*The Einstein Tensor in Orthogonal Coordinates in n Dimensions**

BANESH HOFFMANN

1. Introduction. In an investigation connected with the theory of relativity I needed to know the explicit forms of the components of the Einstein tensor in an orthogonal coordinate system in an n -dimensional Riemannian space of arbitrary signature.

DINGLE [1] has given expressions for the individual components of the tensor for space-time when the line element is written in the form

$$(1) \quad ds^2 = D (dx^4)^2 - A (dx^1)^2 - B (dx^2)^2 - C (dx^3)^2.$$

By using (1), DINGLE was able to present the components of the Einstein tensor in forms well suited for specific applications, but his formulas do not immediately show the compact pattern of the components as a whole. To infer the pattern for the four-dimensional case from DINGLE's results is, of course, easy. One must replace A, B, C, D by their more familiar symbols as components of the metrical tensor, making allowance for minuses arising from the fact that $g_{11} = -A, g_{22} = -B, g_{33} = -C$, but $g_{44} = +D$.

DINGLE does not indicate his mode of calculation. If one works throughout in terms of A, B, C, D , one loses sight of the underlying pattern right from the start and is inclined to perform the calculation by brute force. Since I needed the general pattern of the components of the Einstein tensor in n dimensions, I decided to perform the calculation *ab initio*, keeping the pattern in sight by working throughout in terms of the components of the metrical tensor in the form g_{ab} rather than in terms of symbols like $A, B, C, D, \dots$.

The task began promisingly, but soon became so complicated that all elegance was lost and the chance of making a computational error exceeded prudent levels. After several attempts to force the calculation through, each yielding a different result, I discovered a simple manoeuvre that reduced the problem to one of surprising computational simplicity. Since the details of the method and the compact form of the results may be of interest, I present them here.

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2. Notation. Latin suffixes a, b, c, d, e take on the values $1, 2, 3, \dots, n$. The ranges of certain Greek suffixes will be explained later.

The summation convention proves to be a hindrance so it is not employed in this paper.

We assume that the Riemannian space-time is such that it admits orthogonal coordinate systems. For these systems the metrical tensor is such that

$$(2) \quad g_{ab} = 0 \quad \text{when} \quad a \neq b.$$

We may therefore conveniently write g_a for g_{aa} and g^a for g^{aa} . Thus we take the line element in the form

$$(3) \quad ds^2 = g_1 (dx^1)^2 + g_2 (dx^2)^2 + g_3 (dx^3)^2 + \dots + g_n (dx^n)^2.$$

Note that all signs are written plus in (3), though not in (1). This does not mean, however, that (3) is necessarily positive definite; its signature is left arbitrary.

We denote partial differentiation by a solidus, so that, for example, $f_{/a}$ stands for $\partial f / \partial x^a$.

We write the curvature tensor as

$$(4) \quad R^a_{bcd} = \left\{ \begin{matrix} a \\ bc \end{matrix} \right\}_{/d} - \left\{ \begin{matrix} a \\ bd \end{matrix} \right\}_{/c} + \sum_e \left[\left\{ \begin{matrix} e \\ bc \end{matrix} \right\} \left\{ \begin{matrix} a \\ ed \end{matrix} \right\} - \left\{ \begin{matrix} e \\ bd \end{matrix} \right\} \left\{ \begin{matrix} a \\ ec \end{matrix} \right\} \right],$$

so that the Ricci tensor with the usual sign is given by

$$(5) \quad R_{bd} = \sum_a R^a_{bad} = \sum_a \left[\left\{ \begin{matrix} a \\ ba \end{matrix} \right\}_{/d} - \left\{ \begin{matrix} a \\ bd \end{matrix} \right\}_{/a} \right] + \sum_{a,e} \left[\left\{ \begin{matrix} e \\ ba \end{matrix} \right\} \left\{ \begin{matrix} a \\ ed \end{matrix} \right\} - \left\{ \begin{matrix} e \\ bd \end{matrix} \right\} \left\{ \begin{matrix} a \\ ea \end{matrix} \right\} \right].$$

