*Solution of the Dirichlet Problem for the Ellipse by Interpolating Harmonic Polynomials**

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Since harmonic polynomials are known to possess important properties with reference to approximation to harmonic functions [WALSH, 3, 4], the possibility suggests itself [WALSH, 4, p. 517] that a sequence of such polynomials might be defined by coincidence with the values of a given function $U(x, y)$ in suitably chosen points on the boundary of a given region D , and then might converge in D to the corresponding solution of the Dirichlet problem. The present writer has indicated [3, 4] that if D is, in particular, a circular disc, the Jackson-Bernstein-Montel-de la Vallée Poussin theory of trigonometric interpolation and approximation applies, and various results on such a solution of the Dirichlet problem are immediate.

Recently J. H. CURTISS [1] has raised anew the question of interpolation by harmonic polynomials, and he points out the analogy to interpolation by polynomials in the complex variable. He conjectures that satisfactory points of interpolation may be (for instance) equally spaced on the boundary of D with respect to the conjugate of Green's function for the complement of D . The object of the present note is to confirm this conjecture for the case that D is the interior of an ellipse, by computation as trigonometric polynomials of the harmonic polynomials involved.

Theorem. *Let the boundary B of a bounded region D in the (u, v) -plane be an ellipse, and let the function $U(u, v)$ be defined and satisfy a Lipschitz condition of some positive order on B with respect to arc length. Let the harmonic polynomial $p_n(u, v)$ of degree not greater than n coincide with $U(u, v)$ in $2n + 1$ points equally spaced on B with respect to the conjugate of Green's function for the exterior of $D + B$ with pole at infinity. Then the sequence $p_n(u, v)$ converges to $U(u, v)$ uniformly on B , hence converges uniformly in $D + B$ to the corresponding solution of the Dirichlet problem.*

In an arbitrary simply connected region of the $w (= u + iv)$ -plane a complete

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set of linearly independent harmonic polynomials of degree not greater than n is

$$(1) \quad 1, \quad \operatorname{Re} w, \quad \operatorname{Im} w, \quad \operatorname{Re} w^2, \quad \operatorname{Im} w^2, \quad \dots, \quad \operatorname{Re} w^n, \quad \operatorname{Im} w^n.$$

The determination and uniqueness of a harmonic polynomial $p_n(u, v)$ of degree not greater than n by the property of coinciding with $U(u, v)$ in $2n + 1$ given distinct points of B depend on the solution of a system of $2n + 1$ linear equations in $2n + 1$ unknowns (coefficients), hence depend on the non-vanishing of a certain determinant, whose vanishing is a necessary and sufficient condition that a harmonic polynomial of degree not greater than n and not vanishing identically vanishes in those $2n + 1$ points. The latter condition cannot be satisfied, for the ellipse cannot (principle of the maximum) be represented as a level locus of a harmonic polynomial, and two irreducible algebraic curves of respective degrees 2 and m cannot intersect in more than $2m$ points; thus $p_n(u, v)$ exists and is unique.

The condition of the theorem that the $2n + 1$ points shall be equally spaced on B with respect to a stated parameter determines the spacing uniquely but not the actual points; only the spacing is of significance, even though $p_n(u, v)$ is uniquely determined only when the $2n + 1$ points are chosen.

In the theorem we choose B as the ellipse

$$(2) \quad \frac{4u^2}{(R + 1/R)^2} + \frac{4v^2}{(R - 1/R)^2} = 1, \quad R > 1,$$

which involves no loss of generality. Thus B can be represented parametrically by

$$(3) \quad u = \frac{1}{2}(R + 1/R) \cos \theta, \quad v = \frac{1}{2}(R - 1/R) \sin \theta.$$

Under the transformation

$$(4) \quad w = \frac{1}{2}(z + 1/z), \quad z = r(\cos \theta + i \sin \theta),$$

we may write

$$(5) \quad u = \frac{1}{2}(r + 1/r) \cos \theta, \quad v = \frac{1}{2}(r - 1/r) \sin \theta,$$

so that B corresponds to the two circles $r = R$ and $r = 1/R$ (the counterclockwise sense on B corresponds to the counterclockwise sense on $r = R$ and clockwise sense on $r = 1/R$), whereas D corresponds to the annulus $1/R < |z| < R$. Green's function for the exterior of D with pole at infinity is $\log(|z|/R)$ in $|z| > R$ and $\log(1/R|z|)$ in $|z| < 1/R$, in each case with conjugate θ .

