Trigonometric Series with Gaps

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1.1. A set $\{n_k\}$ of positive integers which for some λ satisfies the inequalities

$$(1.1.1) \quad \frac{n_{k+1}}{n_k} > \lambda > 1 \quad (k = 1, 2, 3, \dots)$$

will be called a Hadamard set, in view of HADAMARD's classical theorem concerning the natural boundary of power series of the form $\sum a_k z^{n_k}$. The present paper contains a study of various problems that are suggested by the following well known theorems [16; pp. 122, 216, 139].

(A) *If f is a Lebesgue-integrable function on the unit circle whose Fourier series has the form*

$$f(e^{i\theta}) \sim \sum_{k=1}^{\infty} a_k e^{in_k \theta},$$

where $\{n_k\}$ is a Hadamard set, then $f \in L^p$ for every $p < \infty$.

(B) *If, in addition, f is bounded, then $\sum |a_k| < \infty$.*

With regard to (A), we shall study sets $\{n_k\}$ such that the conclusion of (A) holds for some p , but not necessarily for all $p < \infty$; these are the sets of type $\Lambda(p)$, defined in 1.5. The sets for which the conclusion of (B) holds will be called Sidon sets.

We now state some definitions, and then give a brief description of the main results of the paper.

1.2. Definitions. (a) If E is a set of integers (not necessarily positive), a function f will be called an E -function if (i) f is a complex Lebesgue-integrable function on the unit circle and (ii) $\hat{f}(n) = 0$ for all n not in E , where $\hat{f}(n)$ is the n^{th} Fourier coefficient of f defined by

$$(1.2.1) \quad \hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta \quad (n = 0, \pm 1, \pm 2, \dots).$$

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For $0 < p < \infty$, we define

$$(1.2.2) \quad \|f\|_p = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta \right\}^{1/p};$$

L^p is the class of all measurable functions f on the circle such that $\|f\|_p < \infty$, and L_E^p will denote the space of all E -functions which belong to L^p .

A trigonometric polynomial which is an E -function will be called an E -polynomial. A measure (i.e., a complex bounded Borel measure) μ on the circle is an E -measure if $\hat{\mu}(n) = 0$ for all n not in E , where

$$(1.2.3) \quad \hat{\mu}(n) = \int_{-\pi}^{\pi} e^{-in\theta} d\mu(\theta) \quad (n = 0, \pm 1, \pm 2, \dots).$$

(b) A set E is said to be a *Sidon set* if there is a finite constant B such that

$$(1.2.4) \quad \sum_{-\infty}^{\infty} |\hat{f}(n)| \leq B \|f\|_{\infty}$$

for every E -polynomial f .

Here $\|f\|_{\infty}$, as usual, stands for $\sup |f(e^{i\theta})|$ ($-\pi \leq \theta \leq \pi$).

(c) Suppose $0 < r < s < \infty$. We say that E is of type (r, s) if there is a finite constant B (depending on E, r, s) such that

$$(1.2.5) \quad \|f\|_s \leq B \|f\|_r$$

for every E -polynomial f .

1.3. If $r \geq 1$ and if E is of type (r, s) , the above definition implies that the completion of the linear space of all E -polynomials in the norm of L^r is the same as its completion in the norm of L^s , and this leads immediately to the observation that E is of type (r, s) if and only if $L_E^r = L_E^s$.

Indeed, if E is of type (r, s) , then the Cesàro means of the Fourier series of any $f \in L_E^r$ are bounded in L^r [16; p. 84] and hence in L^s , by (1.2.5), so that $f \in L^s$, and $L_E^r = L_E^s$. To prove the converse, we note that L_E^r and L_E^s are closed subspaces of L^r and L^s , respectively. Hence if $L_E^r = L_E^s = X$, say, then $\|\cdot\|_r$ and $\|\cdot\|_s$ are two norms on the same Banach space X ; these norms must therefore be equivalent, so that (2.1.5) holds for some B .

Theorem (A) may accordingly be restated by saying that every Hadamard set is of type $(1, p)$, for every $p < \infty$.

We shall now show that the type (r, s) depends only on the larger exponent.

1.4. Theorem. Suppose $0 < r < s < t < \infty$. Then E is of type (r, t) if and only if E is of type (s, t) .

Proof. It is obvious that $(r, t) \subset (s, t)$.

Suppose E is of type (s, t) . The Hölder inequality yields

$$(1.4.1) \quad \left\{ \frac{1}{2\pi} \int |f|^s \right\}^{t-r} \leq \left\{ \frac{1}{2\pi} \int |f|^r \right\}^{t-s} \cdot \left\{ \frac{1}{2\pi} \int |f|^t \right\}^{s-r}.$$

If f is any E -polynomial, then $\|f\|_t \leq B_{s,t} \|f\|_s$, and (1.4.1) implies

$$\|f\|_s^{s(t-r)} \leq B_{s,t}^{t(s-r)} \|f\|_r^{r(t-s)} \|f\|_s^{t(s-r)},$$

so that

$$(1.4.2) \quad \|f\|_s \leq B_{r,s} \|f\|_r,$$

with $B_{r,s} = B_{s,t}^\alpha$, where $\alpha = t(s-r)/r(t-s)$.

It follows that

$$(1.4.3) \quad \|f\|_t \leq B_{s,t} \|f\|_s \leq B_{r,s} B_{s,t} \|f\|_r,$$

so that E is of type (r, t) , and the theorem is proved.

The following definition now suggests itself:

1.5. Definition. Suppose $0 < s < \infty$. We say that E is of type $\Lambda(s)$ (or $E \in \Lambda(s)$) if E is of type (r, s) for some $r < s$.

1.6. So far, we have only defined *analytic* properties of E , that is to say, properties which involve various norms of E -polynomials. The following two *structural* concepts will also be needed:

(a) For any positive integer N , we let $\alpha_E(N)$ be the largest integer α such that some arithmetic progression of N terms contains α terms of E .

In other words, if $\alpha_E(N, a, b)$ is the number of terms which E has in the set

$$(1.6.1) \quad a + b, \quad a + 2b, \quad \dots, \quad a + Nb,$$

then $\alpha_E(N) = \sup_{a,b} \alpha_E(N, a, b)$.

(b) If E is a set of non-negative integers and if $s = 1, 2, 3, \dots$, we let $r_s(E, n)$ be the number of representations of n in the form

$$(1.6.2) \quad n = n_{i_1} + n_{i_2} + \dots + n_{i_s} \quad (n_{i_k} \in E).$$

Different permutations of the terms on the right of (1.6.2) count as different representations. To make the definition unambiguous, we state the relation

$$(1.6.3) \quad \sum_{n=1}^{\infty} r_s(E, n) z^n = \left\{ \sum_{k=1}^{\infty} z^{n_k} \right\}^s \quad (|z| < 1),$$

where $n_1 < n_2 < \dots$ are the terms of E .

1.7. The principal results. Theorem 2.1 gives a list of analytic properties which are necessary and sufficient for E to be a Sidon set; these are not new (see, for instance, [5; p. 254], [6; pp. 310–311], [4]) but our proof is simple and is included for the sake of completeness. One of these properties concerns the following interpolation problem: *Given a bounded function b on E , find a measure μ on the circle so that $\hat{\mu}(n) = b(n)$ for all n in E .* This is possible, for every such b , if and only if E is a Sidon set.

Theorem 2.2 is new and implies the rather surprising result that the above problem can be solved if the following *approximate* interpolation problem can be solved for some fixed $\delta > 0$: Given a function b on E such that $|b(n)| = 1$, find a measure μ on the circle so that

$$(1.7.1) \quad |\hat{\mu}(n) - b(n)| < 1 - \delta \quad (n \in E).$$

Thus we have the following dichotomy: If E is a Sidon set, the first interpolation problem can be solved exactly; if E is not a Sidon set, then there is a function b on E , of absolute value 1, such that

$$(1.7.2) \quad \sup_{n \in E} |\hat{\mu}(n) - b(n)| = 1$$

for every measure μ on the circle.

The approximate interpolation theorem is then used to give a simple proof of a theorem of ŠTŘEČKIN (Theorem 2.4).

