Interpolation with Harmonic and Complex Polynomials to Boundary Values

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1. Introduction. Some thirty years ago, J. L. WALSH [16], p. 517, asked the following question: Let R be a region of the complex z -plane. Let $u(z)$ be given on the boundary of R . Let $\{S_N : z_{1N}, z_{2N}, \dots, z_{NN}\}$ be a sequence of point sets chosen on the boundary of R . When is it possible uniquely to determine the coefficients of a harmonic polynomial* solely by coincidence with $u(z)$ at the points S_N , and what is the asymptotic character of such a polynomial as N becomes infinite?

The hope, of course, would be that the sequence of harmonic interpolation polynomials would converge on R to the solution of the Dirichlet problem** corresponding to the boundary values $u(z)$. This type of constructive solution to a two-dimensional Dirichlet problem would be attractive because it would provide an analytical approximation of a peculiarly simple and manageable form, and because the method of construction would seem to be remarkably straightforward and natural.

At present, perhaps the most widely used practical method for obtaining solutions to Dirichlet problems consists in replacing them by the analogous difference equation problems and solving the resulting linear algebraic equations. In any particular case, this procedure exhibits the solution only at a discrete set of lattice points, and in the form of a numerical table. But quite generally in applied mathematics analytical approximations are more convenient as solutions than numerical tables, and there is a special reason for this to be true in the case of the Dirichlet problem. It is that the conjugate harmonic function†

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*A harmonic polynomial is an expression of the type $\operatorname{Re} p(z)$, where Re means "real part of" and $p(z)$ is a polynomial in z .

**A function $U(x, y)$ is harmonic on a region R of the (x, y) -plane if it is continuous there, together with its first and second derivatives, and if it satisfies the equation $\nabla^2 U = 0$ throughout R , where ∇^2 is the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$. The Dirichlet problem consists in determining (or, in purely theoretical discussions, in proving the existence of) a function $U(x, y)$, ordinarily unique, which is harmonic on R and which assumes given boundary values on the boundary of R .

†Two functions U and V are harmonic conjugates if they satisfy the Cauchy-Riemann equations: $\partial V/\partial x = \partial U/\partial y$, $\partial U/\partial x = -\partial V/\partial y$. The harmonic conjugate of a harmonic function is unique only to within an additive constant.

is often required at the same time, and an approximation to the conjugate lies immediately at hand when an analytic approximation of elementary form is available for the original harmonic function. In fact, let $G(x, y)$ be harmonic on R and let it approximate some harmonic function $U(x, y)$ uniformly on any closed subregion of R . It is easy to show from the theory of harmonic functions that the harmonic conjugate of G will give an approximation to the harmonic conjugate of U which is in a certain sense just as good as the original approximation, at least on closed subregions of R . Now the harmonic conjugate of G theoretically can always be found by integrating the Cauchy-Riemann equations, but an even simpler way to find it when G is an elementary function is as follows: Substitute $x = (z + a)/2$, $y = (z - a)/2i$, $z = x + iy$, for x and y in G , and thereafter take the imaginary part of $2G$. This will be the harmonic conjugate of G . The variable z and the parameter $a = a_1 + ia_2$ must lie on R . (For a rigorous formulation and proof, see [3].) And, of course, if G is merely a polynomial, the harmonic conjugate of G can be written down without any computation whatsoever.

In this paper the discussion will be limited to the case in which the region R is bounded by a Jordan curve C . (A Jordan curve is a one-to-one continuous transform of a circle.) It is known ([18], p. 169) that if $U(x, y)$ is harmonic on such a region R and continuous on $R + C$, then there exists a harmonic polynomial which approximates U as closely as desired on $R + C$. But, given only the values of U on C , how should the approximating polynomial actually be constructed? There are a number of methods which are known to yield successful approximations, of which perhaps only one deserves particular mention here. It consists in the use of orthonormal harmonic polynomials belonging to $R + C$. (See BERGMAN [1] and MERRIMAN [10]. Interesting numerical experiments relating to MERRIMAN's work* have recently been described by DAVIS & RABINOWITZ [6].) In this type of approach, the approximating harmonic polynomials are not required to match the boundary values at given points but rather to approximate them in the sense of least squares. But as DAVIS has pointed out in correspondence with the author, when the orthonormal theory is translated into numerical practice, the inner products involved in the theory are perhaps most conveniently determined by quadratures. If this is done, then, in effect, the fitting of the approximating polynomials to the boundary value is again accomplished through using strictly discrete data, just as in the interpolative approach, and the question might then arise as to whether it would not be better to use simple interpolation right from the outset.

The relative attractiveness of the interpolation approach has not escaped the attention of those who require practical solutions of Dirichlet problems. In

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*It is worth noting in passing that the problem of finding asymptotic formulas for orthonormal harmonic polynomials is apparently still open. The problem was suggested to MERRIMAN by WALSH and is discussed at the end of [10].

fact, the immediate motivation for writing this paper lay in some numerical experiments concerning the torsion problem for a long prism of arbitrary but constant cross section which PORITSKY & DANFORTH described in a paper [14] presented to a session of the Third U. S. National Congress of Applied Mechanics at Brown University on June 12, 1958. Among several graphical, numerical, and analytical methods, PORITSKY & DANFORTH tried approximation by harmonic polynomials of degrees 3 and 7 found by interpolation at 7 and 15 points, respectively, to the boundary value function $\frac{1}{2}|z|^2$, which is appropriate to this problem. The points of interpolation were apparently chosen either intuitively or with regard to convenience for comparison with other numerical methods. The results were not satisfactory.

Unfortunately it would appear that WALSH's question concerning the asymptotic behavior of harmonic interpolation polynomials has never been answered in a satisfactory way for regions more general than the ellipse. In asking the question, WALSH indicated that if R is a circular disc and if the interpolation points S_N are equally spaced on the circumference of the disc, then it can be proved that the sequence of harmonic polynomials found respectively by interpolation at the points S_N , $N = 1, 2, \dots$, to continuous boundary values $u(z)$ will, indeed, tend to the solution of the Dirichlet problem on the interior of the disc. Very recently, WALSH [19] has extended this result to discs of elliptical shape.

Although little is known about the harmonic problem, there is an analogous problem dealing with interpolation to complex boundary values by polynomials in a complex variable which is now rather well understood. This problem is of considerable theoretical interest for itself alone. Almost surely the principal facts about this complex problem are true with only minor qualifications for the harmonic problem. In the present paper an essay toward unifying the theory of the two problems will be made by identifying a certain stability condition (the boundedness of the total variation of the kernel of the interpolation polynomial, considered as a linear operator) as the key necessary and sufficient condition for satisfactory asymptotic behavior. This result will not provide the ultimate answer to the convergence question for harmonic interpolation polynomials because there is no general assurance that sequences $\{S_N\}$ always exist for which the stability condition holds true. But the condition seems to shed new light on the complex interpolation problem, and may point the way toward a complete answer in the case of the harmonic problem. The stability condition is developed chiefly in Sections 4, 5, and 7 below.

Some new results on simultaneous interpolation to complex boundary values on several mutually exterior Jordan curves are given in Section 6. In Section 8 some new formulas relating to the solution of the Dirichlet problem for the unit disc by harmonic polynomial interpolation are given, but they are more or less direct realizations of the remarks made by WALSH [16] in connection with the question stated in the first paragraph above.

Aside from this new material, the present paper is primarily of expository

intent. It is essentially a status report on the subject described in the title, written partly with the idea of informing applied mathematicians as to what is known about the theory (since some of them have shown an interest in the method), and partly with the hope of stimulating further mathematical research on a perhaps somewhat neglected facet of constructive function theory. Incidentally, on the latter score it should be mentioned that even in the case of the complex problem there remain many unanswered theoretical questions, a number of which are identified in the text below.

2. The existence and uniqueness of complex and harmonic polynomials found by interpolation. Let distinct points $z_1, z_2, \dots, z_n$ of the complex z -plane and real or complex values $w_1, w_2, \dots, w_n$ be given. Consider first the problem of proving the existence and uniqueness of a polynomial $L_n(z) = c_0 + c_1z + c_2z^2 + \dots + c_{n-1}z^{n-1}$ of degree not more than $n - 1$ (cases in which some or all of the coefficients are zero are not excluded) which takes on the value w_k at the point z_k , $k = 1, \dots, n$. According to the standard theory of linear algebraic equations, the necessary and sufficient condition that the system of n simultaneous linear equations $L_n(z_k) = c_0 + c_1z_k + c_1z_k^2 + \dots + c_{n-1}z_k^{n-1} = w_k$, $k = 1, \dots, n$, in the n unknowns $c_0, c_1, \dots, c_{n-1}$, shall have a solution for every set w_k is that the matrix

$$(2.1) \quad Z = \begin{bmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{n-1} \\ 1 & z_2 & z_2^2 & \cdots & z_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & z_n^2 & \cdots & z_n^{n-1} \end{bmatrix}$$

shall be non-singular, which is equivalent to saying that its determinant is not zero. If the matrix is non-singular, the solution is unique. The fact that the matrix always is non-singular when the points z_k are distinct can be demonstrated in a variety of ways. For example, the determinant can be evaluated outright; it is a so-called Vandermonde determinant which has the value $\prod_{i>k} (z_i - z_k)$, and clearly this cannot be zero under the circumstances. Alternatively, a necessary and sufficient condition for the singularity of a matrix is that the columns shall be linearly dependent. The condition of linear dependence of the columns in this case is equivalent to the existence of complex numbers $A_0, A_1, \dots, A_{n-1}$, not all zero, such that $A_0 + A_1z + A_2z^2 + \dots + A_{n-1}z^{n-1} = 0$ for $z = z_1, z_2, \dots, z_n$. But a polynomial of degree $n - 1$ or less cannot vanish at n distinct points unless the coefficients are all zero. Therefore the columns of the matrix (2.1) cannot be linearly dependent.

