The Non-isentropic Perturbation of an Arbitrary Simple Wave

ROY GUNDERSEN

Communicated by L. BERS

1. Introduction. In a recent paper [1], a method was presented for discussing the perturbation of a given one-dimensional, non-steady flow, assumed isentropic, of an inviscid, ideal compressible gas. It was not assumed that the perturbed flow was isentropic. Explicit solutions were obtained for a basic flow which was uniform or a centered simple wave, but for an arbitrary simple wave only if the perturbed flow was isentropic. In the present paper, it is shown that the non-isentropic perturbation of an arbitrary simple wave can be solved by quadratures.

The equation of state is

$$P = \exp \left\{ \frac{s - s^*}{c_v} \right\} \rho^\gamma$$

and the equations which characterize the flow are

$$\begin{aligned} c_t + uc_x + \frac{1}{2}(\gamma - 1)cu_x &= 0, \\ u_t + uu_x + \frac{2cc_x}{\gamma - 1} - \frac{c^2 s_x}{c_v \gamma (\gamma - 1)} &= 0, \\ s_t + us_x &= 0, \end{aligned}$$

where P is the pressure, ρ the density, s the specific entropy, s^* the specific entropy at some reference state, γ the ratio of the specific heats at constant pressure c_p and at constant volume c_v , u the particle velocity, c the local speed of sound and the subscripts denote partial derivatives.

A formal linearization in the neighborhood of a known isentropic solution, denoted by the subscript zero, gives the following system of linear equations which govern the first approximation to the disturbed flow:

$$(1) \quad c_{1t} + u_0 c_{1x} + \frac{1}{2}(\gamma - 1)c_0 u_{1x} + u_1 c_{0x} + \frac{1}{2}(\gamma - 1)u_0 x c_1 = 0,$$

$$(2) \quad \frac{1}{2}(\gamma - 1)u_{1t} + \frac{1}{2}(\gamma - 1)u_0 u_{1x} + c_0 c_{1x} + \frac{1}{2}(\gamma - 1)u_0 x u_1 + c_1 c_{0x} = \frac{c_0^2 s_{1x}}{2c_v \gamma},$$

$$(3) \quad s_{1t} + u_0 s_{1x} = 0,$$

where the subscript one denotes the terms of first order.

According to (3), s_1 remains constant along the particle paths of the given flow, *i.e.*, along $dx/dt = u_0$. Also, $\rho_0(dx - u_0 dt)$ is the exact differential of a function ψ_0 which, when equated to a constant, defines the particle paths. Hence the solution of (3) is

$$s_1 = \omega(\psi_0)$$

with ω an arbitrary function. It is convenient to define a new function $H_0(x, t)$ by

$$(4) \quad c_0^2 s_{1x} = \rho_0 c_0^2 \omega'(\psi_0) = \gamma(\gamma - 1) c_x H_0.$$

Thus the two equations which serve to determine u_1 , c_1 are (1) and

$$(5) \quad u_{1t} + u_0 u_{1x} + \frac{2c_0 c_{1x}}{\gamma - 1} + u_{0x} u_1 + \frac{2c_{0x} c_1}{\gamma - 1} = H_0.$$

It is convenient to introduce the characteristic parameters of the basic flow

$$\frac{1}{2}u_0 + \frac{c_0}{\gamma - 1} = \alpha, \quad -\frac{1}{2}u_0 + \frac{c_0}{\gamma - 1} = \beta$$

and the functions

$$\frac{1}{2}u_1 + \frac{c_1}{\gamma - 1} = A, \quad -\frac{1}{2}u_1 + \frac{c_1}{\gamma - 1} = B.$$

The combinations $(1)/(\gamma - 1) \pm \frac{1}{2}(5)$ give

$$(6) \quad A_t + (u_0 + c_0)A_x + \frac{1}{2}[A(\gamma + 1) + (\gamma - 3)B]\alpha_x = \frac{1}{2}H_0,$$

$$(7) \quad B_t + (u_0 - c_0)B_x + \frac{1}{2}[A(3 - \gamma) - (\gamma + 1)B]\beta_x = -\frac{1}{2}H_0.$$

2. Formulation of the Problem. For the simple wave, one of the characteristic parameters (α, β) of the basic flow remains constant, *e.g.*, $\beta = \beta_0$. B will be determined from (7) and then A from (6).