Part III begins with a proof that every Sidon set is of type $\Lambda(q)$, for every $q < \infty$; a quantitative result is obtained, to the effect that $\|f\|_q \leq B \sqrt{q} \|f\|_2$ for every E -polynomial f , where B depends only on E , and it is shown that the factor $\sqrt{q}$ cannot be replaced by any function of q which is $o(\sqrt{q})$, as long as E is infinite (see 3.3). Upper bounds on $\alpha_E(N)$ are obtained which must hold if $E \in \Lambda(q)$ or if E is a Sidon set (Theorems 3.5, 3.6); by 3.7 and 4.8, these estimates are best possible, but Theorem 3.8 shows, on the other hand, that no restriction on the rate of growth of $\alpha_E(N)$ can imply that E is even of type $\Lambda(1)$ (except if $\alpha_E(N)$ is bounded, in which case E is a finite set).

In Part IV we state a sufficient condition (Theorem 4.5 (b)) which assures that $E \in \Lambda(q)$ for $q = 4, 6, 8, \dots$, namely the boundedness of $\{r_s(E, n)\}$, where $q = 2s$. The role of the averages $N^{-1} \sum_1^N r_s^2(E, n)$ and $N^{-1} \sum_1^N r_s(E, n)$ is also discussed. It seems unlikely that Theorem 4.5(b) expresses a condition which is necessary as well as sufficient, but no counterexample is known.* In conjunction with the earlier result on $\alpha_E(N)$, 4.5(b) is used to construct sets which belong to $\Lambda(q)$ ($q = 4, 6, 8, \dots$) but not to $\Lambda(q + \epsilon)$ for any $\epsilon > 0$ (Theorem 4.8).

In this connection, we observe that $\Lambda(p_1) \supset \Lambda(p_2)$ if $p_1 < p_2$ (by Theorem 1.4). It seems likely that the inclusion is always proper, but the cases covered by Theorem 4.8 are the only ones for which the conjecture is proved so far; in particular, no set is known to be of type $\Lambda(2)$ which is not also of type $\Lambda(4)$; in view of 5.3, $\Lambda(2)$ certainly deserves further study.

Modifications of the construction on which the proof of 4.8 is based are used to settle two other problems (4.9, 4.11).

In Part V we find some analytic properties of sets of type $\Lambda(p)$; the case $p = 2$ is due to BANACH [5; p. 250] (for orthonormal systems in general). As in the case of Sidon sets, we find that sets of type $\Lambda(p)$ can be characterized by

*It seems that there is only one gap theorem for trigonometric series, the one due to PALEY, for which the necessary and sufficient structural condition is known at present [10].

the fact that certain interpolation problems always have solutions (Theorem 5.1(d), (e)). The paper concludes with a theorem concerning permutations of Fourier coefficients of functions in L^p (related to the inclusion problem between the various types $\Lambda(p)$), and with a slight extension of the theorem of F. and M. RIESZ.

II. SIDON SETS

2.1. Theorem. *Each of the following five statements about a set E of integers implies the others:*

- (a) E is a Sidon set.
- (b) Every bounded E -function has an absolutely convergent Fourier series.
- (c) Every continuous E -function has an absolutely convergent Fourier series.
- (d) To every bounded function b on E , there is a measure μ on the circle such that $\hat{\mu}(n) = b(n)$ for all $n \in E$.
- (e) If $b(n) \rightarrow 0$ as $|n| \rightarrow \infty$, there is an $f \in L^1$ such that $\hat{f}(n) = b(n)$ for all $n \in E$.

Proof. Suppose (a) holds and B is as in (1.2.4). If f is a bounded E -function, the Fejér means

$$(2.1.1) \quad \sigma_N(f; \theta) = \sum_{-N}^N \left(1 - \frac{|n|}{N}\right) \hat{f}(n) e^{in\theta}$$

are E -polynomials, and $\|\sigma_N(f)\|_\infty \leq \|f\|_\infty$. Hence

$$(2.1.2) \quad \sum_{-N}^N \left(1 - \frac{|n|}{N}\right) |\hat{f}(n)| \leq B \|f\|_\infty \quad (N = 1, 2, 3, \dots).$$

Letting $N \rightarrow \infty$, we see that (a) implies (b).

It is trivial that (b) implies (c). If (c) holds, then the Banach space C_E (the space of all continuous E -functions, with the uniform norm) is isomorphic to the Banach space of all functions ϕ on E such that

$$(2.1.3) \quad \|\phi\| = \sum_{n \in E} |\phi(n)| < \infty$$

by the correspondence $f \leftrightarrow \hat{f}$. It follows that $\|f\|_\infty$ and $\|\hat{f}\|$ are equivalent norms on C_E , which implies the existence of a constant B such that (1.2.4) holds for every $f \in C_E$. Thus (c) implies (a).

Having proved the equivalence of (a), (b), (c), we now show that these conditions imply (d): If E is a Sidon set and if $|b(n)| \leq 1$, the mapping

$$(2.1.4) \quad f \rightarrow \sum_{n \in E} \hat{f}(n) b(n)$$

is a bounded linear functional on C_E of norm at most B , where B satisfies (1.2.4). The Hahn-Banach theorem allows us to extend this functional to C ,

the space of all continuous functions on the circle, with preservation of norm. Hence there is a measure μ , whose total variation $\|\mu\|$ does not exceed B , such that

$$(2.1.5) \quad \sum_{n \in E} \hat{f}(n)b(n) = \int_{-\pi}^{\pi} f(e^{-i\theta}) d\mu(\theta) \quad (f \in C_E).$$

Taking $f(e^{i\theta}) = e^{in\theta}$ in (2.1.5), we obtain $\hat{\mu}(n) = b(n)$ for all $n \in E$, so that (d) is proved.

If (d) holds and if $b(n) \rightarrow 0$, we can find a positive convex sequence $\{c(n)\}$ ($n \geq 0$) such that $c(n) \rightarrow 0$, and such that $b(n)/c(n)$ is bounded, where $c(-n) = c(n)$. By (d) there is a μ with $\hat{\mu}(n) = b(n)/c(n)$ for $n \in E$. Since $c(n) = \hat{f}(n)$ for some $f \in L^1$ [16; p. 109], $b(n) = \hat{f}(n)\hat{\mu}(n)$ for $n \in E$. Since the convolution $f * \mu$, defined by

$$(2.1.6) \quad (f * \mu)(e^{i\theta}) = \int_{-\pi}^{\pi} f(e^{i(\theta-\phi)}) d\mu(\phi),$$

is in L^1 , (d) implies (e).

Finally, if (e) holds and if $f \in C_E$, then the series $\sum \hat{f}(n)g(n)$ is Cesàro-summable for every $g \in L^1$. Since we can choose g so that $\{\hat{g}(n)\}$, restricted to E , is an arbitrary sequence tending to 0, it follows that $\sum |\hat{f}(n)c_n|$ is summable (hence convergent) for every $\{c_n\}$ tending to 0, and it is well known that this implies the convergence of $\sum |\hat{f}(n)|$.

Thus (e) implies (c), and the theorem is proved.

2.2. Theorem. *The set E is a Sidon set if there exists a constant $\delta > 0$ with the following property: To every function b such that $|b(n)| = 1$ for all $n \in E$ there corresponds a measure μ on the circle such that*

$$(2.2.1) \quad |(\hat{\mu}(n) - b(n))| < 1 - \delta \quad (n \in E).$$

Proof. Let f be a continuous E -function, and define $b(n)$ so that $b(n)\hat{f}(n) = \hat{f}(n)$ for all $n \in E$; if $\hat{f}(n) = 0$, put $b(n) = 1$. By hypothesis, there is a measure μ which satisfies (2.2.1). It follows that the number $\hat{\mu}(n)\hat{f}(n)$ lies in the circle with center at $|\hat{f}(n)|$ and radius $(1 - \delta)|\hat{f}(n)|$ if $n \in E$ and $\hat{f}(n) \neq 0$. Hence

$$(2.2.2) \quad \operatorname{Re}[\hat{\mu}(n)\hat{f}(n)] \geq \delta |\hat{f}(n)| \quad (n \in E),$$

where $\operatorname{Re}[z]$ denotes the real part of z . But the series $\sum \hat{\mu}(n)\hat{f}(n)$ is Cesàro-summable [16; p. 88], and so is its real part, which, being a series of non-negative terms, must therefore be convergent. Hence (2.2.2) implies that $\sum |\hat{f}(n)| < \infty$ for every continuous E -function, and, by 2.1(c), E is a Sidon set.