Let $w_k = u_k + iv_k$ and $c_k = a_k + ib_k$, where u_k, v_k, a_k , and b_k are real. Let $z = re^{i\theta} = r \cos \theta + ir \sin \theta$ and $z_k = r_k \cos \theta_k + ir_k \sin \theta_k$. Then, stated entirely

in terms of real numbers and harmonic polynomials, the equations which determine the coefficients of L_n are the following $2n$ equations in $2n$ unknowns:

$$(2.2) \quad \operatorname{Re} L_n(z_k) = a_0 + \sum_{i=1}^{n-1} (a_i r_k^i \cos k\theta_i - b_i r_k^i \sin k\theta_i) = u_k, \\ k = 1, \dots, n,$$

$$(2.3) \quad \operatorname{Im} L_n(z_k) = b_0 + \sum_{i=1}^{n-1} (b_i r_k^i \cos k\theta_i + a_i r_k^i \sin k\theta_i) = v_k, \\ k = 1, \dots, n.$$

Here and in the sequel Re denotes "real part of" and Im "imaginary part of".

Now consider the problem of determining the $2n - 1$ coefficients of a harmonic polynomial $H_n(r, \theta)$ of degree at most $n - 1$ in x and y , $x = r \cos \theta$, $y = r \sin \theta$, by specifying that it shall take on values u_k , $k = 1, \dots, 2n - 1$, at the $2n - 1$ points $z_k = r_k \cos \theta_k + i r_k \sin \theta_k$, $k = 1, \dots, 2n - 1$, *without any restrictions on the conjugate harmonic polynomial*. This is the interpolation process for approximately solving the Dirichlet problem suggested by WALSH's question and described in the introduction. The equations for determining the coefficients look like (2.2), but with k running from 1 to $2n - 1$. Do they always have a unique solution if the points z_k are distinct?

By no means; in fact, they may have no solution at all. To take a simple example, if all the points z_k lie on the x -axis, then all of the terms involving b_i drop out in (2.2), and one is left with the task of trying to make the n unknown coefficients a_i satisfy $2n - 1$ linear equations, which in general is impossible.

The necessary and sufficient condition for the existence and uniqueness of the desired interpolating polynomial is now the non-singularity of the matrix

$$(2.4) \quad \begin{bmatrix} 1 & r_1 \cos \theta_1 & \cdots & r_1^{n-1} \cos(n-1)\theta_1 & r_1 \sin \theta_1 & \cdots & r_1^{n-1} \sin(n-1)\theta_1 \\ 1 & r_2 \cos \theta_2 & \cdots & r_2^{n-1} \cos(n-1)\theta_2 & r_2 \sin \theta_2 & \cdots & r_2^{n-1} \sin(n-1)\theta_2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & r_{2n-1} \cos \theta_{2n-1} & \cdots & r_{2n-1} \cos(n-1)\theta_{2n-1} & r_{2n-1} \sin \theta_{2n-1} & \cdots & r_{2n-1}^{n-1} \sin(n-1)\theta_{2n-1} \end{bmatrix}$$

A necessary and sufficient condition for the *singularity* of this matrix is that the columns shall be linearly dependent, which is equivalent to the existence of a polynomial $p(z)$ in the complex variable z of degree at most $n - 1$ with not all coefficients zero such that $\operatorname{Re} p(z_k) = 0$, $k = 1, \dots, 2n - 1$. In other words, *the necessary and sufficient conditions for the existence of $H_n(r, \theta)$ for all data u_k is that the points z_k shall not all lie on any one locus represented by an equation of the type $\operatorname{Re} p(z) = 0$ where $p(z)$ is a polynomial in z of degree at most $n - 1$* . If this condition is fulfilled, then H_n is unique.

How severe is this restriction? The situation of interest in the present context

is one in which the points z_k all lie on a Jordan curve C . It is known from the theory of harmonic functions that if $\operatorname{Re} p(z) = 0$ everywhere on a Jordan curve, then $p(z) = 0$ everywhere in the interior, and the coefficients of p would then all have to be zero. Therefore it is impossible for all the points of a Jordan curve to lie on a locus given by an equation $\operatorname{Re} p(z) = 0$ with not all of the coefficients of p zero. In fact, there must be non-denumerably many points of C which do not lie on any one such locus. In the special case in which C is the unit circle, it will be shown indirectly in Section 8 that the $2n - 1$ roots of the equation $z^{2n-1} = 1$ —the $(2n - 1)^{\text{th}}$ roots of unity—cannot all lie on a locus given by $\operatorname{Re} p_n(z) = 0$, so the matrix (2.4) is non-singular at least for this choice of the points z_k . WALSH [19] has shown that an analogous set of $2n - 1$ points exists on an ellipse for which the matrix (2.4) is non-singular. But in the general case there is no reason why C cannot coincide with one of the forbidden loci at an uncountably infinite number of points. Indeed, this would be the case if C contained a straight line segment.

From the standpoint of practical numerical work, if the interpolation points are scattered fairly uniformly over a given closed curve C , it is improbable that the matrix (2.4) will turn out to be singular. It might, however, be “poorly conditioned” to such an extent as to give trouble. In this case one could readjust the points and try again.

The above discussion has dealt only with interpolation by harmonic polynomials in an odd number of points. If the last sine term in (2.2) is dropped, a polynomial of degree at most $n - 1$ results which can under suitable restrictions be determined uniquely by interpolation in $2n - 2$ points. The last row and column of the matrix (2.4) are now to be dropped. The order of the matrix is $(2n - 2) \times (2n - 2)$. The necessary and sufficient condition for existence and uniqueness for arbitrary specification of the data now is that the $2n$ points shall not all lie on a locus $\operatorname{Re} p(z) = 0$, where p is now a polynomial of degree at most $n - 1$ in which the coefficient of z^{n-1} is real.

Since L_n always exists for any boundary data and H_n may not always do so, it is tempting to consider the possibility of using L_n rather than H_n to furnish an interpolative approximate solution to the Dirichlet problem for given boundary data $u(z)$. The implication here is that in each case all of the coefficients of the polynomial will be determined solely by interpolation. A full discussion of this suggestion would to some extent have to anticipate later developments in this paper, but the situation is briefly this: Given real boundary values $u(z)$ on a Jordan curve C bounding a region R , the only way to set up L_n so that one could hope that $\operatorname{Re} L_n$ converges on R to the solution of the Dirichlet problem for u and R would be to specify not only that $\operatorname{Re} L_n$ shall coincide with u on C but also that $\operatorname{Im} L_n$ shall coincide either with the function v which is “conjugate” to u on C , or with some sort of average of the values of this function. (The conjugate function of u on C can for present purposes be loosely described as the function with values which are the boundary values of the harmonic conjugate of the solution of the Dirichlet problem for u and C .) Unfortunately,

even in the case in which R is a circular disc and u is continuous on the circumference, the conjugate of u is in general extremely difficult to compute and may be very irregular.* The difficulty is much the same as the one that arises when an attempt is made to use the Cauchy integral formula to solve the Dirichlet problem; the project is usually hopeless unless the conjugate boundary values are somehow known.

3. The coverage theory for complex polynomial interpolation to boundary values: preliminaries. As before, let C be a Jordan curve of the complex z -plane. Let $S_n : z_{1n}, z_{2n}, \dots, z_{nn}$ be a set of n distinct points on C . Let the value of the function $f(z)$ be finite at each of the points of S_n . The unique polynomial $L_n(f; z)$ of degree at most $n - 1$ which coincides with $f(z)$ at the points of S_n may be written in the following form:

$$(3.1) \quad L_n(f; z) = \sum_{k=1}^n f(z_{kn}) \lambda_k(z),$$

where

$$(3.1a) \quad \lambda_k(z) = \frac{\omega_n(z)}{\omega'_n(z_{kn})(z - z_{kn})},$$

$$(3.1b) \quad \omega_n(z) = (z - z_{1n})(z - z_{2n}) \cdots (z - z_{nn}).$$

The polynomial L_n is often called the Lagrange interpolation polynomial. The quickest way to establish (3.1) is merely by *post hoc* verification: Clearly $\lambda_k(z)$ (and therefore L_n) is a polynomial in z of degree at most $n - 1$; and $\lambda_k(z_j) = 0$ if $j \neq k$, and $\lambda_k(z_k) = 1$, so $L_n(f; z_j) = f(z_{jn})$.

From (3.1) it is evident that L_n can be looked upon as a linear operator of a rather simple type. This was also apparent from the existence argument in Section 2 which stated in essence that $L_n = \sum_{k=1}^n f(z_{kn})(Z'^{-1}\hat{\xi})_k$ where Z'^{-1} is the transposed inverse of the matrix Z in (2.1), $\hat{\xi}$ is the vector with components $1, z, z^2, \dots, z^{n-1}$, and $(\hat{\alpha})_k$ denotes the k^{th} component of the vector $\hat{\alpha}$. Clearly $\lambda_k(z) \equiv (Z'^{-1}\hat{\xi})_k$.