Let $(x_0(z), t_0(z))$ be the parametric representation of a curvilinear characteristic of the arbitrary simple wave. The wave may then be represented by

$$(8) \quad x = x_0(z) + [u_0(z) + c_0(z)]\tau, \quad t = t_0(z) + \tau,$$

$$(9) \quad -\frac{1}{2}u_0(z) + \frac{c_0(z)}{\gamma - 1} = \beta_0.$$

The particle paths are given by $\rho_0 c_0 \tau = \text{const.}$ or $y \equiv c_0^2 \tau^{2\theta} = \text{const.}$, where $\theta = (\gamma - 1)/(\gamma + 1)$. From (4),

$$\gamma(\gamma - 1)c_x H_0 = (\rho_0 c_0 \tau) \frac{c_0}{\tau} \omega'(y),$$

so that it is convenient to write

$$H_0 = 2 \frac{c_0}{\tau} \Omega'(y)$$

where Ω is an arbitrary differentiable function.

3. Solution. Equation (7) becomes

$$(10) \quad B_t + (u_0 - c_0)B_x = -\frac{c_0 \Omega'(y)}{\tau}.$$

It is convenient to change from the independent variables (x, t) to (z, τ) in (10). With $B(x, t) \equiv B_1(z, \tau)$, the result is

$$(11) \quad (u'_0 + c'_0)\tau B_{1\tau} - 2c_0 B_{1z} = -\frac{Jc_0 \Omega'(y)}{\tau},$$

where the Jacobian has the value

$$J = \frac{c'_0 \tau}{\theta} - 2c_0 t'_0$$

and primes denote differentiation with respect to the argument. From the definition of the characteristic parameters, $u_0 + c_0 = c_0/\theta - 2\beta_0$, so that (11) becomes

$$(12) \quad \frac{c'_0 \tau B_{1\tau}}{\theta} - 2c_0 B_{1z} = -\frac{Jc_0 \Omega'(y)}{\tau}.$$

The solution to (12) is obtained from the system

$$(13) \quad \frac{\theta}{\tau} \frac{d\tau}{dc_0} = \frac{dz}{-2c_0} = \frac{\tau dB_1}{-Jc_0 \Omega'(y)}.$$

The following first integral is obtained from the first two ratios of (13):

$$(14) \quad c_0 = K\tau^{-2\theta} \quad \text{or} \quad \rho_0 c_0 \tau^2 = \text{const.}$$

where K is a constant. It will be seen that this is a convenient representation of the curvilinear characteristics of the wave.

From the last two ratios of (13)

$$2 \frac{dB_1}{dz} = \left[\frac{\tau}{\theta} \frac{dc_0}{dz} - 2c_0 \frac{dt_0}{dz} \right] \frac{1}{\tau} \Omega'[c_0^2 \tau^{2\theta}]$$

or

$$(15) \quad 2 dB_1 = \frac{dc_0}{\theta} \Omega'[c_0^2 \tau^{2\theta}] - \frac{2c_0}{\tau} dt_0 \Omega'[c_0^2 \tau^{2\theta}].$$

By the use of the first integral (14), (15) yields a second first integral:

$$(16) \quad B_1 = \frac{\Omega[K^2 \tau^{-2\theta}]}{2\theta K} + \int \left\{ \frac{d}{d\tau} \Omega[K^2 \tau^{-2\theta}] \right\} \frac{dt_0}{2\theta K}.$$

The first integral (14) gives a relation between z and τ , so that an integration by parts transforms (16) to

$$(17) \quad B_1 = \frac{\Omega[K^2 \tau^{-2\theta}]}{2\theta} - \frac{t'_0}{c'_0} \tau^{-2\theta-1} \Omega[K^2 \tau^{-2\theta}] + \int \Omega[K^2 \tau^{-2\theta}] d \left[\frac{t'_0}{c'_0} \tau^{-2\theta-1} \right].$$

From the two first integrals (14) and (17), the solution of equation (10) is

$$(18) \quad \begin{aligned} B_1 = & 2(\rho_0 c_0)^{\frac{1}{2}} (t - t_0) F'[\rho_0 c_0 (t - t_0)^2] + \frac{c_0 \Omega(y)}{2\theta y} \\ & - \Omega(y) (t - t_0)^{-2\theta-1} \frac{t'_0}{c'_0} + R(z, \tau), \end{aligned}$$

where $R(z, \tau)$ denotes the result obtained by performing the quadrature indicated in equation (17) and then replacing K by $c_0 \tau^{2\theta}$ from the first integral (9), and F is an arbitrary differentiable function.