2.3. It will be convenient to introduce the following notation: If $n_1, n_2, n_3, \dots$ is an enumeration of the elements of the set E (which may contain positive and negative integers), define $R_s(E, n)$ to be the number of representations of n in the form

$$(2.3.1) \quad n = \pm n_{k_1} \pm n_{k_2} \pm \dots \pm n_{k_s} \quad (k_1 < k_2 < \dots < k_s).$$

2.4. Theorem. Suppose E satisfies the following two conditions:

- (a) If n is in E then $-n$ is not in E .
- (b) There is a finite constant B and a decomposition of E into a union of disjoint sets $E_1, \dots, E_t$, such that

$$(2.4.1) \quad R_s(E_j, n) \leq B^s \quad (1 \leq j \leq t; s = 1, 2, 3, \dots)$$

for all $n \in E$ and for $n = 0$.

Then E is a Sidon set.

This is a slight generalization of STEČKIN's theorem [14; p. 394]; he requires that (2.4.1) holds for all n ; HEWITT & ZUCKERMAN [4] have proved this for $B = 1$.

Proof. Put $\beta = (3t B^2)^{-1}$, and let b be an arbitrary function on E , subject only to $|b(n)| = \beta$.

Fix j ($1 \leq j \leq t$), let $n_1, n_2, n_3, \dots$ be the terms of E_j , and form the Riesz products

$$(2.4.2) \quad P_N(e^{i\theta}) = \prod_{k=1}^N (1 + b(n_k)e^{in_k\theta} + \overline{b(n_k)}e^{-in_k\theta}).$$

Since $\beta < \frac{1}{2}$, $P_N(e^{i\theta}) \geq 0$. Multiplying out, we obtain

$$P_N(e^{i\theta}) = 1 + \sum_{k=1}^N b(n_k)e^{in_k\theta} + \sum_{k=1}^N \overline{b(n_k)}e^{-in_k\theta} + \sum_{-\infty}^{\infty} c_n^{(N)}e^{in\theta},$$

where

$$|c_n^{(N)}| \leq \sum_{s=2}^N \sum |b(n_{k_1}) \cdots b(n_{k_s})| \leq \sum_{s=2}^{\infty} (B\beta)^s = \frac{B^2\beta^2}{1 - B\beta} \leq \frac{1}{6t^2B^2}$$

if $n \in E$ or if $n = 0$; here the inner summation extends over all $n_{k_1}, \dots, n_{k_s}$ in E_j which satisfy (2.3.1); hence this sum has at most B^s terms.

Since $P_N \geq 0$,

$$\|P_N\|_1 = 1 + c_0^{(N)} \leq 1 + \frac{1}{6t^2B^2},$$

so that $\{\|P_N\|_1\}$ is bounded as $N \rightarrow \infty$, and a subsequence of $\{P_N\}$ converges weakly (as elements of the conjugate space of C) to a measure μ_j such that

$$(2.4.3) \quad \begin{cases} |\hat{\mu}_j(n_k) - b(n_k)| \leq \frac{1}{6t^2B^2} & (n_k \in E_j), \\ |\hat{\mu}_j(n)| \leq \frac{1}{6t^2B^2} & (n \in E, n \notin E_j). \end{cases}$$

We now put $\mu = \mu_1 + \dots + \mu_t$. Adding the inequalities (2.4.3), we see that

$$(2.4.4) \quad |\hat{\mu}(n) - b(n)| = \frac{t}{6t^2B^2} < \frac{\beta}{2} \quad (n \in E).$$

Since $|b(n)| = \beta$, E satisfies the hypothesis of Theorem 2.2 with $\delta = \frac{1}{2}$, and this completes the proof.

2.5. Remarks. (1) Since the measure μ constructed in the proof of Theorem 2.4 is positive, the proof of Theorem 2.2 shows that the conclusion of Theorem 2.4 can be strengthened (as is done in [14] for real f):

If E satisfies the hypotheses of Theorem 2.4 and if f is an E -function whose real part is bounded above, then $\sum |f(n)| < \infty$.

(2) Hypothesis (a) of Theorem 2.4 is needed if one wishes to solve the interpolation problem 2.1(d) with a real measure μ , for then $\hat{\mu}(n)$ is the complex conjugate of $\hat{\mu}(-n)$. It is not clear whether (a) can be dropped from Theorem 2.4.

In this connection, it seems to be an open problem whether the union of any two Sidon sets is a Sidon set.

(3) If E is a Hadamard set with $n_{k+1}/n_k > 3$, then $R_i(E, n) \leq 1$ for every n . Since every Hadamard set is a finite union of sets with $n_{k+1}/n_k > 3$, Theorem 2.4 implies that every finite union of Hadamard sets is a Sidon set [13; p. 207].

On the other hand, there exist Sidon sets (consisting of positive integers) which are not finite unions of Hadamard sets. STEČKIN [14; p. 391] has given such an example, and so have HEWITT & ZUCKERMAN [4]; their proof can be made very simple:

For $m = 0, 1, 2, \dots$ set $M = 2^m$, and let E be the set of all numbers

$$(2.5.1) \quad 3^{4M} + 3^{M+i} \quad (j = 0, \dots, M-1; m = 0, 1, 2, \dots).$$

Since E contains M terms between $a_m = 3^{4M} + 3^M$ and $b_m = 3^{4M} + 3^{2M-1}$ and since $b_m < 2a_m$, E is not a finite union of Hadamard sets. (Any such union has a bounded number of elements between x and $2x$ as $x \rightarrow \infty$.)

If some n had more than one representation of the form (2.3.1) with $n_{k_i} \in E$, there would be a relation

$$(2.5.2) \quad 0 = \epsilon_1 n_{k_1} + \epsilon_2 n_{k_2} + \dots + \epsilon_i n_{k_i} \quad (n_{k_i} \in E)$$

with $n_{k_1} < n_{k_2} < \dots < n_{k_i}$ and $\epsilon_i = \pm 1$ or ± 2 . But n_{k_1} is divisible by a lower power of 3 than are $n_{k_2}, \dots, n_{k_i}$, which makes (2.5.2) impossible. Hence E satisfies the hypotheses of Theorem 2.4 with $B = 1$ and $t = 1$.

III. RELATIONS BETWEEN SIDON SETS AND SETS OF TYPE $\Lambda(q)$

3.1. Theorem. *Every Sidon set E is of type $\Lambda(q)$, for all $q < \infty$. More precisely, if B is the constant in (1.2.4), then every E -polynomial f also satisfies the inequalities*

$$(3.1.1) \quad \|f\|_q \leq B \sqrt{q} \|f\|_2 \quad (2 < q < \infty),$$

$$(3.1.2) \quad \|f\|_2 \leq 2B \|f\|_1.$$

Proof. If $\phi_k(t)$ are the Rademacher functions and if $g(t) = \sum a_k \phi_k(t)$, then it is known that

$$(3.1.3) \quad \int_0^1 |g(t)|^{2m} dt \leq m^m \left\{ \sum |a_k|^2 \right\}^m \quad (m = 1, 2, 3, \dots).$$

The proof in [16; p. 124] is given for real a_k , but applies equally well to complex a_k . Applying (1.4.1) to g , with $r = 1$, $s = 2$, $t = 4$, (3.1.3) yields

$$(3.1.4) \quad \left\{ \sum |a_k|^2 \right\}^{\frac{1}{2}} \leq 2 \int_0^1 |g(t)| dt.$$

Suppose now that E is a Sidon set, and that

$$f(e^{i\theta}) = \sum c_k e^{i n_k \theta}$$

is an E -polynomial. Define

$$(3.1.5) \quad g(t, e^{i\theta}) = \sum c_k \phi_k(t) e^{i n_k \theta} \quad (0 \leq t \leq 1, -\pi \leq \theta \leq \pi).$$

The proof of the implication (c) $\rightarrow$ (d) in Theorem 2.1 shows that we can associate with each $t \in [0, 1]$ a measure μ_t on the circle such that $\|\mu_t\| \leq B$ and $\hat{\mu}_t(n_k) = \phi_k(t)$. Hence $f = g_t * \mu_t$ (see (2.1.6)), where

$$g_t(e^{i\theta}) = g(t, e^{i\theta}),$$

so that

$$(3.1.6) \quad \|f\|_a \leq \|g_t\|_a \|\mu_t\| \leq B \|g_t\|_a \quad (0 \leq t \leq 1).$$

But we also have $g_t = f * \mu_t$, so that

$$(3.1.7) \quad \|g_t\|_1 \leq B \|f\|_1 \quad (0 \leq t \leq 1).$$

(These two inequalities contain the idea of the proof: No matter how we change the signs of the coefficients of f , the norms of f do not change much.)