In the special case in which $f(z) = 1/(z - t)^m$, where m is a positive integer and t is an arbitrarily chosen parameter (but not equal to any of the numbers of S_n), L_n can be written in the convenient forms

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*When C is the unit circle and if u is continuous, then all that seems to be known about the regularity of the conjugate v is that $|v|^p$ is integrable in the sense of Lebesgue for every p , and

$$\int_0^{2\pi} \exp(\lambda |v(e^{i\theta})|) d\theta < \infty \quad \text{for every real } \lambda.$$

(see ZYGMUND [20], p. 164.) Professor WALTER RUDIN pointed out to the author that examples can be constructed in which $u(re^{i\theta})$ is harmonic for $r < 1$, continuous for $r \leq 1$, and the harmonic conjugate $v(re^{i\theta})$ is unbounded in every sector of the unit disc.

$$(3.2a) \quad L_n\left(\frac{1}{z-t}; z\right) = \frac{1}{z-t} \left(1 - \frac{\omega_n(z)}{\omega_n(t)}\right),$$

$$(3.2b) \quad L_n\left(\frac{1}{(z-t)^m}; z\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial t^{m-1}} L_n\left(\frac{1}{z-t}; z\right).$$

If $f(z)$ is a polynomial of degree at most $n-1$, then $L_n(f; z) \equiv f(z)$.

The central questions in the convergence theory for L_n can be formulated as follows: Consider a sequence of point sets S_n , $n = 1, 2, \dots$, and the corresponding sequence $\{L_n(f; z)\}$. If the values on C of the function f are assumed to be reasonably "smooth"—say continuous on C —what are necessary and sufficient conditions on the sequence of sets $\{S_n\}$ that $\{L_n(f; z)\}$ shall converge for z inside C for *all* such functions f ? Does a sequence $\{S_n\}$ surely exist which fulfills the conditions? If so, what is its structure?

These questions will be answered first for the class of functions f each of which is defined not only on C but throughout the region R inside C and is analytic and single-valued on $R + C$, and second for the class of functions defined and continuous merely on C . The answer to the first question has been known for a quarter of a century or more. The answer given here to the second question is partly new.

First of all, one might well ask this: When $\{L_n\}$ does converge on R , what limit function does it normally select? Formula (3.1), taken with due regard to the factor $1/(z - z_{kn})$ appearing in (3.1a), looks as if it might be some sort of quadrature formula for the integral

$$(3.3) \quad F(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{t-z} dt.$$

And, indeed, it does turn out that this function $F(z)$ may be regarded as the natural limit of $\{L_n\}$.

Doubtless the reader is aware that unless the boundary values $f(t)$ which appear in the above integral are coincident with the boundary values of a function which is analytic everywhere inside C , then the physical connection, so to speak, between $f(t)$ and $F(z)$ may be rather remote. (For a rigorous formulation and discussion of the relationship between f and F , see, for example, WALSH [18], pp. 39–42 and pp. 141 ff.) Thus if $f(t) = 1/t^m$, where m is a positive integer, and if the origin of the z -plane lies in R , then it turns out that $F(z) \equiv 0$ on R . In 1884 MERAY [11] noted that if C is the unit circle and S_n consists of the n^{th} roots of unity, then $\{L_n(1/z; z)\} = \{z^{n-1}\}$, which converges to zero for $|z| < 1$.

Turning to conditions on $\{S_n\}$ which will cause convergence, one might well wonder if it will not suffice merely to require that $\max_k |z_{k,n} - z_{k+1,n}| \rightarrow 0$, $k = 1, \dots, n-1$, when the points are placed on C in the order of their first indices. The answer is that this condition is not strong enough, even in the simplest cases. The fact is illustrated by the following example.

Let C be the unit circle and let $f(z) = 1/(a-z)$, where a is real and $a > 1$. Here f is analytic on the closed region $C + R$, and on R it can be represented

by the Cauchy integral formula operating on its values on C . Therefore one might expect to obtain convergence of $\{L_n\}$ to the function itself on R . Let the points of S_n be

$$(3.4a) \quad S_n : z_{kn} = \frac{1 - ae^{2\pi i k/n}}{a - e^{2\pi i k/n}}, \quad k = 1, \dots, n.$$

The “linear transformation”

$$(3.4b) \quad t = \frac{1 - az}{a - z}$$

gives a one-to-one conformal mapping of the unit disc in the z -plane onto the unit disc in the t -plane under which the circumference in one plane goes into the circumference in the other. Thus the points of S_n certainly lie on C , and it is not hard to see that they get closer and closer together as n increases. A brief calculation shows that

$$L_n\left(\frac{1}{a-z}; z\right) = \frac{1}{a-z} \left\{ 1 - \left(\frac{z-a}{a^2-1}\right)^n \left[\left(\frac{1-az}{a-z}\right)^n - 1 \right] \right\}.$$

The convergence behavior of $\{L_n\}$ for $|z| < 1$ can be read off directly from this formula. The modulus of the first term in the square brackets is less than one for all $|z| < 1$. Thus if z lies in the lune common to $|z| < 1$ and the circular region $|z-a| < a^2-1$, $\{L_n\}$ converges to $1/(a-z)$, but if z lies in $|z| < 1$ but outside the lune, $\{L_n\}$ diverges to infinity. On the boundary of the lune $\{L_n\}$ oscillates in a bounded manner. If $1 < a < 2$, there is a part of the interior of the unit circle for which divergence takes place.

To describe what turns out to be the successful method of spacing the points S_n on C , the following notation and definitions are useful. Let the analytic function $z = \phi(w)$ map the *exterior* K of C conformally and in a one-to-one manner onto the region $|w| > 1$ so that the points of infinity correspond. The mapping function is to be defined so as to be continuous and univalent for $1 \leq |w| < \infty$. Its Laurent series is

$$(3.5) \quad \phi(w) = cw + c_0 + \frac{c_1}{w} + \frac{c_2}{w^2} + \dots, \quad |w| \geq 1,$$

where $|c|$ is the “transfinite diameter” or “capacity” of C . Let the points $S_n : z_{1n}, z_{2n}, \dots, z_{nn}$ be ordered on C with respect to their first indices. Let the real numbers $\Sigma_n : \theta_{1n}, \theta_{2n}, \dots, \theta_{nn}$ be such that $0 \leq \theta_{kn} \leq 2\pi$ and $\phi(e^{i\theta_{kn}}) = z_{kn}$, $k = 1, \dots, n$. Finally, rotate the map (if necessary) so that $0 \leq \theta_{1n} < \theta_{2n} < \dots < \theta_{nn} < 2\pi$. Let $N_n(\theta)$ be the number of the numbers Σ_n which fall on the closed interval $(0, \theta)$. If a non-negative integrable function $p(\theta)$ exists such that

$$\int_0^{2\pi} p(\theta) d(\theta) = 1$$

and such that for each θ , $0 \leq \theta \leq 2\pi$,

$$\lim_{n \rightarrow \infty} \frac{N_n(\theta)}{n} = \int_0^\theta p(\alpha) d\alpha,$$

then the points $\{S_n\}$ will be said to be ϕ -distributed on C according to the frequency function $p(\theta)$. An alternative terminology to “ ϕ -distributed” is “distributed with respect to harmonic measure”.

It is not difficult to calculate that the points (3.4a) are distributed on $|z| = 1$ (or in the present notation, $|w| = 1$) according to the frequency function

$$p(\theta) = \frac{1}{2\pi} \frac{a^2 - 1}{1 + a^2 - 2a \cos \theta}.$$

It will be brought out in the sequel that the special case in which $p(\theta) = 1/2\pi$ is the key one in the present study. This case will be called a uniform distribution on $|w| = 1$, and, after the mapping by $z = \phi(w)$, a uniform ϕ -distribution. It is fairly clear that the sequence of n^{th} roots of unity (that is, the sequence of sets of roots of the equations $w^n = 1$, $n = 1, 2, \dots$) are uniformly distributed on $|w| = 1$. Other classical examples of uniformly distributed sequences of points on $|w| = 1$ (see PÓLYA & SZEGÖ [13], vol. 1, pp. 70–73) are

$$(3.6a) \quad w_{kn} = \epsilon^k, \quad |\epsilon| = 1, \quad \epsilon \text{ not a root of unity}, \quad k = 1, \dots, n;$$

$$(3.6b) \quad w_{kn} = e^{2\pi i a k^s}, \quad a > 0, \quad 0 < s < 1, \quad k = 1, \dots, n.$$

It can be seen from these examples that reference is being made here to deterministic, not probabilistic, distributions of the interpolation points. However, the results described below all must have probabilistic analogues which would provide “Monte Carlo” approaches to the convergence problems under consideration. The probabilistic analogues have not yet been worked out, so far as the author is aware. It would seem that there might be some intriguing problems in this direction.

4. The convergence theory for complex polynomial interpolation to boundary values: necessary and sufficient conditions. It is now possible to give a complete and succinct answer to the question of necessary and sufficient conditions on $\{S_n\}$ that the sequence $\{L_n(f; z)\}$, z on R , shall converge for all functions f analytic on $C + R$. It is that the points $\{S_n\}$ must be uniformly ϕ -distributed on C .

