate in the sense that

$$2^{-n} r \leq \rho(P, Q) < r.$$

Since $r > 0$ and P is bounded, it is clear that only finitely many integral cubes can enclose a portion of P with moderate density. In particular, there are only finitely many critical cubes.

I claim that the class $\mathfrak{Q}$ of all *maximal* critical cubes is a moderate boxing of P . Since there are only finitely many such cubes, all dyadic blocks of moderate partial density, it remains to show that they are nonoverlapping and cover P .

Any two distinct maximal critical cubes, being dyadic blocks, are either nonoverlapping or comparable. They cannot be comparable, else not both would be maximal, so are nonoverlapping.

Finally, the maximal critical cubes cover P . It suffices to show that any unit block Q^0 composing P is contained in a critical cube and hence in a maximal critical cube. Let Q^k be that block of $\mathfrak{G}^k$ which contains Q^0 . Thus

$$Q^0 \subset \dots \subset Q^k \subset Q^{k+1} \subset \dots$$

Since τP is finite, $\rho(P, Q^k) \rightarrow 0$ as $k \rightarrow \infty$. Moreover, $\rho(P, Q^0) = 1 \geq r > 0$, so there is a last index k such that $\rho(P, Q^k) \geq r$; hence $\rho(P, Q^{k+1}) < r$. Therefore Q^{k+1} is a critical cube containing Q^0 , as was to be shown.

The maximal critical cubes thus constitute a moderate boxing of the given polycube P .

Proof of area lemma.* A (unit) block g may be identified with its center and so represented by coordinates $g = (g_1, \dots, g_n)$, each such coordinate value

*The proof here given is due to J. KEMPERMAN. It replaces my original proof whereby the blocks of P in Q are first favorably rearranged to form a polycube P_Q with diminished partial area by pushing them successively against n faces of Q meeting at a corner, then the area of P_Q inside Q is estimated by projecting toward the opposite corner upon the boundary of a small cube in that corner. Thus my proof is partly combinatorial and partly geometric, whereas KEMPERMAN'S is wholly combinatorial.

being a midinteger, that is, half an odd integer. Likewise a facet f may be identified with its center and so given coordinates $f = (f_1, \dots, f_n)$. If f lies on the plane with constant integral m^{th} coordinate, then f_m is integral but all other coordinate values f_ν are midintegral.

Given two blocks p and q we define $L(p, q)$ to be the path of blocks from p to q consisting of n straight segments $L^1, \dots, L^n$ respectively parallel to the $x_1, \dots, x_n$ coordinate axes and consecutively joining the $n + 1$ blocks

$$p = q^0, q^1, \dots, q^{n-1}, q^n = q;$$

where the first m coordinates of q^n are those of q and the last $n - m$ those of p . The segment L^m thus consists of those blocks g such that $g_\nu = q_\nu$ for $\nu < m$, $g_\nu = p_\nu$ for $\nu > m$, and g_m assumes all midintegral values between p_m and q_m inclusive. Thus L^m degenerates into a single block if $p_m = q_m$; otherwise we assign as direction of L^m the sign of $q_m - p_m$.

If $L(p, q)$ pierces a facet f , both blocks facing on f belong to a single segment L^m , whence $f_\nu = q_\nu$ for $\nu < m$, $f_\nu = p_\nu$ for $\nu > m$, and f_m is an integer between p_m and q_m . We define the direction of L through f as that of L^m through f .

Let Q be an integral cube with edge length or diameter δ whose $2n$ faces we may for convenience assume to lie on the coordinate planes $x_\nu = 0, \delta$ for $\nu = 1, \dots, n$. Clearly any path joining two blocks of Q lies wholly in Q , so any facet pierced by the path is inside Q .

Consider for a given facet f inside Q the number w_f of pairs (p, q) of blocks, both in Q , for which the path $L(p, q)$ pierces f in a given direction, say positive. Let m be the coordinate index of the plane of f , the index m for which f_m is integral. Then $p_\nu = f_\nu$ for $\nu > m$, $q_\nu = f_\nu$ for $\nu < m$, and $p_m < f_m < q_m$. Midintegral coordinates between 0 and δ can thus be chosen for p in $\delta^{m-1}f_m$ ways and independently for q in $(\delta - f_m)\delta^{n-m}$ ways; hence

$$w_f \leq \frac{1}{4} \delta^{n+1} = w.$$

The same estimate w holds for the negative direction.