Put $q = 2m$ in (3.1.6), where $m = 1, 2, 3, \dots$:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^{2m} d\theta \leq B^{2m} \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(t, e^{i\theta})|^{2m} d\theta \quad (0 \leq t \leq 1).$$

Integrate this with respect to t over $[0, 1]$, interchange the order of integration, and apply (3.1.3) with $c_k e^{i n_k \theta}$ in place of a_k . The result is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^{2m} d\theta \leq B^{2m} \cdot m^m \cdot \left\{ \sum |c_k|^2 \right\}^m$$

or

$$(3.1.8) \quad \|f\|_{2m} \leq B \sqrt{m} \|f\|_2 \quad (m = 1, 2, 3, \dots).$$

If now $2m - 2 \leq q \leq 2m$ ($m = 2, 3, \dots$), then $m \leq q$, and

$$\|f\|_q \leq \|f\|_{2m} \leq B \sqrt{m} \|f\|_2 \leq B \sqrt{q} \|f\|_2.$$

This is (3.1.1).

Similarly, (3.1.4) gives

$$\left\{ \sum |c_k|^2 \right\}^{\frac{1}{2}} \leq 2 \int_0^1 |g(t, e^{i\theta})| dt \quad (-\pi \leq \theta \leq \pi);$$

if we integrate this with respect to θ over $[-\pi, \pi]$ and use (3.1.7), we obtain (3.1.2), and the proof is complete.

3.2. As a corollary of (3.1.1), we note that

$$\int_{-\pi}^{\pi} \exp \{ |f(e^{i\theta})|^2 \} d\theta < \infty$$

for every E -function f , if E is a Sidon set (compare [16; p. 124]).

We shall see in 4.11 that there are sets which are of type $\Lambda(q)$ for all $q < \infty$ but which are not Sidon sets; no example is known, however, of a set which is not a Sidon set but which satisfies (3.1.1).

3.3. The estimate (3.1.1) is a best possible result, in the following strong sense: *For every infinite set E and for every positive integer k , there is an E -polynomial f such that*

$$\|f\|_k \geq \frac{1}{4} \sqrt{k} \|f\|_2.$$

This is a consequence of a quite general theorem about orthonormal systems:

3.4. Theorem. *Let μ be a positive measure on a measure space X with $\mu(X) = 1$. Let $\{\psi_n\}$ be a set of complex functions on X , orthonormal with respect to μ , and suppose there is a constant C such that*

$$(3.4.1) \quad \int_X |\psi_n| d\mu \geq 4C > 0 \quad (n = 1, 2, 3, \dots).$$

Then, for every positive integer k , we can choose the signs $\pm$ so that the function

$$(3.4.2) \quad f(x) = \sum_{i=1}^k \pm \psi_i(x)$$

satisfies

$$(3.4.3) \quad \|f\|_k \geq C \sqrt{k} \|f\|_2.$$

(In this theorem, $\|f\|_p$ stands for $\{\int_X |f|^p d\mu\}^{1/p}$.)

Proof. We fix k , and consider the function

$$g(t) = \sum_1^k c_i \phi_i(t) \quad (0 \leq t \leq 1)$$

where $\phi_i(t)$ are the Rademacher functions. It is easy to see that $\sum |c_i| \leq$

$2 \max_t |g(t)|$ (for real c_i the factor 2 can be dropped). Let I be an interval of length 2^{-k} on which $|g(t)|$ is equal to its maximum. Then

$$(3.4.4) \quad \sum_1^k |c_i| \leq \frac{2}{|I|} \int_I |g(t)| dt \leq 2 \left\{ 2^k \int_I |g(t)|^k dt \right\}^{1/k},$$

so that

$$(3.4.5) \quad \sum_1^k |c_i| \leq 4 \left\{ \int_0^1 |g(t)|^k dt \right\}^{1/k}.$$

We now apply (3.4.5) to

$$(3.4.6) \quad g(x, t) = \sum_1^k \psi_i(x) \phi_i(t) \quad (x \in X, 0 \leq t \leq 1),$$

and obtain

$$(3.4.7) \quad \left\{ \sum_1^k |\psi_i(x)| \right\}^k \leq 4^k \int_0^1 |g(x, t)|^k dt.$$

Integrating (3.4.7), we use (3.4.1):

$$(4Ck)^k \leq \left\{ \int_X \sum_1^k |\psi_i| d\mu \right\}^k \leq \int_X \left\{ \sum_1^k |\psi_i| \right\}^k d\mu \leq 4^k \int_0^1 dt \int_X |g(x, t)|^k d\mu(x).$$

Hence there is a value of t , say t_0 , such that

$$(3.4.8) \quad (Ck)^k \leq \int_X |g(x, t_0)|^k d\mu(x).$$

Setting $f(x) = g(x, t_0)$, f is of the form (3.4.2), and $\|f\|_k \geq Ck$, by (3.4.8). Since

$$(3.4.9) \quad \|f\|_2 = \left\{ \sum_1^k \phi_i^2(t_0) \right\}^{\frac{1}{2}} = \sqrt{k},$$

f satisfies (3.4.3).

We now turn our attention to intersections with finite arithmetic progressions; see 1.6(a) for the definition of $\alpha_E(N)$.

3.5. Theorem. Suppose E is of type $\Lambda(q)$, for some $q > 2$, and B is such that

$$(3.5.1) \quad \|f\|_q \leq B \|f\|_2$$

for every E -polynomial f . Then

$$(3.5.2) \quad \alpha_E(N) \leq 4B^2 N^{2/q} \quad (N = 1, 2, 3, \dots).$$

Proof. If K_N is the Fejér kernel

$$K_N(e^{i\theta}) = \sum_{-N}^N \left(1 - \frac{|n|}{N}\right) e^{in\theta},$$

then $\|K_N\|_1 = 1$ and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_N^2(e^{i\theta}) d\theta = \sum_{-N}^N \left(1 - \frac{|n|}{N}\right)^2 = \frac{2N^2 + 1}{3N} \leq N.$$

Let p, q be conjugate exponents ($1/p + 1/q = 1$; $p < 2$) and apply (1.4.1) to K_N , with $r = 1, s = p, t = 2$. This yields

$$(3.5.3) \quad \|K_N\|_p \leq N^{1/q}.$$

Consider the arithmetic progression

$$A = \{a + b, a + 2b, \dots, a + Nb\},$$

set $m = \frac{1}{2}N$ or $\frac{1}{2}(N + 1)$, depending on whether N is even or odd, and define

$$Q(e^{i\theta}) = e^{im\theta} K_N(e^{ib\theta}).$$

Then

$$(3.5.4) \quad \hat{Q}(n) \geq \frac{1}{2} \quad (n \in A).$$

If $A \cap E$ consists of the numbers $n_1, \dots, n_\alpha$, consider the E -polynomial

$$f(e^{i\theta}) = \sum_1^\alpha e^{in_k\theta}.$$

By (3.5.4), (3.5.1) and (3.5.3), we obtain

$$\begin{aligned} \frac{\alpha}{2} &\leq \sum_1^\alpha \hat{Q}(n_k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{-i\theta}) Q(e^{i\theta}) d\theta \\ &\leq \|f\|_q \|Q\|_p \leq B \|f\|_2 \|K_N\|_p \leq B \sqrt{\alpha} \cdot N^{1/q}. \end{aligned}$$

Thus $\alpha \leq 4B^2 N^{2/q}$, and the theorem follows.

3.6. Theorem. Suppose E is of type $\Lambda(q)$, for every $q < \infty$, and suppose there is a constant B such that

$$(3.6.1) \quad \|f\|_q \leq B \sqrt[q]{q} \|f\|_2 \quad (2 < q < \infty)$$

for every E -polynomial f . Then

$$(3.6.2) \quad \alpha_E(N) \leq 8eB^2 \log N \quad (N = 2, 3, 4, \dots).$$

Proof. By (3.5.2), we have

$$(3.6.3) \quad \alpha_E(N) \leq 4B^2 q N^{2/q} \quad (2 < q < \infty).$$

Fix N and take $q = 2 \log N$; when $N \geq 3$, this makes $q > 2$. This choice of q makes $N^{2/q} = e$, and (3.6.2) follows.

Corollary. (3.6.2) holds for every Sidon set (by (3.1.1)).

That $\alpha_E(N) = O(\log N)$ when E is a Sidon set has also been proved by STEČKIN [14; p. 401] and independently by HELSON (oral communication,

1956); their proofs are alike, but they differ from the one given here. An analogue of this result, in a more general context, may be found in [6; p. 312].