If this condition is fulfilled, the sequence $\{L_n(f, z)\}$ converges uniformly to f for z on $R + C$ and, indeed, on any closed region bounded by a level curve $C_\rho : z = \phi(\rho e^{i\theta})$, $0 \leq \theta \leq 2\pi$, ρ fixed, $\rho > 1$, provided that $f(z)$ is analytic on and interior to C_ρ . Such a curve C_ρ contains $C + R$ in its interior. There is a greatest number ρ , say ρ_0 , finite or infinite, such that f is analytic on interior to C_ρ , $\rho < \rho_0$, but not on and interior to C_{ρ_0} . The error sequence $\{|L_n(f; z) - f(z)|\}$ is $O(1/\rho^n)$ for every ρ , $1 < \rho < \rho_0$, uniformly for z on C . The sequence $\{L_n\}$ is said to converge on C maximally under these circumstances. (The terminology is that of WALSH.)

The sufficiency of the condition was published by FEJÉR [7] in 1918. He showed that it implied uniform convergence in $C + R$ for all functions f analytic in $C + R$. Starting with the hypothesis of uniform convergence for all f analytic on $C + R$. KALMÁR [9] published the proof of the necessity in 1926. The results of FEJÉR and KALMÁR have been generalized in various ways by WALSH [18]. The theorem will here be called the FEJÉR-KALMÁR-WALSH theorem.

The necessary condition was derived from weaker hypotheses by the present author [4] in 1941:

Theorem 1. *If at just one point z_0 of R ,*

$$\overline{\lim}_{n \rightarrow \infty} |L_n(1/(t - z_0); z_0)|^{1/n} \leq 1$$

for all choices of t in K , then the following relations all hold true:

$$(a) \quad \lim_{n \rightarrow \infty} |\omega_n(z)|^{1/n} = |c|$$

uniformly on any closed point set interior to R .

$$(b) \quad \lim_{n \rightarrow \infty} |\omega_n(z)|^{1/n} = |c| |w|, \quad z = \phi(w),$$

uniformly on any closed point set interior to K .

(c) *The points $\{S_n\}$ are uniformly ϕ -distributed on C .*

(The chief new contribution of this theorem lay in conclusions (a) and (b). WALSH ([18], pp. 167–170) had earlier shown that (c) was a consequence of (b).)

Thus, in the analytic case, knowing that $\{S_n\}$ yields convergent interpolation at just a single fixed interior point to all analytic functions of a very special class is equivalent to knowing that the points $\{S_n\}$ are uniformly ϕ -distributed on C and therefore yield uniformly convergent interpolation *throughout* $C + R$ to all functions analytic on $C + R$.

The proofs of the above results are easily accessible in the literature and will not be sketched here.

The question of necessary and sufficient conditions that $\{S_n\}$ shall give convergent interpolation to all functions f merely defined and continuous on C will now be taken up. It was remarked earlier that here one should be looking for convergence on R to the function $F(z)$ given by (3.3), for which $f(z)$ may or may not seem to have any boundary value significance.

An esthetically satisfying simple necessary and sufficient condition describing directly the spacing of the points $\{S_n\}$ on C , analogous to the uniform ϕ -distribution condition for the analytic case, seems to be difficult to obtain for the present case, even for the unit circle. This is because apparently some detailed specifications as to the asymptotic “fine structure” of the interpolation points are required. Perhaps the most fruitful way of looking at the matter consists in treating $\{L_n\}$ as a sequence of linear operators and applying Helly’s theorem. A form of this theorem given by SZEGÖ ([15], p. 13) for real functions and real

operators can be translated into the complex domain as follows so as to make it applicable to the present situation.

Let $\{\alpha_n(\theta)\}$ be a sequence of complex functions of bounded variation on the interval $0 \leq \theta \leq 2\pi$, and let $\alpha(\theta)$ be another such function.* A necessary and sufficient condition that

$$(4.1) \quad \lim_{n \rightarrow \infty} \int_0^{2\pi} f(e^{i\theta}) d\alpha_n(\theta) = \int_0^{2\pi} f(e^{i\theta}) d\alpha(\theta),$$

where $f(w)$ is an arbitrary continuous function of w for $|w| = 1$, is that both of the following two relations shall hold true:

$$(4.2a) \quad \lim_{n \rightarrow \infty} \int_0^{2\pi} e^{ik\theta} d\alpha_n(\theta) = \int_0^{2\pi} e^{ik\theta} d\alpha(\theta), \quad k = \dots -2, -1, 0, 1, 2, \dots,$$

$$(4.2b) \quad \int_0^{2\pi} |d\alpha_n(\theta)| d\theta < M, \quad n = 1, 2, \dots,$$

where M is independent of n .

Moreover, if (4.2b) is not satisfied, a continuous function $f(w)$ exists such that the sequence of operators appearing on the left side of (4.1) is unbounded.

When the parametrization given by $t = \phi(e^{i\theta})$, t on C (with C now rectifiable), is substituted into (3.1), this formula may be written in the following form for a continuous f :

$$(4.3) \quad L_n(f; z) = \int_0^{2\pi} f(\phi(e^{i\theta})) d\alpha_n(\theta), \quad \alpha_n(0) = 0, 0 \leq \theta \leq \theta_{1n},$$

$$(4.4) \quad \alpha_n(\theta) = \sum_{i=1}^k \lambda_i(z) = \sum_{i=1}^k \frac{\omega_n(z)}{z - \phi(e^{i\theta_{i,n}})} \cdot \frac{\frac{d\phi(e^{i\theta_{i,n}})}{d\theta}}{\frac{d\omega_n(\phi(e^{i\theta_{i,n}}))}{d\theta}},$$

$$\theta_{kn} < \theta \leq \theta_{k+1,n}, \quad k = 1, \dots, n, \quad \theta_{n+1} = 2\pi.$$

(Here the same assumptions are being made about the ordering of the points z_{kn} and the orientation of the map as were made above in introducing the concept of the distribution of $\{S_n\}$.) Also, (3.3) becomes

$$(4.5) \quad F(z) = \int_0^{2\pi} f(\phi(e^{i\theta})) d\alpha(\theta), \quad \alpha(\theta) = -\frac{1}{2\pi i} \int_0^\theta \frac{1}{z - \phi(e^{i\xi})} \frac{d\phi(e^{i\xi})}{d\xi} d\xi.$$

(The substitution of $(d\phi/d\xi) d\xi$ for dt is made legitimate by the fact that if C is rectifiable, as here, then $\phi(e^{i\theta})$ is absolutely continuous—i.e., is the Lebesgue indefinite integral of its derivative. See ZYGMUND [20], pp. 157 ff. The integral by which $\alpha(\theta)$ is being defined here is to be taken in the sense of Lebesgue.) Helly's theorem in the form stated above is thus seen to be directly applicable

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*A normalization of the functions α_n and α of a type such as $\operatorname{Re} \alpha(\theta - 0) \leq \operatorname{Re} \alpha(\theta) \leq \operatorname{Re} \alpha(\theta + 0)$, $\operatorname{Im} \alpha(\theta - 0) \leq \operatorname{Im} \alpha(\theta) \leq \operatorname{Im} \alpha(\theta + 0)$ is assumed.

to the study of the convergence of $\{L_n\}$ for continuous functions. The appropriate specialization can be stated as follows primarily in terms of the z -plane without conformal mapping:

Theorem 2. *Let the rectifiable curve C contain the origin in the z -plane. A necessary and sufficient condition that the sequence of polynomials $\{L_n(f; z)\}$, found respectively by interpolation on the points $\{S_n\}$ on C to the arbitrary function $f(z)$ continuous on this curve, shall converge to the function*

$$F(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{t - z} dt$$

at the single point z_0 on R for all continuous functions is that both of the following relations shall hold true:

$$(4.6a) \quad \lim_{n \rightarrow \infty} L_n(z^{-k}; z_0) = 0, \quad k = 1, 2, \dots,$$

$$(4.6b) \quad \sum_1^n |\lambda_k(z_0)| < M, \quad n = 1, 2, \dots,$$

where M is independent of n . The convergence of $\{L_n(f; z)\}$ to $F(z)$ is uniform on any closed point set of R on which (4.6a) and (4.6b) hold uniformly.

The sufficiency can easily be proved without explicitly mapping the problem back into the w -plane. The argument runs as follows. There is a well-known generalization of Weierstrass' polynomial approximation theorem due to WALSH ([18], pp. 38–39) which states that any function continuous on a Jordan curve of the finite z -plane containing the origin in its interior can be uniformly approximated on that curve by polynomials in z and $1/z$. Let $p(z)$ be a polynomial in z of degree not more than $n - 1$. Now

$$L_n(p; z) \equiv p(z) \equiv \frac{1}{2\pi i} \int_C \frac{p(t)}{t - z} dt, \quad z \text{ on } R.$$

Also, (4.6a) implies that if $q(z)$ is any polynomial in $1/z$ without a constant term, then

$$\lim_{n \rightarrow \infty} L_n(q; z_0) = 0 \equiv \frac{1}{2\pi i} \int_C \frac{q(t)}{t - z_0} dt, \quad z_0 \text{ on } R.$$

The rest of the proof can be given by applying the standard estimation techniques of elementary analysis to the following identity:

$$(4.7) \quad \begin{aligned} L_n(f; z_0) - F(z_0) &\equiv \sum_{k=1}^n [f(z_{kn}) - p(z_{kn}) - q(z_{kn})] \lambda_k(z_0) \\ &\quad + \frac{1}{2\pi i} \int_C \frac{p(t) + q(t) - f(t)}{t - z_0} dt + L_n(q; z_0). \end{aligned}$$

First p and q are chosen so that $|f - p - q|$ is arbitrarily small on C , and then n is allowed to become infinite.