Thanks to the special properties of (4), the function $z^n + 1/z^n$ is a polynomial in w of degree n , as is readily shown by induction, and so the harmonic polynomials (1) in u and v are equivalent to the system composed of $\operatorname{Re}(z^n + 1/z^n)$ and $\operatorname{Im}(z^n + 1/z^n)$, namely

$$(6) \quad 1, \quad (r + 1/r) \cos \theta, \quad (r - 1/r) \sin \theta, \quad (r^2 + 1/r^2) \cos 2\theta, \\ (r^2 - 1/r^2) \sin 2\theta, \dots, \quad (r^n + 1/r^n) \cos n\theta, \quad (r^n - 1/r^n) \sin n\theta,$$

which are also linearly independent in any region.

When interpreted in the z -plane on the circle $r = R$, the harmonic polynomial of interpolation of degree not greater than n in u and v has the form

$$(7) \quad p_n(u, v) \equiv A_{n0} + \sum_{k=1}^n [A_{nk}(R^k + 1/R^k) \cos k\theta + B_{nk}(R^k - 1/R^k) \sin k\theta],$$

with suitably determined A_{nk} and B_{nk} , and when interpreted on the circle $r = 1/R$ has the form

$$(8) \quad p_n(u, v) \equiv A_{n0} + \sum_{k=1}^n [A_{nk}(R^k + 1/R^k) \cos k\theta - B_{nk}(R^k - 1/R^k) \sin k\theta];$$

in D (when interpreted in the annulus $1/R < |z| < R$) this harmonic polynomial has the form

$$(9) \quad p_n(u, v) \equiv A_{n0} + \sum_{k=1}^n [A_{nk}(r^k + 1/r^k) \cos k\theta + B_{nk}(r^k - 1/r^k) \sin k\theta].$$

According to the theorem, the $p_n(u, v)$ are defined by interpolation on $r = R$ in $2n + 1$ values of θ which are equally spaced, and so [JACKSON 2, p. 123, Corollary II] the sequence $p_n(u, v)$ converges uniformly on $r = R$ and to $U(u, v)$ on B , and the conclusion of the theorem follows. We have used here the fact that a Lipschitz condition of positive order on B implies such a condition on $|z| = R$. If $U(u, v)$ is known to have further continuity properties on B , such as the existence of a p^{th} derivative which satisfies a Lipschitz condition, JACKSON's corollary yields further precise results on the degree of convergence of the sequence $p_n(u, v)$ on B and in $D + B$. Indeed, the entire theory of trigonometric approximation can be interpreted so as to apply to approximation on B by harmonic polynomials.

Moreover, JACKSON has studied [2, p. 145] a special interpolation formula for which the interpolating trigonometric sums converge uniformly to the given function whenever the latter is continuous. If such sums are used instead of the sequence $p_n(u, v)$, the sequence of interpolating harmonic polynomials converges uniformly in $D + B$ to the solution of the Dirichlet problem.

Under the transformation $w = \frac{1}{2}(z + 1/z)$, the ellipse B corresponds to the two circles $|z| = R$ and $|z| = 1/R$, and the point (u, v) on B , as in (3), corresponds to $(r = R, \theta)$ and $(r = 1/R, -\theta)$. When interpreted in the z -plane, the sequence $p_n(u, v)$ converges uniformly both on $|z| = R$ and on $|z| = 1/R$, hence also in the annulus bounded by those circles, and is in a form suitable for numerical computation.

With the hypothesis of the theorem concerning D and B , but with $U(u, v)$ merely continuous on B , there may be readily determined, by methods other than interpolation, harmonic polynomials which converge in D to the solution of the Dirichlet problem. For instance, one may expand $U(u, v)$ in harmonic polynomials orthogonal on B (it is clear that the harmonic polynomials (6) are simultaneously mutually orthogonal on every circle $|z| = r$ with weight function unity, hence with suitable weight function on every ellipse of the family

(5), each ellipse with fixed r and variable θ [WALSH & MERRIMAN, 5]); convergence throughout D uniformly on any closed set in D follows by Green's formula involving Green's function for D ; alternatively, one may use harmonic polynomials of best approximation to $U(u, v)$ on B in the sense of least p^{th} powers ($p \geq 1$) or in the sense of TCHEBYCHEFF.

Indeed, GHIZZETTI [6] has recently published a solution of the Dirichlet problem for the ellipse closely related to the method just suggested. He maps the interior of the ellipse, slit along the line segment joining the foci, onto a circular annulus in the (x, y) -plane, whereas we have mapped the two-sheeted ellipse having as branch line the segment joining the foci onto an annulus. He then uses Fourier series in the (x, y) -plane to expand the solution of the Dirichlet problem, whereas we have used interpolating trigonometric polynomials, with emphasis on the fact that the interpolating functions in the (x, y) -plane become interpolating harmonic polynomials in u and v when transformed to the (u, v) -plane.

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