3.7. The set $E = \{2^k\}$ ($k = 0, 1, 2, \dots$) is a Sidon set, and

$$\alpha_E(2^k) \geq k + 1 > \frac{\log 2^k}{\log 2}.$$

Hence $\log N$ cannot be replaced in (3.6.2) by any function of N which is $o(\log N)$. Also, (3.5.2) is a best possible estimate, at least when $q = 4, 6, 8, \dots$, as shown by Theorem 4.8. Nevertheless, neither of these results has a converse:

3.8. Theorem. Suppose $\alpha(x) \geq 3$ for $x > 0$ and $\alpha(x) \rightarrow \infty$ monotonically as $x \rightarrow \infty$. Then there exists a set E , which is not even of type $\Lambda(1)$, such that $\alpha_E(N) \leq \alpha(N)$ for all positive integers N .

Proof. Define positive integers $w(k)$ such that $w(k+1) > w(k)$, and such that

$$(3.8.1) \quad \alpha(w(k)) > k + 1 \quad (k = 1, 2, 3, \dots).$$

Put $n_1 = 1$; having defined $n_1, \dots, n_p$ ($p \geq 1$), let n_{p+1} be an integer such that

$$(3.8.2) \quad n_{p+1} > 2(n_1 + \dots + n_p) \cdot w(3^{p+1}).$$

Let E_p be the set which consists of the 3^p distinct numbers

$$(3.8.3) \quad n = \epsilon_1 n_1 + \epsilon_2 n_2 + \dots + \epsilon_p n_p \quad (\epsilon_i = 1, 0, -1),$$

and let E be the union of these sets E_p .

Note that E_{p+1} is a union of three sets, which we shall call its components and which are translates of E_p :

$$(3.8.4) \quad E_{p+1} = (E_p - n_{p+1}) \cup E_p \cup (E_p + n_{p+1}).$$

We claim that if an arithmetic progression A intersects at least 2 of these components, and if $A \cap E_{p+1}$ has at least 4 elements, then A has at least $w(3^{p+1})$ elements. Indeed, some 2 points of A must be in the same component, so that the distance between consecutive points of A does not exceed $2(n_1 + \dots + n_p)$; since A intersects 2 components, and since the distance between these components is at least $n_{p+1} - 2(n_1 + \dots + n_p)$, our assertion follows from (3.8.2).

Suppose now that $k \geq 4$ and that A_0 is an arithmetic progression such that $A_0 \cap E$ consists of k points. Let p be the smallest integer such that $A \cap E_{p+1}$ has k points, for some translate A of A_0 . Since p is the smallest such integer, A intersects 2 components of E_{p+1} , and since E_{p+1} has 3^{p+1} points, we have $k \leq 3^{p+1}$. The preceding paragraph thus shows that A has at least $w(k)$ terms.

In other words,

$$(3.8.5) \quad \alpha_E(w(k) - 1) < k \quad (k = 4, 5, 6, \dots).$$

With $k = 4$, this yields $\alpha_E(N) \leq 3 \leq \alpha(N)$ if $N < w(4)$. If $w(k) \leq N \leq w(k+1) - 1$, for some $k \geq 4$, (3.8.5) and (3.8.1) imply

$$\alpha_E(N) \leq \alpha_E(w(k+1) - 1) < k+1 < \alpha(w(k)) \leq \alpha(N),$$

and one half of the theorem is proved.

To prove that E is not of type $\Lambda(1)$, we form the Riesz product

$$(3.8.6) \quad \prod_{p=1}^{\infty} (1 + \tfrac{1}{2}e^{in_p\theta} + \tfrac{1}{2}e^{-in_p\theta}),$$

which, when multiplied out formally, is equal to a Fourier-Stieltjes series $\sum_{-\infty}^{\infty} c_n e^{in\theta}$ with $c_{n_p} = \frac{1}{2}$ ($p = 1, 2, 3, \dots$) and $c_n = 0$ if $n \notin E$. It follows [16; p. 153] that the series $\sum_0^{\infty} c_n z^n$ ($|z| < 1$) is a function of class H_{δ} , for every $\delta < 1$. This function can be approximated, in the metric of L^{δ} , by E -polynomials, and if E were of type $\Lambda(1)$, these polynomials would be bounded in the L^1 -norm. Hence $\sum_0^{\infty} c_n e^{in\theta}$ would be a Fourier-Stieltjes series of some measure which, by the theorem of F. and M. RIESZ (see 5.7), must be absolutely continuous, so that $c_n \rightarrow 0$ as $n \rightarrow +\infty$. But $c_{n_p} = \frac{1}{2}$, and this contradiction shows that E is not of type $\Lambda(1)$.

IV. STRUCTURAL PROPERTIES OF SETS OF TYPE $\Lambda(q)$

We begin with another theorem on arithmetic progressions.

4.1. Theorem. *If E is of type $\Lambda(1)$, then E does not contain arbitrarily long arithmetic progressions.*

In other words, $\alpha_E(N) < N$ for all sufficiently large N .

Proof. Choose $p < 1$ and B_p such that

$$(4.1.1) \quad \|f\|_1 \leq B_p \|f\|_p$$

for every E -polynomial f . Suppose E contains the numbers $a+b, a+2b, \dots, a+Nb$, where $b > 0$. Then

$$f(e^{i\theta}) = e^{ia\theta} \sum_{n=1}^N e^{inb\theta}$$

is an E -polynomial, and

$$|f(e^{i\theta})| = \left| \frac{\sin \frac{1}{2}Nb\theta}{\sin \frac{1}{2}b\theta} \right|.$$

Hence $\|f\|_1 \geq A \log N$ and $\|f\|_p < C_p$, where A is an absolute constant and $C_p < (1-p)^{-1}$. Thus (4.1.1) implies

$$A \log N \leq B_p C_p,$$

and this puts an upper bound on the possible values of N .

4.2. It seems plausible that every set of type $\Lambda(p)$ has density zero (when

$p > 2$, Theorem 3.5 asserts much more, in fact), but a direct proof of this might be easier than one based on Theorem 4.1. ROTH [8] has proved that a set of integers must have density zero if it contains no arithmetic progression of 3 terms, but it is not known whether the same is true if 3 is replaced by a larger integer.

4.3. Although Theorem 4.1 is rather crude, it does give some information. For instance, SIDON [13; p. 230] conjectured that a set E of positive integers is of type $\Lambda(2)$ if no term of E is the sum of two terms of E .

This conjecture is false. To disprove it, let E be the union of the sets E_m ($m = 1, 2, 3, \dots$), where E_m is the arithmetic progression consisting of the numbers

$$(4.3.1) \quad (2m)! + (2m-1)!k \quad (k = 0, 1, \dots, m).$$

It is easily checked that no term of E_m is the sum of two terms of E_m ; furthermore, the difference between any two terms of E_m is greater than the largest term of E_{m+1} , and the smallest term of E_m is more than twice as large as the largest term of E_{m-1} . It follows that E has the property postulated by SIDON, but Theorem 4.1 shows that E is not even of type $\Lambda(1)$.

Theorem 4.1 will also be used in the proof of Theorem 4.9.

4.4. Theorem. Suppose E_1 and E_2 are of type $\Lambda(p)$ and $E = E_1 \cup E_2$.

- (a) If $p > 2$, then E is of type $\Lambda(p)$.
- (b) If E_1 is a set of non-negative integers and if E_2 is a set of negative integers, then the conclusion of (a) holds also for $p > 1$.

Proof. We assume, without loss of generality, that E_1 and E_2 are disjoint. If $p > 2$, Theorem 1.4 shows that there are constants B_i such that $\|f_i\|_p \leq B_i \|f_i\|_2$ for every E_i -polynomial f_i ($i = 1, 2$). If f is an E -polynomial, then $f = f_1 + f_2$, where f_i ($i = 1, 2$) is an E_i -polynomial; hence

$$\|f\|_p \leq \|f_1\|_p + \|f_2\|_p \leq B_1 \|f_1\|_2 + B_2 \|f_2\|_2 \leq (B_1 + B_2) \|f\|_2,$$

and (a) follows.

To prove (b), choose s so that $1 < s < p$. By 1.4, there are constants B_i such that $\|f_i\|_p \leq B_i \|f_i\|_s$ for E_i -polynomials f_i , and M. RIESZ' theorem on conjugate functions [16; p. 147] implies the existence of a constant A_s such that $\|f_i\|_s \leq A_s \|f\|_s$ if $f = f_1 + f_2$, as above. Hence

$$\|f\|_p \leq \|f_1\|_p + \|f_2\|_p \leq B_1 \|f_1\|_s + B_2 \|f_2\|_s \leq A_s (B_1 + B_2) \|f\|_s,$$

for every E -polynomial f .