The fact that (4.6b) is a *necessary* condition for convergent interpolation for all continuous f is perhaps most easily established by using (4.3) and (4.4) and referring the matter back to Helly's theorem. The relation (4.6a) is now just a special case of the hypothesis.

The author showed some years ago that if the points z_{kn} are the transforms under $z = \phi(w)$ of the n^{th} roots of unity, then (4.6a) is valid for any rectifiable C and any z_0 in R , and (4.6b) holds true if C is such that $\phi'(w)$ is non-vanishing and of bounded variation for $|w| = 1$. Actually, the following more specific result was proved [5]:

Theorem 3. *If C is rectifiable and S_n is the map under the transformation, (3.5) of the n^{th} roots of unity, then*

$$(4.8a) \quad \lim_{n \rightarrow \infty} \frac{\omega_n(z)}{-c^n} = 1$$

uniformly on any closed point set of R and

$$(4.8b) \quad \lim_{n \rightarrow \infty} \frac{\omega_n(z)}{c^n(w^n - 1)} = 1, \quad z = \phi(w),$$

uniformly on any closed point set of K . If $\phi'(w)$ is non-vanishing and of bounded variation for $|w| = 1$, and if the quotient in (4.8b) is defined by continuity whenever it is indeterminate, then (4.8b) is valid uniformly on $C + K$.

It is clear from (3.2a) that (4.8a) implies (4.6a). To establish that (4.8a) and (4.8b) imply (4.6b) when ϕ' is of bounded variation, first write

$$\pi_n(w) = \frac{\omega_n(\phi(w))}{c^n(w^n - 1)} = \frac{\omega_n(\phi(w)) - \omega_n(\phi(w_{kn}))}{c^n(\phi(w) - \phi(w_{kn}))} \cdot \frac{\phi(w) - \phi(w_{kn})}{w - w_{kn}} \cdot \frac{w - w_{kn}}{w^n - 1}.$$

Note that

$$\pi_n(w) \rightarrow \frac{1}{c^n} \omega'(\phi(w_{kn})) \phi'(w_{kn}) \cdot \frac{1}{nw_{kn}^{n-1}}, \quad w \rightarrow w_{kn}.$$

Then

$$(4.9) \quad \lambda_k(z_0) = \frac{\omega_n(z_0)}{\omega'(\phi(w_{kn}))(z_0 - z_{kn})} = \frac{\omega_n(z_0) \phi'(w_{kn})}{nc^n \pi_n(w_{kn}) w_{kn}^{n-1} (z_0 - z_{kn})}.$$

The relations (4.8a) and (4.8b) insure the boundedness of $\omega_n(z_0)/c^n \pi_n(w_{kn})$ for z_0 on R , and the fact that ϕ' is of bounded variation implies that $|\phi'(w)|$ is uniformly bounded for $|w| = 1$. Therefore $\lambda_k(z_0) = O(1/n)$ uniformly in k . The validity of (4.6b) now follows from this. More explicitly, if n is large enough so that for a certain value of n —call it n_ϵ —and for all greater values, the expressions after the limit signs in (4.8a) and (4.8b) are within an ϵ distance of their limits, then

$$(4.9a) \quad |\lambda_k(z_0)| \leq \frac{1 + \epsilon}{1 - \epsilon} \frac{\text{bound } |\phi'(e^{i\theta})|}{n} \frac{1}{\min_{t \text{ on } C} |z_0 - t|},$$

$$k = 1, \dots, n, \quad n \geq n_\epsilon.$$

A sufficient condition for (4.6a) to hold true is that

$$(4.10) \quad \lim_{n \rightarrow \infty} \frac{\omega_n(z_0)}{\omega_n(t)} = 1, \quad \text{uniformly, } |t| < \rho > 0,$$

for some ρ . This follows from (3.2a) and (3.2b). For $\{L_n(1/(z - t); z)\}$ is a sequence of analytic functions of t , and (4.10) implies that the sequence approaches zero uniformly in some neighborhood of the origin. Therefore the derivatives of the terms of the sequence approach zero in this neighborhood and the italicized assertion is now established by reference to (3.2b).

With some attention to detail it can be shown that (4.6a) and the relations

$$(4.6c) \quad |\lambda_k(z_0)| < \frac{M}{n}, \quad k = 1, \dots, n; \quad n = 1, 2, \dots,$$

where M is independent of n , are sufficient conditions for convergence to $F(z)$ for all boundary-value functions which are bounded and have a finite number of discontinuities.

5. Sufficient conditions when f is analytic on R and continuous on $R + C$. Consider again the sufficiency part of Theorem 2. What can be said if (4.6a) is dropped and (4.6b) is retained?

The first thing to observe is that (4.6b) implies that for any function f which is uniformly bounded on C , $|L_n(f; z_0)|$ is bounded for all n , and uniformly on any set of values of z_0 for which (4.6b) holds uniformly. Therefore, by the Bolzano-Weierstrass theorem and MONTEL's generalization of it to the space of bounded analytic functions ([12], pp. 21 ff.), in every infinite subsequence of the sequence $\{L_n(f; z_0)\}$ there exists a subsequence which converges to some limit, and if (4.6b) holds uniformly for z_0 in some subregion of R , then the convergence is uniform in this subregion and the limit is an analytic function. But is there any assurance that the limit functions have any relation to $F(z)$? Apparently not in general, and not even for boundary value functions f of the type $1/(t - z)$, with t on R .

Nevertheless (4.6b) all by itself carries a rather strong implication regarding $\{S_n\}$ and also regarding an important subclass of the functions f continuous on C :

Theorem 4. *Let C be any Jordan curve, not necessarily rectifiable. Let $f(z)$ be any function which is analytic on R and continuous on $C + R$. A sufficient condition that the sequence of polynomials $\{L_n(f; z)\}$, found respectively by interpolation in the points $\{S_n\}$ on C to the function $f(z)$, shall converge to $f(z)$ at the point z_0 in R is that (4.6b) shall hold true. The convergence is uniform on any closed point set of R on which (4.6b) holds true uniformly. Moreover, given that (4.6b) holds*

true at but a single point z_0 , it must follow that relations (a) and (b) of Theorem 1 are valid and that the points $\{S_n\}$ are uniformly distributed on C .

The last sentence is a direct consequence of Theorem 1. The rest of the theorem follows from the fact that now $f(z)$ can be approximated uniformly on C by a polynomial in z only ([18], p. 36), and q and $L_n(q; z_0)$ in (4.7) can be set equal to zero.

It is worth noticing that, under the hypotheses of Theorem 4, if a sequence of polynomials $\{p_n(z)\}$ of respective degrees at most $n - 1$ exists such that $|f(z) - p_n(z)| \leq \chi(n)$ for all z on C , then the asymptotic degree of convergence of $\{L_n(f; z_0)\}$ is $O(\chi(n))$ uniformly for z_0 on any closed point set of R on which (4.6b) is valid uniformly. For example, it is known [2] that if C is an analytic Jordan curve and if the p^{th} derivative of f exists on C and satisfies a uniform Lipschitz condition with exponent α , $0 < \alpha \leq 1$, then there exists a sequence $\{p_n\}$ such that $\chi(n) = M/n^{p+\alpha}$, where M is independent of n . The interpolative sequence will then exhibit the same degree of convergence on R .

These remarks can be extended only with some qualification to the more general situation involved in Theorem 2, because there the degree of convergence of the sequence $\{L_n(f; z_0)\}$ will depend in part on how rapidly $\{L_n(q; z_0)\}$ approaches zero. The truth of this can be seen at a glance from (4.7). The speed of convergence of $\{L_n(q; z_0)\}$ to zero would seem to depend on the spacing of the points S_n on C , and also on the smoothness of C , but rather less on the properties of f , although these must enter into consideration because of their effect on the coefficients of q . It can at least be said that, under the hypotheses of Theorem 2, if polynomials $p_n(z)$ in z and $q_n(z)$ in $1/z$ exist such that $|f(z) - p_n(z) - q_n(z)| \leq \chi(n)$ from z on C , then $L_n(f; z_0) - L_n(q_n z_0) - F(z_0) = O(\chi(n))$, z_0 on R .

Perhaps the most important question which is left unanswered by the results of this section is whether for a general rectifiable Jordan curve C there always exist point sets $\{S_n\}$ for which (4.6b) is valid. Theorem 3 affirms that there do exist such point sets $\{S_n\}$ when C is such that $\phi'(w)$ is nonvanishing and of bounded variation for $|w| = 1$, but these restrictions on C are not satisfied even by a polygon. However, certain developments in Section 6 suggest that (4.6b) might well be valid for the point sets $\{S_n\}$ of Theorem 3 when C is a polygon, and more generally when C is any Jordan curve which is analytic except for a finite number of corners.

Other questions of some interest which seem to remain open are these:

(1) In Theorem 4 it was stated that (4.6b) is by itself a sufficient condition for convergence at z_0 to all functions analytic on R and continuous on $R + C$. But is it necessary? If it does not hold true, could a function f of this class be constructed such that $\{L_n(f; z_0)\}$ diverges?