Let now a polycube P be given. For any pair (p, q) of blocks both in Q , but with p in P and q not, the path $L(p, q)$ starts at p in P , stays in Q , and ends at q not in P , so first passes out of P through a bounding facet $f(p, q)$ of P inside Q . Putting $t = \tau PQ$, the number of blocks of P in Q , we see that p can be chosen in t ways and q independently in $\delta^n - t$ ways. By hypothesis the partial density $\bar{r} = \rho(P, Q)$ of P in Q is moderate in the sense that $\theta r \leq \bar{r} \leq r$ where θ and r are numbers between 0 and 1. Therefore

$$t = \bar{r} \delta^n \geq \theta r \delta^n,$$

$$\delta^n - t = (1 - \bar{r}) \delta^n \geq (1 - r) \delta^n,$$

so the pairs (p, q) can be chosen in at least

$$\theta r (1 - r) \delta^{2n} = W$$

ways, and to each such pair a definite bounding facet $f(p, q)$ of P inside Q is associated through which $L(p, q)$ passes out from P . However, as we have seen, at most w paths in Q can pass through any one of these facets $f(p, q)$ in the direction out from P . Hence there must be at least W/w bounding facets of P inside Q . Therefore, since $\sigma Q = 2n \delta^{n-1}$, we conclude that

$$\sigma(P, Q) \geq \frac{W}{w} = 4\theta r(1-r) \delta^{n-1} = B^{-1} \sigma Q,$$

where $B = n/2\theta r(1-r)$, as was to be shown.

Isoperimetric corollary. A slight but easy consequence of the boxing inequality is the following crude cubic combinatorial isoperimetric inequality. Note that the proof uses the strong form of the boxing inequality guaranteeing a *non-overlapping* cover.

Isoperimetric inequality. *There is a finite constant a , depending only on the dimension n , such that*

$$(\tau P)^{(n-1)/n} \leq a \sigma P$$

for any polycube P .

To prove this cover P with finitely many nonoverlapping cubes Q satisfying the boxing inequality with constant b . Since volume is additive on nonoverlapping polycubes,

$$\tau P = \sum \tau PQ \leq \sum \tau Q.$$

Hence, applying JENSEN's inequality with exponent $e = (n-1)/n < 1$, we get

$$(\tau P)^e \leq \sum (\tau Q)^e.$$

Since a cube Q has volume $\tau Q = |Q|^n$ and area $\sigma Q = 2n |Q|^{n-1}$,

$$(\tau Q)^e = \frac{1}{2n} \sigma Q.$$

Finally, according to the boxing inequality,

$$\sum \sigma Q \leq b \sigma P.$$

Thus, chaining these results together, we obtain the isoperimetric inequality

$$(\tau P)^e \leq a \sigma P$$

with constant $a = b/2n$.

The conjectured minimal boxing constant $b_n = n$ therefore yields as an isoperimetric constant $a = \frac{1}{2}$; whereas the minimal isoperimetric constant should be $a_n = 1/2n$ with isoperimetric equality holding if and only if P is a cube.

Boxing bounded sets. For convenience of notation I have called the (ele-

mentary) area σP of a polycube P what should more properly be termed the area of its boundary ∂P . The same argument used in proving that $b_n \geq n$ shows that fine cubic areal measure of a λ facet is $2n\lambda^{n-1}$, that is, $2n$ times larger than its elementary area. Hence

$$\sigma P = \frac{1}{2n} q_0 \partial P$$

for every polycube P . Thus the combinatorial boxing inequality, rephrased, asserts the existence of a finite constant c such that every polycube P admits a cubic cover $\mathfrak{Q}$ with total area

$$\sigma \mathfrak{Q} \leq c q_0 \partial P.$$

We now seek an analogue for an arbitrary bounded set X of this combinatorial result for a polycube P . We cannot just replace P by X , for ∂X may be a non-void null set, so the right side must be augmented at least by an arbitrarily small positive ϵ . Such an increment, however, suffices. Moreover, the right side can then be diminished by replacing fine cubic areal measure q_0 by coarse cubic areal measure q_∞ .

Coarse measure boxing inequality. *There is a finite constant c , depending only on the dimension n , such that, for any $\epsilon > 0$ and any bounded set X , the closure $\bar{X}$ of X admits a finite nonoverlapping cubic cover $\mathfrak{Q}$ with total elementary area*

$$\sigma \mathfrak{Q} < c q_\infty \partial X + \epsilon.$$

In proving this we first note that, since X is bounded, ∂X is bounded as well as closed, so is compact. Being bounded, ∂X can be covered by a single cube, so $q_\infty \partial X$ is finite. Hence, for given $\epsilon_1 > 0$, there exists a countable cubic cover $\mathfrak{Q}_1$ of ∂X such that