The curvature scalar, in view of (2), is given by

$$(6) \quad R = \sum_b g^b R_{bb}.$$

For present purposes the mixed form of the Einstein tensor turns out to be the most convenient. It is given by

$$(7) \quad G^b_a = R^b_a - \frac{1}{2} \delta^b_a R.$$

3. The Calculation of the Einstein Tensor. Because of (2),

$$(8) \quad g^a = 1/g_a,$$

and thus

$$(9) \quad g^a_{/b} = -g^a g^a g_{a/b}.$$

One easily finds that

$$(10) \quad \left\{ \begin{matrix} a \\ bc \end{matrix} \right\} = 0 \quad \text{if} \quad a \neq b \neq c,$$

$$(11) \quad \left\{ \begin{matrix} a \\ bb \end{matrix} \right\} = -\frac{1}{2} g^a g_{b/a} \quad \text{if} \quad a \neq b,$$

$$(12) \quad \left\{ \begin{matrix} a \\ ba \end{matrix} \right\} = \left\{ \begin{matrix} a \\ ab \end{matrix} \right\} = \frac{1}{2} g^a g_{a/b} ,$$

a special case of (12) being

$$(13) \quad \left\{ \begin{matrix} a \\ aa \end{matrix} \right\} = \frac{1}{2} g^a g_{a/a} .$$

It is convenient to divide the work into two parts, calculating G_d^b first for the case $b \neq d$, and second for the case $b = d$.

Case I. $b \neq d$. Here we consider b, d as having fixed values, and introduce the Greek index σ to take on the $n - 2$ values that differ from the particular values of b and d .

Since $b \neq d$, (7) reduces to

$$(14) \quad G_d^b = R_d^b$$

and by (5) we have

$$(15) \quad G_d^b = \sum_a g^a g^b R_{abad} .$$

It is well known that

$$(16) \quad R_{abed} = -R_{baed} = -R_{abde} .$$

Therefore in (15) we may ignore the terms with $a = b$ or $a = d$ and write

$$(17) \quad G_d^b = g^b \sum_{\sigma} g^{\sigma} R_{\sigma b \sigma d} = g^b \sum_{\sigma} \left[\left\{ \begin{matrix} \sigma \\ b\sigma \end{matrix} \right\}_{/d} - \left\{ \begin{matrix} \sigma \\ bd \end{matrix} \right\}_{/\sigma} \right] \\ + g^b \sum_{\sigma, e} \left[\left\{ \begin{matrix} e \\ b\sigma \end{matrix} \right\} \left\{ \begin{matrix} \sigma \\ ed \end{matrix} \right\} - \left\{ \begin{matrix} e \\ bd \end{matrix} \right\} \left\{ \begin{matrix} \sigma \\ e\sigma \end{matrix} \right\} \right] .$$

By (12) and (9),

$$(18) \quad \sum_{\sigma} \left\{ \begin{matrix} \sigma \\ b\sigma \end{matrix} \right\}_{/d} = \frac{1}{2} \sum_{\sigma} (g^{\sigma} g_{\sigma/bd} - g^{\sigma} g^{\sigma} g_{\sigma/b} g_{\sigma/d}) .$$

By (10), since $\sigma \neq b \neq d$,

$$(19) \quad \left\{ \begin{matrix} \sigma \\ bd \end{matrix} \right\} = 0 .$$

By (10), and then by (12),

$$(20) \quad \sum_{\sigma, e} \left\{ \begin{matrix} e \\ b\sigma \end{matrix} \right\} \left\{ \begin{matrix} \sigma \\ ed \end{matrix} \right\} = \sum_{\sigma} \left\{ \begin{matrix} \sigma \\ b\sigma \end{matrix} \right\} \left\{ \begin{matrix} \sigma \\ \sigma d \end{matrix} \right\} = \frac{1}{4} \sum_{\sigma} g^{\sigma} g^{\sigma} g_{\sigma/b} g_{\sigma/d} ,$$