(2) If C is sufficiently regular, will the mapping under ϕ of some of the well-known sets of uniformly distributed points on $|w| = 1$ other than the roots of unity (such as the points (3.6a) or (3.6b)) yield universally convergent inter-

polation for every continuous f ? As far as the author knows, even for the case of the function $1/z$ and interpolation on the unit circle, it is not known how $\{L_n(1/z; z)\}$ behaves when found by interpolation in the points (3.6a). This is equivalent to saying that the behavior of the sequence

$$\left\{ \prod_{k=1}^n \left(\frac{z - \epsilon^k}{-\epsilon^k} \right) \right\}, \quad |\epsilon| = 1, \quad \epsilon \text{ not a root of unity},$$

seems never to have been studied.

6. Simultaneous approximation on two or more mutually exterior regions; more open problems. WALSH [18] successfully generalized the Fejér-Kalmár-Walsh theorem to the case in which the sequence $\{L_n\}$ is found by interpolation on several mutually exterior Jordan curves $C_1, C_2, \dots, C_m$ to functions $f_1, f_2, \dots, f_m$ which are respectively analytic on the closed regions bounded by these curves. With the proper generalization of a uniform ϕ -distribution, the sequence $\{L_n\}$ converges uniformly to the expected limits simultaneously in all of the closed regions. The first part of Theorem 4 (but not the last sentence) can also be so generalized. The key to the extension is that the functions f_i can all simultaneously be approximated by a single polynomial in z (WALSH [18], p. 47).

It seems doubtful if there is any such direct generalization of Theorem 2, where f_i is defined and continuous merely on C_i . The trouble seems to be this: In proving the sufficiency part of Theorem 2, f was approximated by a function in z and $1/z$. The direct generalization of this process to simultaneous approximation on several curves involves the use of a rational function with poles at points a_i interior to the curves. Such a function can be broken up by the method of partial fractions into the sum of a polynomial and a function q which is a linear combination of terms of the type $1/(z - a_i)^k$, $k = 1, 2, \dots$. When applied to such a term on a curve C_i , the Cauchy integral operator appearing in (3.3) has the value zero for z inside C_i only if a_i lies inside C_i ; otherwise the operator reproduces the term at the point z . This would be all right if $\{L_n(1/(z - a_i); z_0)\}$ then converged to the same value. (See (4.7).) But unfortunately the sequence $\{L_n(1/(z - a_i); z_0)\}$ in the present case normally seems to converge to zero inside *each one* of the curves $C_1, C_2, \dots$, whenever a_i lies inside one of them.

The following example illustrates this behavior. A suitably chosen branch of the function $w = (z^2 - 1)^{1/2}/\mu^{1/2}$, $0 < \mu < 1$, maps the exterior of the ovals of Cassini, of which the equation is $|z^2 - 1| = \mu$, onto $|w| > 1$ conformally (but not uniformly) so that the points at infinity correspond. This mapping function is the direct analogue of the function $z = \phi(w)$ appearing in (3.5). Let $z_{1,2n}, \dots, z_{n,2n}$ denote those roots of the polynomial $w_{2n}(z) = (z^2 - 1)^n - \mu^n$ which lie on the right-hand oval, and let $z_{n+1,2n}, \dots, z_{2n,2n}$ denote the roots which lie on the left-hand oval. With appropriate identification of the branches of the mapping function, these points $z_{k,2n}$ can be interpreted as the images of the $2n$ roots of $w^{2n} = 1$. It is easily determined that (4.6c), and therefore (4.6b),

is valid on any closed point set inside the ovals. Substituting into (3.2a), one obtains

$$L_{2n}\left(\frac{1}{z-t}; z\right) = \frac{1}{z-t} \left[1 - \frac{\left(\frac{z^2-1}{\mu}\right)^n - 1}{\left(\frac{t^2-1}{\mu}\right)^n - 1} \right].$$

If z lies inside the ovals, then $|z^2 - 1| < \mu$, and if z lies outside, then $|z^2 - 1| > \mu$. It follows at once that if the parameter t lies inside one of the ovals, then $\{L_{2n}\}$ converges to zero inside both ovals, and if t lies outside both ovals (so that $1/(z-t)$ is analytic in each one), then $\{L_{2n}\}$ converges to $1/(z-t)$ inside each oval. But if t lies inside one of the ovals, $1/(z-t)$ is analytic for z inside the other one, and there one would hope to get convergence not to zero but to $1/(z-t)$ if Theorem 2 could be directly generalized to the case of two curves.

By inspection of $\omega_{2n}(z)$ above it can be seen that Theorem 3 with n replaced by $2n$ and c by $\mu^{\frac{1}{2}}$ carries over to the ovals of Cassini. Indeed, the following extension of Theorem 3 to a certain special class of multi-component loci which includes the ovals of Cassini lies immediately at hand: *Let C be a lemniscate of which the equation is $|p(z)| = |\mu|$, where μ may be complex and p is a polynomial of degree h with leading coefficient unity. Let R denote the interior of the lemniscate, identified by $|p(z)| < |\mu|$. Let $S_{nh} : z_{1,nh}, \dots, z_{nh,nh}$ be the nh roots of the equation $[p(z)]^n = \mu^n$. Then*

$$\lim_{n \rightarrow \infty} \frac{\omega_{nh}(z)}{\mu^n} = 1$$

uniformly on any closed point set of R and

$$\frac{\omega_{nh}(z)}{\mu^n(w^{nh} - 1)} \equiv 1, \quad w = \left(\frac{p(z)}{\mu}\right)^{1/n}.$$

The result is obvious because in this case $\omega_{nh}(z) = [p(z)]^n - \mu^n$.

In view of the central importance of (4.6b), it is natural to inquire whether this relation is valid for the points of S_{nh} defined above and with the point z_0 lying on the interior R of the lemniscate $C: |p(z)| = |\mu|$. For these points it is easy to verify that

$$(6.1) \quad \lambda_k(z_0) = \frac{[p(z_0)]^n - \mu^n}{n(z_0 - z_{k,nh})} \left(\frac{p(z_{k,nh})}{p'(z_{k,nh})} \right)$$

If no zero of $p'(z)$ lies on C , then clearly (4.6b) is satisfied for z_0 on R . The locus C under these circumstances consists of at most h mutually exterior Jordan curves. (For a thorough discussion of lemniscates see WALSH, [18], pp. 54 ff.)

If one or more zeros of $p'(z)$ do lie on C , then C still may consist of as many as h mutually exterior Jordan curves except perhaps for common points at the zeros of $p'(z)$, which are multiple points of the locus C as a whole. Here an initial difficulty arises from a fact which is obvious from (6.1): If a particular

interpolation point, say z_{k_0, n_h} , falls on a root of $p'(z)$, then the corresponding $\lambda_{k_0}(z_0)$ is infinite. In specific instances such a misfortune can be avoided by rotating the sequence $\{S_{nh}\}$ on C (which means changing the argument of μ) and perhaps by skipping over some of the values of n . But after all necessary revisions of this sort have been made, the problem of establishing the validity of (4.6b) in the presence of zeros of $p'(z)$ on C still remains. It can easily be shown that the points of $\{S_{n_{jh}}\}$ are everywhere dense* on C , where $\{n_j\}$ is any infinite subsequence of $\{n\}$, and therefore the factor $p'(z_{k, n_{jh}})$ in the denominator of λ_k , as given by (6.1), cannot be bounded from zero. The problem is essentially one of showing that, in spite of this, $\sum_{k=1}^{n_{jh}} (1/n \mid p'(z_{k, n_{jh}}) \mid)$ is uniformly bounded for all n_j . The author conjectures that a properly rotated infinite subsequence of $\{S_{nh}\}$ always exists for which it is possible to prove that this sum is indeed uniformly bounded.

These remarks about interpolation to boundary values on lemniscates probably point the way to the discussion of the analogous results for configurations consisting of mutually exterior general analytic curves which are not necessarily components of lemniscates. The conjecture at the end of the preceding paragraph, if correct, would seem to carry an implication for the single component case: namely, that (4.6b) is valid when the points of S_n are the transforms of the n^{th} roots of unity under the mapping function $z = \phi(w)$ of Sections 3 and 4, and when C is a Jordan curve which is analytic except for a finite number of corners (in particular, a polygon). The plausibility of such a result was mentioned above in Section 5.

7. Interpolation by harmonic polynomials. The basic coverage problem for harmonic polynomials interpolating to boundary values will now be formulated in terms analogous to those used for the complex interpolation problem.

Let C be a Jordan curve of the z -plane. Let R be the region interior to C . Install polar coordinates in the z -plane, and let S_{2n-1} be the set of points

$$(r_{k, 2n-1} \cos \theta_{k, 2n-1}, r_{k, 2n-1} \sin \theta_{k, 2n-1}), \quad k = 1, \dots, 2n-1.$$

The points of S_{2n-1} are assumed to lie on C and to be distinct. Let $u(z) = u(re^{i\theta}) = U(x, y)$ be real and continuous on C . The points of S_{2n-1} must not all lie on any locus represented by an equation of the type $\operatorname{Re} p(z) = 0$, where $p(z)$ is a polynomial of degree at most $n-1$. (See Section 2.) Let the harmonic polynomial $H_n(u; r, \theta)$ of degree at most $n-1$ be found by interpolation to the values of u at the points of S_{2n-1} . What is a necessary and sufficient condition on $\{S_{2n-1}\}$ that the corresponding sequence $\{H_{2n-1}\}$ converges to the solution of the Dirichlet problem on R for the boundary values u ? Do sequences $\{S_{2n-1}\}$ actually exist which satisfy the conditions?