The solution of (6) is obtained from the system

$$(19) \quad dt = \frac{dx}{u_0 + c_0} = \frac{2 dA}{H_0 + [(3 - \gamma)B - (\gamma + 1)A]\alpha_x}.$$

The integral of the first equation of (19) has the first family of characteristics (rectilinear characteristics) for level curves. Let this first integral be

$$(20) \quad z(x, t) = a, \quad \text{a constant.}$$

This function is defined implicitly by

$$x - x_0(z) = [u_0(z) + c_0(z)][t - t_0(z)].$$

From the definition of the characteristic parameters

$$(\gamma + 1)\alpha = 2[u_0 + c_0 - \frac{1}{2}(\gamma - 3)]\beta_0.$$

Substitution in (19) gives a differential equation for A . To integrate this equation, B is substituted from (18) and the first integral (20) is utilized, *i.e.*, z may be treated as a constant. This equation is

$$(21) \quad \begin{aligned} \frac{d}{dt} \left[\left(t - t_0 - \frac{2\theta c_0 t'_0}{c'_0} \right) A \right] - \frac{3 - \gamma}{\gamma + 1} \frac{d}{dt} \left\{ \frac{F[\rho_0 c_0 (t - t_0)^2]}{(\rho_0 c_0)^{\frac{1}{2}}} \right\} \\ - \frac{3 - \gamma}{\gamma + 1} R(a, t - t_0) \\ = \left(t - t_0 - \frac{2c_0 \theta t'_0}{c'_0} \right) \left[\frac{c_0 \Omega'(y)}{t - t_0} + \frac{3 - \gamma}{2(\gamma - 1)} \frac{c_0}{y} \frac{\Omega(y)}{(t - t_0)} \right]. \end{aligned}$$

Integrating (21) and performing an integration by parts yields a second first integral:

$$\begin{aligned}
 (22) \quad \left(t - t_0 - \frac{2c_0\theta t'_0}{c'_0}\right)A &= \frac{3 - \gamma}{\gamma + 1} \frac{F[\rho_0 c_0(t - t_0)^2]}{(\rho_0 c_0)^{\frac{1}{2}}} \\
 &+ \frac{3 - \gamma}{\gamma + 1} \int R(a, t - t_0) dt + \frac{c_0}{2\theta y} \Omega(y) \left(t - t_0 - \frac{2c_0\theta t'_0}{c'_0}\right) \\
 &- \frac{1}{2\theta c_0} \int \Omega(y) \frac{d}{dt} \left[\left(t - t_0 - \frac{2c_0\theta t'_0}{c'_0}\right) (t - t_0)^{-2\theta} \right] dt \\
 &+ \frac{3 - \gamma}{2(\gamma - 1)c_0} \int \Omega(y) \left(t - t_0 - \frac{2c_0\theta t'_0}{c'_0}\right) (t - t_0)^{-2\theta-1} dt.
 \end{aligned}$$

Hence the solution for A is

$$\begin{aligned}
 (23) \quad \left(t - t_0 - \frac{2c_0\theta t'_0}{c'_0}\right)A &= \frac{3 - \gamma}{\gamma + 1} \frac{F[\rho_0 c_0(t - t_0)^2]}{(\rho_0 c_0)^{\frac{1}{2}}} \\
 &+ \frac{c_0 \Omega(y)}{2\theta y} \left(t - t_0 - \frac{2c_0\theta t'_0}{c'_0}\right) + G(z) + \left\{ \frac{3 - \gamma}{\gamma + 1} \int R(a, t - t_0) dt \right. \\
 &- \frac{1}{2\theta c_0} \int \Omega(y) \frac{d}{dt} \left[\left(t - t_0 - \frac{2c_0\theta t'_0}{c'_0}\right) (t - t_0)^{-2\theta} \right] dt \\
 &\left. + \frac{3 - \gamma}{2(\gamma - 1)c_0} \int \Omega(y) \left(t - t_0 - \frac{2c_0\theta t'_0}{c'_0}\right) (t - t_0)^{-2\theta-1} dt \right\}_{a=z},
 \end{aligned}$$

where the notation $\{ \}_{a=z}$ indicates that the integrations are to be carried out with z replaced by the constant a , and then a is to be replaced by the variable z to give the solution (23), and G is an arbitrary function.

Equations (18) and (23) give the solution of equations (6) and (7) in terms of three arbitrary functions F , G and Ω . Theoretically these functions may be determined in an actual problem, as is easily seen by a generalization of the discussion given in paragraph 6 of [1].

To obtain the solution for a wave centered at $x = 0$, $t = 0$, let $z = x/t$ and $t_0 = 0$. This is seen to agree with results presented in [1].

This work was supported by a grant from the National Science Foundation.

REFERENCE

- [1] R. M. GUNDERSEN, The flow of a compressible fluid with weak entropy changes, *J. Fluid Mech.* **3** (1958).

Illinois Institute of Technology
Chicago, Illinois