In view of this theorem we can restrict ourselves, without loss of generality, to sets of non-negative integers in the theorems which follow.

We refer to 1.6(b) for the definition of $r_s(E, n)$.

4.5. Theorem. Let E be a set of non-negative integers, and let s be an integer, $s > 1$.

(a) If E is of type $\Lambda(2s)$, then the averages

$$(4.5.1) \quad \frac{1}{N} \sum_{n=0}^N r_s^2(E, n)$$

are bounded as $N \rightarrow \infty$.

(b) If E is a union of sets $E_1, \dots, E_t$ such that $r_s(E_i, n)$ is a bounded function of n , for each j , then E is of type $\Lambda(2s)$.

Proof. Suppose $E = \{n_k\}$, $0 \leq n_1 < n_2 < \dots$, and consider the E -polynomial

$$f(e^{i\theta}) = e^{in_1\theta} + \dots + e^{in_k\theta}.$$

Writing $r(n)$ in place of $r_s(E, n)$, we find

$$f^s(e^{i\theta}) = r(0) + r(1)e^{i\theta} + r(2)e^{2i\theta} + \dots + r(n_k)e^{in_k\theta} + \dots;$$

the coefficient of $e^{in\theta}$, for $n > n_k$, will in general be different from $r(n)$.

If $\|f\|_{2s} \leq B \|f\|_2$ (that is, if $E \in \Lambda(2s)$), the Bessel inequality implies

$$\sum_{i=1}^{n_k} r^2(i) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^{2s} d\theta \leq B^{2s} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 d\theta \right\}^s = B^{2s} k^s.$$

By Theorem 3.5, $k \leq 4B^2 n_k^{1/s}$, so that $n_k \geq B_1 k^s$. (B, B_1, B_2 are positive numbers which depend only on E and s .) Hence

$$(4.5.2) \quad \frac{1}{B_1 k^s} \sum_{i=0}^{\lfloor B_1 k^s \rfloor} r^2(i) \leq B_2.$$

If N is given, choose k so that $B_1(k-1)^s \leq N \leq B_1 k^s$; (4.5.2) then implies that the averages (4.5.1) are bounded.

By Theorem 4.4 it is enough to prove 4.5(b) for the case $t = 1$. As in [16; p. 217], we apply PARSEVAL's theorem to f^s , where f is an E -polynomial, to obtain an upper bound for $\|f\|_{2s}$. If

$$f(e^{i\theta}) = \sum a(k)e^{in_k\theta},$$

where $E = \{n_k\}$ as above, then

$$f^s(e^{i\theta}) = \sum b_m e^{im\theta},$$

where

$$b_m = \sum a(i_1) \cdots a(i_s),$$

the summation being extended over all representations of m in the form $m = n_{i_1} + \dots + n_{i_s}$. Among these representations there is one (we denote the corresponding subscripts by $i_1(m), \dots, i_s(m)$) which maximizes $|a(i_1) \cdots a(i_s)|$. Then if R is an upper bound for $r_s(E, m)$, we have

$$|b_m|^2 \leq R^2 |a(i_1(m)) \cdots a(i_s(m))|^2,$$

so that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^{2s} d\theta = \sum |b_m|^2 \leq R^2 \left\{ \sum |a(k)|^2 \right\}^s = R^2 \|f\|_2^{2s},$$

or $\|f\|_{2s} \leq R^{1/s} \|f\|_2$.

This completes the proof.

4.6. In connection with the preceding theorem, it may be of interest to note that the boundedness of the averages

$$(4.6.1) \quad \frac{1}{N} \sum_{n=0}^N r_2(E, n)$$

does not imply that E is of type $\Lambda(4)$ (*i.e.*, the exponent 2 cannot be replaced by 1 in (4.5.1)).

An example of this is obtained by taking for E the set of all squares ($n_k = k^2$). Then $\sum_0^N r_2(E, n)$ is the number of lattice points in the first quadrant of the plane, inside the circle of radius $\sqrt{N}$ with center at the origin, so that (4.6.1) tends to $\frac{1}{4}\pi$ as $N \rightarrow \infty$. However, the set of those integers which are sums of two squares has density zero [8; vol. II, p. 257]. Setting $\beta(n) = 1$ or 0 according as n is or is not a sum of two squares, we obtain

$$\frac{\pi}{4} N \sim \sum_0^N r_2(n) = \sum_0^N \beta(n) r_2(n) \leq \left\{ \sum_0^N \beta(n) \cdot \sum_0^N r_2^2(n) \right\}^{\frac{1}{2}},$$

or

$$N^2 = o(N) \cdot \sum_1^N r_2^2(n),$$

so that the averages

$$\frac{1}{N} \sum_0^N r_2^2(E, n)$$

tend to infinity as $N \rightarrow \infty$. We thus conclude from Theorem 4.5(a) that the set of all squares is not of type $\Lambda(4)$.

This suggests an interesting (and apparently difficult) number-theoretic question: Does the set E of all squares satisfy the conclusion of Theorem 3.5 with $q = 4$? In other words, is $\alpha_E(N) = O(N^{\frac{1}{4}})$? It is not even known whether $\alpha_E(N) = o(N)$; and it is also not known whether E is of type $\Lambda(p)$ for any p .

4.7. In preparation for Theorem 4.8, we now construct certain finite sets $F = F(s, p)$, where $s = 2, 3, 4, \dots$ and p is a prime, $p > s$. The object is to keep $r_s(F, n)$ down to its minimum (which, by 1.6(b), is $s!$), while at the same time packing the terms of F as closely as possible to make $\alpha_F(N)$ large. For $s = 2$, the construction which follows is due to ERDÖS & TURAN ([3]; see also [15; pp. 132–134] where this problem is discussed in some detail).

If s and p are as above, we define $\lambda(k)$, for any integer k , by

$$(4.7.1) \quad \lambda(k) \equiv k \pmod{p}, \quad 0 \leq \lambda(k) \leq p-1,$$

and we let $F = F(s, p)$ be the set consisting of the p integers

$$(4.7.2) \quad x_k = \sum_{i=0}^{s-1} \lambda(k^{s-i})(sp)^i \quad (k = 0, \dots, p-1).$$

This set has the following properties:

$$(4.7.3) \quad \alpha_F(s^{s-1}p^s) = p.$$

$$(4.7.4) \quad r_s(F, n) \leq s! \quad \text{for all } n.$$

Since

$$x_k = k(sp)^{s-1} + \lambda(k^2)(sp)^{s-2} + \dots + \lambda(k^{s-1})(sp) + \lambda(k^s),$$

we see that $0 = x_0 < x_1 < \dots < x_{p-1}$, and

$$x_{p-1} \leq (p-1) \sum_{i=0}^{s-1} (sp)^i < s^{s-1}p^s.$$

Thus the p terms of F lie in the arithmetic progression $0, 1, 2, \dots, s^{s-1}p^s - 1$, and (4.7.3) is established.

Suppose now that

$$(4.7.5) \quad y = x_{k_1} + x_{k_2} + \dots + x_{k_s},$$

and express y in the number system whose base is sp :

$$(4.7.6) \quad y = \sum_{i=0}^{s-1} y_i(sp)^i \quad (0 \leq y_i < sp).$$

Since $\lambda(k) < p$, (4.7.5) implies

$$\sum_{j=1}^s \lambda(k_j^{s-i}) = y_i \quad (i = 0, \dots, s-1),$$

so that

$$(4.7.7) \quad \sum_{j=1}^s k_j^n \equiv y_{s-n} \pmod{p} \quad (n = 1, \dots, s).$$

Thus the digits y_i of y determine the first s power sums of $k_1, \dots, k_s$ in the cyclic field of p elements; hence they determine the elementary symmetric functions of $k_1, \dots, k_s$ in this field [2; pp. 145–147], and this in turn means that $k_1, \dots, k_s$ are determined by (4.7.5), up to permutations. This proves (4.7.4).

Since 0 is one of the terms of F , we note that we have actually proved a little more than (4.7.4), namely:

If $t \leq s$, the representation of any y in the form

$$(4.7.8) \quad y = x_{k_1} + x_{k_2} + \cdots + x_{k_t}$$

is unique, up to permutations of the x_{k_i} .

4.8. Theorem. Suppose s is an integer, $s > 1$. There exists a set E , consisting of non-negative integers, such that

- (i) $r_s(E, n) \leq s!$ for all n ;
- (ii) $\limsup N^{-1/s} \alpha_E(N) > 0$.

By (i) and 4.5(b), E is of type $\Lambda(2s)$. By (ii) and 3.5, E is not of type $\Lambda(2s + \epsilon)$ if $\epsilon > 0$.