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*The points of $S_{n_{jh}}$ are the roots of the n_j equations $p(z) = \mu e^{2\pi i k/n_j}$, $k = 1, \dots, n_j$. A point z_0 on the lemniscate satisfies the equation $p(z_0) = \mu e^{i\phi_0}$ for some angle ϕ_0 . The roots of $p(z) = \mu e^{i\phi}$ are continuous functions of ϕ . Therefore, since ϕ_0 is a limit point of $2\pi k/n_j$, $k = 1, \dots, n_j$, $j = 1, \dots$, it follows that z_0 must be a limit point of $\{S_{n_{jh}}\}$.

It will doubtless be noted that these questions are being asked only for the case of an odd number of interpolation points. The case of an even number seems so similar that it does not appear to be worth while to consider it further here.

A primary difficulty in the study of these questions is the apparent lack of any manageable formula for H_n , such as the Lagrange interpolation polynomial (3.1). It was noted in connection with (3.1) and (3.1a) that $\lambda_k(z)$ is the k^{th} component of the vector $Z'^{-1}\xi$, where Z is the matrix (2.1), Z'^{-1} is the transpose of the inverse of Z , and ξ is the vector $(1, z, \dots, z^{n-1})$. In the harmonic case, if the matrix (2.4) be denoted by T and the $(2n-1)$ -vector

$$(1, r \cos \theta, r^2 \cos 2\theta, \dots, r^{n-1} \cos (n-1)\theta, r \sin \theta, \dots, r^{n-1} \sin (n-1)\theta)$$

by $\hat{\eta}(r, \theta)$, or simply by $\hat{\eta}$, then

$$(7.1) \quad H_n(u; r, \theta) = \sum_{k=1}^{2n-1} u(r_{k, 2n-1} e^{i\theta_{k, 2n-1}}) (T'^{-1} \hat{\eta})_k,$$

where $(\hat{\alpha})_k$ means the k^{th} component of the vector $\hat{\alpha}$ and T'^{-1} is the transposed inverse of T . Just as $\lambda_k(z) = (Z'^{-1}\xi)_k$ is a polynomial in z of degree $n-1$ which vanishes at the points $z_{1n}, \dots, z_{k-1, n}, z_{k+1, n}, \dots, z_{nn}$ and takes on the value one at z_{kn} , so is $(T'^{-1}\hat{\eta})_k$ a harmonic polynomial of degree $n-1$ which vanishes at the points

$$z_{1, 2n-1}, \dots, z_{k-1, 2n-1}, z_{k+1, 2n-1}, \dots, z_{2n-1, 2n-1}$$

and takes on the value one at $z_{k, 2n-1}$.

If T is non-singular, which is here being assumed, and if U is a harmonic polynomial of degree $n-1$ or less, then $H_n \equiv U$. This is because the difference $H_n - U$ is then a harmonic polynomial of degree at most $n-1$ which vanishes at $2n-1$ points, and in such a case the linear equation theory which is the basis for the determination of the coefficients by interpolation assigns the value zero uniquely to each of the coefficients, provided that the conditions on S_{2n-1} are fulfilled.

The analogue of Theorem 2 can now be stated, and it turns out to be the analogue of Theorem 4 also.

Theorem 5. *Let C be any Jordan curve, not necessarily rectifiable. A necessary and sufficient condition that the sequence of harmonic polynomials $\{H_n(u; r, \theta)\}$, found respectively by interpolation in the points $\{S_{2n-1}\}$ on C to the function u , shall converge at the point (r, θ) to the solution of the Dirichlet problem for any continuous u is that*

$$(7.2) \quad \sum_{k=1}^{2n-1} |(T'^{-1} \hat{\eta}(r, \theta))_k| < M,$$

where M is independent of n . The convergence is uniform for (r, θ) on any point

set of $R + C$ on which (7.2) holds uniformly. If (7.2) is not satisfied at a point (r, θ) , there is a continuous u for which $\{H_n(u; r, \theta)\}$ is unbounded.

The necessity is a direct result of Helly's theorem. The sufficiency hinges on the fact that given any continuous function u on C , there exists a harmonic polynomial which approximates arbitrarily closely on C to u and at the same time approximates uniformly on R to the corresponding solution of the Dirichlet problem with boundary values given by u (WALSH [18], p. 169). The proof of the sufficiency can then be carried out directly by writing out the identity

$$H_n(u; r, \theta) - u(re^{i\theta}) = \sum_1^{2n-1} [u(r_{k,2n-1}e^{i\theta_{k,2n-1}}) - P(r_{k,2n-1}, \theta_{k,2n-1})](T'^{-1}\hat{\eta})_k \\ + P(r, \theta) - u(re^{i\theta}),$$

where P is a harmonic polynomial of degree at most $n - 1$ which approximates U on $R + C$. The right-hand side can now be estimated in the standard manner.

In a way this could be viewed as a definitive solution of the central asymptotic harmonic polynomial interpolation problem. The difficulty is that the meaning of (7.2) is far from clear. For a general curve C there is as yet no theorem like Theorem 3 which provides a simple criterion for spacing the points of S_{2n-1} which will insure that (7.2) holds true. The recent result of WALSH [19] for the case in which C is an ellipse, which was mentioned in the introduction, doubtless is indicative of the general situation.

8. Interpolation by harmonic polynomials on the unit circle. If C is the unit circle, then $r_{k,2n-1} = 1$, $k = 1, \dots, 2n - 1$, and the expression $(T'^{-1}\eta(1, \theta))_k$ is a trigonometric sum of the $(n - 1)^{\text{th}}$ order. It is known that this sum can be represented as follows (see (JACKSON [8], pp. 109-111):

$$(8.1) \quad (T'^{-1}\hat{\eta}(1, \theta))_k = \frac{s(\theta)}{s'(\theta_{k,2n-1}) \sin \frac{\theta - \theta_{k,2n-1}}{2}},$$

where

$$s(\theta) = \prod_{j=1}^{2n-1} \sin \frac{\theta - \theta_{j,2n-1}}{2}.$$

The implication in this case is that T is never singular as long as interpolation points are distinct.

This formula is reminiscent of the formula (3.1a) for $\lambda_k(z)$. Its validity can be established directly by noting that the factors in the numerator can be grouped into $n - 1$ pairs after cancellation of one of them with the denominator, and that each pair is a trigonometric sum of the *first* order, so their product is a trigonometric sum of the $(n - 1)^{\text{th}}$ order. It is quite apparent that the right-hand side of (8.1) assumes the values 1 and 0 at the points where it ought to do so.

At this point it is in order to recall the Poisson integral, which provides a solution to the Dirichlet problem for the unit disc and boundary values $u(e^{i\theta})$:

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\beta}) \frac{1-r^2}{1-2r \cos(\theta-\beta)+r^2} d\beta, \quad 0 \leq r < 1.$$

The Poisson integral applied to (8.1) yields the expression

$$(8.2) \quad (T'^{-1}\eta(r, \theta))_k = \frac{1}{2\pi} \int_0^{2\pi} \frac{s(\beta)}{s'(\theta_{k,2n-1}) \sin \frac{\beta - \theta_{k,2n-1}}{2}} \cdot \frac{1-r^2}{1-2r \cos(\theta-\beta)+r^2} d\beta, \quad 0 \leq r < 1,$$

and the right-hand side of (8.2) may now be substituted into (7.1) to obtain a formula for the unit circle in which the inverse of the matrix T' is explicitly evaluated.

On the unit circle there does exist a sequence of interpolation points $\{S_{2n-1}\}$ such that (7.2) holds true for (r, θ) on R . This sequence is the one in which the points S_{2n-1} are the $(2n-1)^{\text{th}}$ roots of unity, which means that $\theta_{k,2n-1} = 2\pi k/(2n-1)$. In this case ([8], pp. 112-116) it turns out that

$$(8.3) \quad \frac{s(\theta)}{s'(\theta_{k,2n-1}) \sin \frac{\theta - \theta_{k,2n-1}}{2}} = \frac{1 + 2 \sum_{j=1}^{n-1} \cos j(\theta - \theta_{k,2n-1})}{2n-1} = \frac{\sin \frac{2n-1}{2}(\theta - \theta_{k,2n-1})}{(2n-1) \sin \frac{\theta - \theta_{k,2n-1}}{2}},$$

as may be verified directly. The Poisson integral operating on $\cos j(\theta - \theta_{k,2n-1})$ transforms it into the corresponding circular harmonic function, which is $r^j \cos j(\theta - \theta_{k,2n-1})$. Therefore by substituting (8.3) into (8.2) one obtains

$$(8.4) \quad (T'^{-1}\eta(r, \theta))_k = \frac{1 + 2 \sum_{j=1}^{n-1} r^j \cos j(\theta - \theta_{k,2n-1})}{2n-1} = \operatorname{Re} \frac{1 + 2(t_k + t_k^2 + \cdots + t_k^{n-1})}{2n-1} = \operatorname{Re} \frac{1 + t_k - 2t_k^n}{(2n-1)(1-t_k)}, \quad t_k = re^{i(\theta - \theta_{k,2n-1})}.$$

It is clear from the last member of (8.4) that (7.2) holds true with this sequence $\{S_{2n-1}\}$ for $r < 1$, uniformly for $0 \leq r \leq r_0 < 1$, where r_0 is fixed.