$$\sigma \mathfrak{Q}_1 < q_\infty \partial X + \epsilon_1.$$

By suitably slightly enlarging each cube of $\mathfrak{Q}_1$ we can get another countable cubic cover $\mathfrak{Q}_2$ with the same upper bound for $\sigma \mathfrak{Q}_2$. Since the interiors of the enlarged cubes cover the compact set ∂X , the cover $\mathfrak{Q}_2$ admits a finite subcover $\mathfrak{Q}_3$. Let P_3 be the union of the cubes of $\mathfrak{Q}_3$. Thus P_3 is a polycube containing ∂X in its interior. Moreover, $\sigma P_3 \leq \sigma \mathfrak{Q}_3$. To P_3 adjoin all its bounded residual components to form the polycube P . Evidently P contains P_3 and $\sigma P \leq \sigma P_3$. Furthermore, P has no bounded residual components and contains the boundary ∂X of the bounded set X in its interior, so must also contain the entire set $\bar{X}$. Now, according to the combinatorial boxing inequality, the polycube P admits a finite nonoverlapping cubic cover $\mathfrak{Q}$ with total area $\sigma \mathfrak{Q} \leq b \sigma P$. Thus, since

$$\sigma P \leq \sigma P_3 \leq \sigma \mathfrak{Q}_3 < q_\infty \partial X + \epsilon_1,$$

we see, by taking $\epsilon_1 = \epsilon/b$ and $c = b$, that

$$\sigma\Omega < c \, q_\infty \, \partial X + \epsilon,$$

as was to be shown.

Appendix: proof that $b_2 = 2$. Since we have already shown that $b_2 \geq 2$, it remains to prove that 2 is a 2 dimensional boxing constant.

For any integral polysquare P we define $R(P)$ to be the minimal rectangle containing P . If C is a connected polysquare, every midintegral coordinate line cutting $R(C)$ must meet a block of C , so must cut at least as many bounding facets of C as of $R(C)$; hence

$$\sigma R(C) \leq \sigma C.$$

Let P be resolved into its components—maximal connected subsets. These components C are disjoint, and, since P is a polysquare, they are also polysquares finite in number. Thus $P = \sum C$ and

$$\sigma P = \sum \sigma C,$$

since σ is additive on disjoint polysquares.

Although the components C of P are disjoint, the enveloping rectangles $R(C)$ need not be. Let $S(P)$ be the union of all such rectangles.

Each of the sets P , C , $R(C)$ is contained in the rectangle $R(P)$, so $S(P)$ is also; that is, $S(P) \subset R(P)$.

Since $C \subset R(C)$, it is clear that $P \subset S(P)$. If each component C of P is a rectangle, so that $R(C) = C$, then P is obviously fixed under the operation S , that is, $S(P) = P$. Conversely, if P is fixed under S , then every component of P is a rectangle. For, otherwise, if some component C of P were not a rectangle, there would be in $R(C)$ a block which though not in C would meet C . If, however, $S(P) = P$, such a block would have to be in some other component of P , so could not meet C .

Under operation S a polysquare increases in the weak sense that $P \subset S(P)$, and its perimeter decreases in the weak sense that $\sigma P \geq \sigma S(P)$, for

$$\sigma S(P) \leq \sum \sigma R(C) \leq \sum \sigma C = \sigma P.$$

Consider now the S iterates of a polysquare P :

$$P \subset S(P) \subset S^2(P) \subset \dots,$$

with respective perimeters

$$\sigma P \geq \sigma S(P) \geq \sigma S^2(P) \geq \dots.$$

Since the rectangle $R(P)$ contains $S(P)$, it contains all iterates $S^k(P)$ of P . Thus these iterates eventually become constant, say P_s , so P_s is fixed under S , $P_s \supset P$, and $\sigma P_s \leq \sigma P$. Let $\mathfrak{R}$ be the class of all components of P_s . Then $\mathfrak{R}$ consists of finitely many rectangles covering P , and

$$\sigma \mathfrak{R} = \sigma P_s \leq \sigma P.$$

We obtain a square cover $\mathfrak{Q}$ of P by dissecting each rectangle R of $\mathfrak{R}$ into a square cover $\mathfrak{Q}_R$ with total perimeter

$$\sigma\mathfrak{Q}_R < 2\sigma R;$$

whereupon, taking $\mathfrak{Q} = \sum \mathfrak{Q}_R$, we get

$$\sigma\mathfrak{Q} < 2\sigma\mathfrak{R} \leq 2\sigma P,$$

thereby proving $b_2 = 2$. That any integral rectangle $R = R(a, b)$ with sides a and b admits a dissection $\mathfrak{Q}_R$ such that

$$\sigma\mathfrak{Q}_R = 4(a + b - d) = 2\sigma R - 4dR < 2\sigma R,$$

where $d = dR = d(a, b)$ is the greatest common divisor of a and b , can easily be shown by the Euclidean algorithm or by induction on the perimeter σR by splitting off from R , unless R is already a square, a square whose diameter (side) is the width (small side) of R .

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