and also

$$(21) \quad - \sum_{\sigma, e} \left\{ \begin{matrix} e \\ bd \end{matrix} \right\} \left\{ \begin{matrix} \sigma \\ e\sigma \end{matrix} \right\} = - \sum_{\sigma} \left[\left\{ \begin{matrix} b \\ bd \end{matrix} \right\} \left\{ \begin{matrix} \sigma \\ b\sigma \end{matrix} \right\} + \left\{ \begin{matrix} d \\ bd \end{matrix} \right\} \left\{ \begin{matrix} \sigma \\ d\sigma \end{matrix} \right\} \right] \\ = - \frac{1}{4} g^b g_{b/d} \sum_{\sigma} g^{\sigma} g_{\sigma/b} - \frac{1}{4} g^d g_{d/b} \sum_{\sigma} g^{\sigma} g_{\sigma/d} .$$

So from (17) to (21) we have

$$(22) \quad G_d^b = g^b \sum_{\sigma} [\tfrac{1}{2} g^{\sigma} g_{\sigma/bd} - \tfrac{1}{4} g^{\sigma} g^{\sigma} g_{\sigma/b} g_{\sigma/d} - \tfrac{1}{4} g^b g_{b/d} g^{\sigma} g_{\sigma/b} - \tfrac{1}{4} g^d g_{d/b} g^{\sigma} g_{\sigma/d}],$$

with $\sigma \neq b, d$.

Case II. $b = d$. Here we replace d by b and consider b as having a fixed value. We introduce the Greek letters λ, μ , each to take on the $n - 1$ values that differ from the particular value of b .

From (2), (5), (6), (7), (14), and (17) we have

$$(23) \quad \begin{aligned} G_b^b &= R_b^b - \tfrac{1}{2} R = R_b^b - \tfrac{1}{2} \sum_c R_c^c = \tfrac{1}{2} R_b^b - \tfrac{1}{2} \sum_{\lambda} R_{\lambda}^{\lambda} \\ &= \tfrac{1}{2} R_b^b - \tfrac{1}{2} \sum_{\substack{\lambda, a \\ (\lambda \neq a)}} g^a g^{\lambda} R_{a\lambda a\lambda} = \tfrac{1}{2} R_b^b - \tfrac{1}{2} \sum_{\lambda} g^b g^{\lambda} R_{b\lambda b\lambda} - \tfrac{1}{2} \sum_{\substack{\lambda, \mu \\ (\lambda \neq \mu)}} g^{\mu} g^{\lambda} R_{\mu\lambda\mu\lambda} \\ &= \tfrac{1}{2} R_b^b - \tfrac{1}{2} \sum_{\lambda} g^b g^{\lambda} R_{\lambda b\lambda b} - \tfrac{1}{2} \sum_{\substack{\lambda, \mu \\ (\lambda \neq \mu)}} g^{\mu} g^{\lambda} R_{\mu\lambda\mu\lambda} \\ &= \tfrac{1}{2} R_b^b - \tfrac{1}{2} R_b^b - \tfrac{1}{2} \sum_{\substack{\lambda, \mu \\ (\lambda \neq \mu)}} g^{\mu} g^{\lambda} R_{\mu\lambda\mu\lambda} = -\tfrac{1}{2} \sum_{\substack{\lambda, \mu \\ (\lambda \neq \mu)}} g^{\mu} g^{\lambda} R_{\lambda\mu\lambda\mu}. \end{aligned}$$

With G_b^b in the form (23) the computation becomes extremely simple. Using (4), we may write (23) in the form

$$(24) \quad G_b^b = -\tfrac{1}{2} \sum_{\substack{\lambda, \mu \\ (\lambda \neq \mu)}} \sum_e g^{\mu} \left[\left\{ \begin{matrix} \lambda \\ \mu\lambda \end{matrix} \right\}_{/\mu} - \left\{ \begin{matrix} \lambda \\ \mu\mu \end{matrix} \right\}_{/\lambda} + \left\{ \begin{matrix} e \\ \mu\lambda \end{matrix} \right\}_{/e\mu} - \left\{ \begin{matrix} e \\ \mu\mu \end{matrix} \right\}_{/e\lambda} \right].$$