Proof. Let $\{p_i\}$ be a sequence of primes such that $s < p_1 < p_2 < \cdots$. Our set E is the union of the sets G_i which are obtained from $F(s, p_i)$ (see 4.7) by homothetic transformations: G_i is the set of all $a_i x$, for $x \in F(s, p_i)$, where the "stretching factors" a_i are positive integers, chosen inductively so that a_i is larger than s times the largest number in G_{i-1} .

If we put $N_i = s^{s-1} p_i^s$, (4.7.3) implies that

$$(4.8.1) \quad \alpha_E(N_i) \geq p_i = s^{(1/s)-1} N_i^{1/s} \quad (i = 1, 2, 3, \dots)$$

and (ii) is proved.

On the other hand, suppose we have

$$(4.8.2) \quad n_1 + \cdots + n_s = m_1 + \cdots + m_s$$

with m_i and n_i in E . Let H_0 be the set of all m_i and n_i which are 0, and let H_i ($i \geq 1$) be the set of all m_i and n_i which are in G_i but not in H_{i-1} . There is a largest integer i_0 such that H_{i_0} is not empty. Transpose (4.8.2) so that the terms of H_{i_0} are on the left and the others are on the right. On the left we have a multiple of a_{i_0} , and our choice of $\{a_i\}$ shows that the left side must be 0. That is to say, the sum of the n_i in H_{i_0} equals the sum of the m_i in H_{i_0} , and we conclude from (4.7.8) that the n_i in H_{i_0} are identical with the m_i in H_{i_0} . We can therefore replace them by 0 in (4.8.2) and repeat the argument.

The conclusion is that, up to permutations, the sets $\{n_1, \dots, n_s\}$ and $\{m_1, \dots, m_s\}$ are identical. This implies (i), and the proof is complete.

4.9. In [12; p. 91] SIDON raised a question which may be stated in the following way: If $\{r_2(E, n)\}$ is bounded, does it follow that $E + E$ is of type $\Lambda(2)$? Here $E + E$ is the set of all $m + n$ with $m \in E, n \in E$.

The answer is negative: There exists a set E of non-negative integers, with $r_2(E, n) \leq 2$, such that $E + E$ contains arbitrarily long arithmetic progressions and is therefore not even of type $\Lambda(1)$.

We can construct such a set as in 4.8, taking $s = 2$. The sets $F = F(2, p)$ of 4.7 now consist of the integers

$$(4.9.1) \quad 2kp + \lambda(k^2) \quad (k = 0, \dots, p-1),$$

so that $F + F$ contains the numbers

$$2kp + \lambda(k^2) + 2(p - k)p + \lambda((p - k)^2) = 2p^2 + 2\lambda(k^2)$$

for $k = 1, \dots, \frac{1}{2}(p - 1)$.

Let $q_1, \dots, q_i$ be the first i primes, and let p_i be a prime of the form

$$(4.9.2) \quad p_i = 4m_i q_i \cdots q_i + 1,$$

where the positive integer m_i is chosen so that $p_i > p_{i-1}$ (this is possible, by DIRICHLET'S theorem on primes in arithmetic progressions).

By the quadratic reciprocity law [7; vol. I, p. 72], $q_1, q_2, \dots, q_i$ are quadratic residues (mod p_i), and it follows that each of the integers $1, 2, \dots, q_i - 1, q_i$ is a quadratic residue (mod p_i) since products of residues are residues.

Thus if $F_i = F(2, p_i)$, then $F_i + F_i$ contains the numbers $2p_i^2 + 2\lambda(k^2)$, for $k = 1, \dots, \frac{1}{2}(p_i - 1)$ (where λ is defined with p_i in place of p); the numbers $\lambda(k^2)$ run through the quadratic residues (mod p_i); hence $F_i + F_i$ contains each of the numbers

$$(4.9.3) \quad 2p_i^2 + 2m \quad (m = 1, 2, \dots, q_i).$$

If we use only primes of the form (4.9.2) in the construction of E in 4.8, we therefore obtain a set with the required properties.

4.10. As a corollary of Theorem 3.5, we observe that

$$(4.10.1) \quad \lim_{N \rightarrow \infty} N^{-\epsilon} \alpha_E(N) = 0$$

for every $\epsilon > 0$, if E is of type $\Lambda(q)$ for all $q < \infty$. We conclude this section with a proof that (4.10.1) is a best possible result; in particular, there exist sets which are of type $\Lambda(q)$ for all $q < \infty$ without being Sidon sets (we may take $\phi(N) = \log^2 N$, for instance).

4.11. Theorem. *Suppose*

$$(4.11.1) \quad \lim_{N \rightarrow \infty} \phi(N)/N^\epsilon = 0$$

for every $\epsilon > 0$. Then there exists a set E , which is of type $\Lambda(q)$ for every $q < \infty$, such that

$$(4.11.2) \quad \alpha_E(N) > \phi(N)$$

for infinitely many N .

Proof. By (4.11.1), $\phi(s^{*-1}x^*)/x \rightarrow 0$ as $x \rightarrow \infty$, for every $s > 0$. Hence there is a sequence $\{p_s\}$ of primes $p_2 < p_3 < \dots$ such that

$$(4.11.3) \quad p_s > \phi(s^{*-1}p_s) \quad (s = 2, 3, 4, \dots).$$

If $F_s = F(s, p_s)$, in the notation of 4.7, and $N_s = s^{*-1}p_s^*$, then

$$(4.11.4) \quad \alpha_{F_s}(N_s) = p_s > \phi(N_s) \quad (s = 2, 3, 4, \dots)$$

by (4.7.3).

We now proceed as in 4.8, setting $E = \cup G_s$ where $G_s = a_s F_s$; the stretching factors a_s are chosen so that a_s is larger than s times the largest term of G_{s-1} . Then

$$(4.11.5) \quad \alpha_E(N_s) \geq \alpha_{F_s}(N_s) > \phi(N_s) \quad (s = 2, 3, 4, \dots).$$

Let E_s be the set which is obtained by removing $G_2, \dots, G_s$ from E . The argument used in the proof of 4.8 can be repeated with E_s in place of E , and we conclude that

$$(4.11.6) \quad r_s(E_s, n) \leq s! \quad (s = 2, 3, 4, \dots; n = 1, 2, 3, \dots).$$

Hence E_s is of type $\Lambda(2s)$. Since E differs from E_s by a finite set, E is also of type $\Lambda(2s)$. This completes the proof.

V. ANALYTIC PROPERTIES OF SETS OF TYPE $\Lambda(p)$

In this section, p and q will always denote conjugate exponents; that is to say, $1/p + 1/q = 1$. We shall also assume that $1 < p < \infty$.

5.1. Theorem. *Each of the following five statements about a set E of integers implies the others:*

- (a) E is of type $\Lambda(p)$.
- (b) If μ is an E -measure, then there is an $f \in L_p$ such that $d\mu(\theta) = f(e^{i\theta})d\theta$.
- (c) Every E -function in L^1 belongs to L^p .
- (d) To every $g \in L^q$ there exists a bounded function h such that $\hat{h}(n) = \hat{g}(n)$ for all $n \in E$.
- (e) To every $g \in L^q$ there exists a continuous function h such that $\hat{h}(n) = \hat{g}(n)$ for all $n \in E$.

Proof. Suppose (a) holds and μ is an E -measure. The Cesàro means of the Fourier-Stieltjes series of μ are bounded in L^1 , hence they are bounded in L^p , and thus the series is the Fourier series of a function in L^p . It follows that (a) implies (b).

It is trivial that (b) implies (c), and the implication (c) $\rightarrow$ (a) is established as in 1.3, if we observe that L_E^1 and L_E^p are identical Banach spaces if (c) holds.

Having proved the equivalence of (a), (b), and (c), let us suppose again that (a) holds, and choose B such that

$$(5.1.1) \quad \|f\|_p \leq B \|f\|_1$$

for every E -polynomial. If now $g \in L^q$, the mapping

$$(5.1.2) \quad f \rightarrow Tf = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{-i\theta})g(e^{i\theta}) d\theta$$

is a bounded linear functional on the space of all E -polynomials endowed with the L^1 -norm:

$$(5.1.3) \quad |Tf| \leq \|f\|_p \cdot \|g\|_q \leq B \cdot \|g\|_q \cdot \|f\|_1.$$

By the Hahn-Banach theorem, we can extend T to a bounded linear functional on L^1 , and we conclude that there is a bounded function h such that

$$(5.1.4) \quad Tf = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{-i\theta}) h(e^{i\theta}) d\theta \quad (f \in L^1).$$

Take $f(e^{i\theta}) = e^{in\theta}$ in (5.1.2) and in (5.1.4), where $n \in E$; the two expressions for Tf so obtained yield $g(n) = \hat{h}(n)$. Thus (a) implies (d).