It is instructive in the roots of unity case to write out H_n and compare it with

the Lagrange polynomial L_n . The boundary value function will in each case be denoted by u . The points of interpolation are

$$S_N : w_{kN} = e^{2\pi i k/N} = e^{i\theta_{kN}}, \quad k = 1, \dots, N.$$

In the harmonic case $N = 2n - 1$; in the complex case $N = n$. Then with $z = re^{i\theta}$

$$(8.5) \quad H_n = h_n(u; z) = \frac{1}{2\pi} \sum_{k=1}^{2n-1} u(w_{k,2n-1}) \operatorname{Re} \frac{1 + z\bar{w}_{k,2n-1} - 2(z\bar{w}_{k,2n-1})^n}{1 - z\bar{w}_{k,2n-1}} \cdot \frac{2\pi}{2n-1} \\ = \frac{1}{2\pi} \sum_{k=1}^{2n-1} \left[u(w_{k,2n-1}) \cdot \frac{1 - r^2 - 2r^n [\cos n(\theta - \theta_{k,2n-1}) - r \cos (n-1)(\theta - \theta_{k,2n-1})]}{1 - 2r \cos (\theta - \theta_{k,2n-1}) + r^2} \right] \frac{2\pi}{2n-1},$$

$$(8.6) \quad L_n(u; z) = \frac{1}{2\pi} \sum_{k=1}^n u(w_{kn}) \frac{1 - (z\bar{w}_{kn})^n}{1 - z\bar{w}_{kn}} \cdot \frac{2\pi}{n}.$$

The bars over complex numbers here denote the conjugate complex numbers. The extra factors 2π and $1/2\pi$ are introduced here to bring out the fact that each of these expressions is essentially an approximating Riemann sum for a certain definite integral with respect to θ over the interval $(0, 2\pi)$, corresponding to the division of the interval into equal parts. To be sure, each integrand contains an infinitesimal of uniformly higher order in the form of a term involving r^n , with $0 \leq r < 1$. But by Duhamel's theorem in elementary calculus, these infinitesimals do not affect the limits, which may be read off directly. The limit of (8.5) for $0 \leq r < 1$ is obviously the Poisson integral for boundary values $u(z)$, and the limit of (8.6) for $0 \leq r < 1$ is

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{u(e^{i\beta})}{1 - ze^{-i\beta}} d\beta = \frac{1}{2\pi i} \int \frac{u(w)}{w - z} dw,$$

as indeed it should be according to Theorems 2 and 3.

This result for complex polynomial interpolation was proved by FEJÉR [7] in the case in which the boundary values are those of a function continuous on $|z| \leq 1$ and analytic on $|z| < 1$. The result was extended by WALSH ([18], pp. 179–180) to the case in which $u(z)$ is defined only on $|z| = 1$ and is only integrable in the sense of Riemann.

The behavior of (8.5) and (8.6) on the boundary of the unit disc is worthy of attention. FEJÉR [7] showed that there exists a complex function $u(z)$, analytic for $|z| < 1$ and continuous for $|z| \leq 1$, and such that $\{L_n(u; z)\}$, as given by (8.6), is unbounded for z at a certain point of the unit circle. In the case of $\{h_n(u; z)\}$, as given by (8.5), MARCINKIEWICZ has exhibited an example of a continuous u for which this sequence of interpolation polynomials is unbounded at *all* points of $|z| = 1$. (See SZEGÖ [15], p. 323, for a discussion and primary references.)

For $|z| = r = 1$ the multiplier of the values of u in (8.5) under the summation

sign (with the factor 2π omitted)—which might for any r be called the kernel of the operator H_n or h_n —reduces to (8.3). The latter looks like the kernel of DIRICHLET's formula for the n^{th} partial sum of a Fourier series, and it is well known that there exist continuous functions such that the partial sums of their Fourier series diverge. On the other hand, it is also well known that the sequence of arithmetic means of the partial sums of a Fourier series generated by a continuous function always converges uniformly to that function. The kernel of FEJÉR's integral formula for the n^{th} arithmetic mean is

$$\frac{\sin^2 \frac{n}{2} (\theta - \alpha)}{n^2 \sin^2 \frac{1}{2} (\theta - \alpha)}.$$

This suggests that by replacing the kernel (8.3) of $h_n(u; e^{i\theta})$ by

$$\frac{\sin^2 \frac{n}{2} (\theta - \theta_{k, 2n-1})}{n^2 \sin^2 \frac{\theta - \theta_{k, 2n-1}}{2}}$$

and finding the corresponding harmonic polynomial for $r < 1$, one might obtain a modification of $\{h_n(u; z)\}$ which will converge uniformly for $|z| \leq 1$ for any continuous u , and still be a sequence of interpolation polynomials. The result of following this suggestion is

$$(8.7) \quad h_n^*(u; z) = \frac{1}{2\pi} \sum_{j=1}^n u(w_{kn}) \operatorname{Re} \left[\frac{1 + z\bar{w}_{kn}}{1 - z\bar{w}_{kn}} - \frac{2}{n} \frac{z\bar{w}_{kn} - (z\bar{w}_{kn})^{n+1}}{(1 - z\bar{w}_{kn})^2} \right] \frac{2\pi}{n}.$$

This is a harmonic polynomial of degree $n - 1$ which coincides with u at the equally spaced points w_{kn} identified above, and further satisfies the auxiliary conditions

$$\frac{\partial h^*}{\partial \theta}(u; w_{kn}) = 0, \quad k = 1, \dots, n.$$

It is not the arithmetic mean of $h_1, h_2, \dots, h_n$. By inspection of the formula it is apparent that for $r < 1$ the sequence $\{h_n^*\}$ converges to the Poisson integral formula operating on u , as does $\{h_n\}$. It is also known ([8], pp. 143–145) that if u is continuous, then $\{h_n\}$ converges uniformly to u on $|z| = 1$. This fact, when taken with the principle of the maximum for harmonic functions, can be used to show that for every continuous u the sequence $\{h_n^*(u; z)\}$ converges uniformly to the solution of the Dirichlet problem on the closed region $|z| \leq 1$.

The results in this section concerning harmonic polynomial interpolation in the unit circle are an implementation of brief suggestions made by WALSH [16] in connection with the question stated at the beginning of the introduction. In [17] WALSH established the convergence of (8.5) for the case in which $u(z)$ coincides on $|z| = 1$ with a function harmonic on a disc $|z| \leq \rho > 1$.

9. Conclusion. This paper discussed in a peripheral way the theory of a possible method for getting polynomial approximations to the solution of Dirichlet problems by interpolation to the boundary value data. Since there is not much known at present about the main convergence questions for the method, an analogous but more accessible problem for complex polynomial approximation was examined in some detail, partly for its own interest and partly to serve as a basis for conjectures.

The complex problem suggests that the following surmises concerning the harmonic problem may be correct:

(1) If the boundary curve C is sufficiently regular, then there almost certainly do exist universal sequences $\{S_N\}$ of sets of interpolation points on C such that the corresponding sequences of harmonic interpolation polynomials converge to the solution of the Dirichlet problem for any continuous boundary value data. The sequences of interpolation points are universal in the sense that they do not depend on the boundary value data—only on C . To put the matter another way, the sequences of harmonic interpolation polynomials “belong” to C .

(2) But it also seems probable that a basic characteristic of the successful sequences $\{S_N\}$ is that they must be the images on C of sequences of certain types of uniformly distributed points on the unit circle under a conformal transformation which maps the exterior of C onto the exterior of the unit circle so that the points at infinity correspond. Therefore to construct a universal sequence $\{S_N\}$ must one not know this conformal map? If one did, there are other possibly better ways to solve Dirichlet problems, at least for the exterior region.

These conjectures are somewhat unfavorable insofar as the possibility of practical applications of harmonic polynomial interpolation are concerned. On the favorable side this can be said: The paper identified a certain type of stability condition (involving the boundedness of the total variation of the kernel of the interpolation polynomial, considered as a linear operator) as the key necessary and sufficient characterization of a successful universal sequence of interpolation points, both in the complex and in the harmonic case. In the harmonic case, the condition appears in (7.2). For any given set of points S_N it would be not difficult to program an electronic computer to calculate the left side of (7.2). It does not seem to be impossible to set up a reasonable numerical bound for the left hand member of (7.2). (In the complex case, (4.9a) shows that if something is known about the derivative of the mapping function, such a bound can be specified.) For each of various conveniently chosen values of N , the computer could calculate the left side of (7.2) over and over for different configurations of interpolation points until a rough minimum lying below the bound is attained. By this empirical method perhaps a reasonably effective universal sequence of points $\{S_N\}$ extending to, say, $N = 25$ or $N = 51$, might be built up once and for all for a given curve C , which should then yield good harmonic polynomial approximations to any Dirichlet problem for that curve.

Finally, what seems to be a completely unexplored question will be mentioned.

This paper has dealt only with the characterization of universal sequences of interpolation points which would yield effective approximations no matter what the function being interpolated to is. It was shown that the convergence problems for interpolation polynomials are somewhat similar to mechanical quadrature problems. In the latter type of problem it can be well worth-while to adjust the quadrature points to the particular integrand. Therefore it might be interesting to see what can be done with developing a theory of non-universal, data-oriented sequences of interpolation points both in the complex case and in the harmonic case.

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