Consider the third term on the right. Since $\lambda \neq \mu$, it will vanish, by (10), unless $e = \lambda$ or $e = \mu$. So, using (11) and (12), we see that

$$(25) \quad \begin{aligned} \sum_{\substack{\lambda, \mu \\ (\lambda \neq \mu)}} \sum_e g^{\mu} \left\{ \begin{matrix} e \\ \mu\lambda \end{matrix} \right\}_{/e\mu} &= \sum_{\substack{\lambda, \mu \\ (\lambda \neq \mu)}} g^{\mu} \left[\left\{ \begin{matrix} \lambda \\ \mu\lambda \end{matrix} \right\}_{/\lambda\mu} + \left\{ \begin{matrix} \mu \\ \mu\lambda \end{matrix} \right\}_{/\mu\mu} \right] \\ &= \tfrac{1}{2} \sum_{\substack{\lambda, \mu \\ (\lambda \neq \mu)}} [g^{\mu} g^{\lambda} g^{\lambda} g_{\mu/\mu} g_{\lambda/\mu} - g^{\lambda} g^{\mu} g^{\mu} g_{\mu/\lambda} g_{\mu/\mu}] = 0. \end{aligned}$$

The last term on the right side of (24), unlike the third, does not vanish when $e \neq \lambda, \mu$. However, for that part of the summation for which $e = \lambda$ and $e = \mu$ the sum is, by (11) and (12),

$$(26) \quad \begin{aligned} -\tfrac{1}{2} \sum_{\substack{\lambda, \mu \\ (\lambda \neq \mu)}} g^{\mu} \left[\left\{ \begin{matrix} \lambda \\ \mu\mu \end{matrix} \right\}_{/\lambda\lambda} + \left\{ \begin{matrix} \mu \\ \mu\mu \end{matrix} \right\}_{/\mu\lambda} \right] \\ = -\tfrac{1}{8} \sum_{\substack{\lambda, \mu \\ (\lambda \neq \mu)}} (-g^{\mu} g^{\lambda} g^{\lambda} g_{\mu/\lambda} g_{\lambda/\lambda} + g^{\lambda} g^{\mu} g^{\mu} g_{\mu/\mu} g_{\mu/\lambda}) = 0. \end{aligned}$$

Thus we may sum the last term on the right of (24) for $\lambda \neq \mu \neq e$ only. Using

this fact, together with (9), (11), (12), and (25), we may now at once obtain from (24) the required result

$$(27) \quad G_b^b = -\frac{1}{2} \sum_{\substack{\lambda, \mu \\ (\lambda \neq \mu)}} g^\lambda g^\mu g_{\lambda/\mu\mu} + \frac{1}{4} \sum_{\substack{\lambda, \mu \\ (\lambda \neq \mu)}} g^\lambda g^\lambda g^\mu (g_{\lambda/\mu} g_{\lambda/\mu} + g_{\lambda/\mu} g_{\mu/\mu}) \\ - \frac{1}{8} \sum_{\substack{\lambda, \mu, e \\ (\lambda \neq \mu \neq e)}} g^e g^\lambda g^\mu g_{\lambda/e} g_{\mu/e} ,$$

with $\lambda, \mu \neq b$.

The formulas (22), (27) give the explicit compact forms of the components of the Einstein tensor for the n -dimensional orthogonal line element (3). For the space-time case they agree with the individual formulas for these components given by DINGLE for the orthogonal line element (1) when the appropriate changes of notation are made. In using (27) one should note that the $-\frac{1}{8}$ before the $\sum$ yields coefficients $-\frac{1}{4}$ since each type of term in the summation occurs twice.

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Queens College
Flushing, New York