Suppose (d) holds, and choose $g \in L^q$. By Lemma 5.2 which follows, there exist $f \in L^1$ and $g_1 \in L^q$ so that $g = f * g_1$. By (d), there is a bounded function h_1 with $\hat{h}_1(n) = \hat{g}_1(n)$ for all $n \in E$. Put $h = f * h_1$. Then h is continuous, and for $n \in E$ we have

$$\hat{h}(n) = \hat{f}(n) \hat{h}_1(n) = \hat{f}(n) \hat{g}_1(n) = \hat{g}(n).$$

Thus (d) implies (e).

Finally, suppose (e) holds. Let C_0 and L_0^q denote the subspaces of C and L^q , respectively, which consist of those functions f for which $\hat{f}(n) = 0$ for all $n \in E$. Then (e) says precisely that the two quotient spaces C/C_0 and L^q/L_0^q are identical, so that their two quotient space norms are equivalent. In other words, there is a constant B with the following property: To every $g \in L^q$ there exists a continuous function h such that $\hat{h}(n) = \hat{g}(n)$ and such that $\|h\|_{\infty} \leq B \|g\|_q$. If now f is an E -polynomial, we have

$$\begin{aligned} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{-i\theta}) g(e^{i\theta}) d\theta \right| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{-i\theta}) h(e^{i\theta}) d\theta \right| \\ &\leq \|f\|_1 \cdot \|h\|_{\infty} \leq B \cdot \|f\|_1 \cdot \|g\|_q, \end{aligned}$$

for every $g \in L^q$, so that $\|f\|_p \leq B \|f\|_1$.

Thus (e) implies (a), and the proof is complete.

5.2. Lemma. Suppose $1 < q < \infty$ and $h \in L^q$. Then there exist $f \in L^1$, $g \in L^q$ such that $h = f * g$; i.e.,

$$h(e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i(\theta-\phi)}) g(e^{i\phi}) d\phi,$$

for almost all θ .

Proof. Put $s_k(e^{i\theta}) = \sum_{-k}^k \hat{h}(n) e^{in\theta}$ and $\epsilon_k = \|h - s_k\|_q$. Then $\epsilon_k \rightarrow 0$ [16; p. 153], and we can choose $\{a_k\}$ such that $a_0 > a_1 > \dots > 0$, $\sum_0^{\infty} a_k = \infty$, but $\sum_1^{\infty} a_k \epsilon_{k-1} < \infty$. Then the series

$$\sum_{k=0}^{\infty} a_k (h - s_{k-1}) \quad (s_{-1} = 0)$$

converges in the norm of L^q to a function $g \in L^q$, and we have

$$g(n) = \sum_{k=0}^{\infty} a_k [\hat{h}(n) - \hat{s}_{k-1}(n)] = \hat{h}(n)(a_0 + a_1 + \cdots + a_{|n|})$$

$$(n = 0, \pm 1, \pm 2, \cdots).$$

Setting $C_n = (a_0 + a_1 + \cdots + a_{|n|})^{-1}$, $\{C_n\}$ is an even sequence which tends to 0 and is convex for positive n , so that $C_n = \hat{f}(n)$ for some $f \in L^1$ [16; p. 109]. Since $\hat{h}(n) = \hat{f}(n) g(n)$, the lemma is proved.

Remark. The lemma is also true for $q = 1$, but the proof is a little more involved since the s_k have to be replaced by their averages [11].

5.3. The case $p = q = 2$ of Theorem 5.1. In this case 5.1(e) becomes

(e') If $\sum |a(n)|^2 < \infty$ ($n \in E$), there exists a continuous function h such that $\hat{h}(n) = a(n)$ for all $n \in E$.

The equivalence of 5.1(c) with (e') is the theorem of BANACH which was mentioned in the introduction.

It is clear that 5.1(d) can be similarly rephrased.

5.4. The case $p > 2$ of Theorem 5.1. In this case we can add a sixth statement to our list:

(f) If $g \in L^q$, then $\sum_{n \in E} |g(n)|^2 < \infty$.

Indeed, it is clear that 5.1(e) implies (f), and our proof of the implication (e) $\rightarrow$ (a) is readily adapted to show that (f) implies (a). (The equivalence of (f) and (a) is known; see for instance [5; p. 245].)

5.5. Permutations. By a *permutation* we shall mean a one-to-one mapping of the set of all integers onto itself. A permutation τ is said to carry L^p to L^p if the series $\sum a(\tau(n))e^{in\theta}$ is the Fourier series of a function in L^p whenever $\sum a(n)e^{in\theta}$ is such a series.

With every p there is thus associated a group $G(p)$ of permutations; it is easy to see that $G(p)$ is precisely the group of all automorphisms of the convolution algebra L^p .

The elements of $G(1)$ were explicitly determined in [9]. Consideration of the adjoint map shows that $G(p) = G(q)$ if p and q are conjugate exponents, and the interpolation theorem of M. RIESZ [1; p. 177] implies that $G(p_1) \supset G(p_2)$ if $2 \leq p_1 < p_2$; $G(2)$, of course, contains all permutations.

The conjecture that $G(p_2)$ is a *proper* subgroup of $G(p_1)$, under the above conditions, is supported by the following theorem.

5.6. Theorem. If $2 \leq p_1 < p_2 < \infty$ and if $\Lambda(p_2)$ is properly contained in $\Lambda(p_1)$, then $G(p_2)$ is a proper subgroup of $G(p_1)$.

Proof. By hypothesis, there is a set E which is of type $\Lambda(p_1)$ but not of type

$\Lambda(p_2)$. By 5.4, there is a function $g \in L^{q^*}$ such that

$$(5.6.1) \quad \sum_{n \in E} |\hat{g}(n)|^2 = \infty.$$

Decompose E into the union of two infinite disjoint sets E_1 and E_2 such that E_2 is of type $\Lambda(p_2)$. We can then find a permutation τ such that $\tau(n) = n$ for all n not in E and such that

$$(5.6.2) \quad \sum_{n \in E_2} |\hat{g}(\tau(n))|^2 = \infty.$$

By 5.4, this implies that the series $\sum \hat{g}(\tau(n))e^{in\theta}$ is not in L^{q^*} , so that τ is not in $G(q_2)$.

But if $f \in L^{q^*}$, 5.4 shows that

$$(5.6.3) \quad \sum_{n \in E} |\hat{f}(n)|^2 < \infty,$$

so that τ carries L^{q^*} to L^{q^*} (the transformed function differs from the original one by a function in L^2 , by (5.6.3)). Thus τ is in $G(q_1)$, and this group contains $G(q_2)$ properly.

Since $G(p) = G(q)$, the theorem is proved.

Remarks. (a) The proof of 5.6 can be adapted to show that $G(p)$ is a proper subgroup of $G(2)$ if $p \neq 2$.

(b) The proof of 5.6 shows that $G(p)$ has the power of the continuum if $1 < p < \infty$. $G(1)$, on the other hand, is a countable group [9].

5.7. Extension of the Theorem of F. and M. Riesz. This theorem states that if μ is a measure on the circle such that $\hat{\mu}(n) = 0$ for all $n > 0$, then μ is absolutely continuous [16; p. 158]. The hypothesis can be weakened:

Theorem. Suppose E is a set of positive integers of type $\Lambda(1)$ and μ is a measure whose Fourier-Stieltjes series has the form

$$(5.7.1) \quad \sum_{-\infty}^0 c_n e^{in\theta} + \sum_{n \in E} c_n e^{in\theta}.$$

Then μ is absolutely continuous.

Proof. The function $f(z) = \sum_{n \in E} c_n z^n$ ($|z| < 1$) belongs to H_p , for every $p < 1$ [16; p. 153], and f can be approximated in the metric of H_p by E -polynomials. Since $E \in \Lambda(1)$, these polynomials are bounded in the L^1 -norm, so that the series

$$(5.7.2) \quad \sum_{n \in E} c_n e^{in\theta}$$

is a Fourier-Stieltjes series.

Subtracting (5.7.2) from (5.7.1), we see that each of the series in (5.7.1) is a Fourier-Stieltjes series, and the theorem of F. and M. RIESZ implies that each of these series is actually a Fourier series. The theorem follows.

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