Polynomials Associated with Divisors*

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Introduction. Let X be an irreducible, normal, projective variety of dimension r with sheaf of local rings L , and defined over an arbitrary groundfield. We select n , *not necessarily distinct*, divisors $D_1, \dots, D_n$ of X which are all locally linearly equivalent to zero, and we select an algebraic coherent sheaf F of X . We proved in Theorem 9.1 of [3] (square brackets refer to the references) that the Euler-Poincaré characteristic $\chi(X, F(x_1 D_1 + \dots + x_n D_n))$ is a rational polynomial in $x_1, \dots, x_n$ for all $x_1, \dots, x_n \in \mathbb{Z}$. The purpose of the present paper is to investigate the coefficients of this polynomial. (*Rational polynomial* means a polynomial with rational numbers as coefficients.)

Let Q be the field of rational numbers. We select n nonzero elements $h_1, \dots, h_n \in Q$, define the polynomials $C(x_1, i_1, h_1), \dots, C(x_n, i_n, h_n)$ as in Definition 1.1, and denote the product of these n rational polynomials by $[i_1, \dots, i_n]$. The polynomials $[i_1, \dots, i_n]$, for $i_1, \dots, i_n \geq 0$, form a linearly independent Q -basis of $Q[x_1, \dots, x_n]$, *i.e.* if $\alpha \in Q[x_1, \dots, x_n]$, then $\alpha = \sum a(i_1, \dots, i_n)[i_1, \dots, i_n]$, where the rational numbers $a(i_1, \dots, i_n)$ are uniquely determined by α . The rational polynomials α we are concerned with evidently have the special property that they take on integral values for integral values of the variables. We conclude easily from Chapter II of [1] that, for polynomials with this property, all $a(i_1, \dots, i_n)$ become integers if each "increment" h_i is chosen equal to $+1$ or -1 .

If $F = L$ and $n = 1$ and D is a divisor cut out on X by a hyperplane of the ambient projective space, the polynomial $\chi(X, L(xD))$ is the classical Hilbert polynomial of X . It is customary to expand this polynomial in terms of the "binomial coefficients" $\binom{x}{0} = 1, \binom{x}{1}, \dots, \binom{x}{r}$. Precisely then, it is customary to choose the increment $h = 1$, and to put $\chi(X, L(xD)) = \sum a(i)[i]$, where the summation index i runs from 0 to r . (Observe that now $[i] = C(x, i, 1) = \binom{x}{i}$.)

The result is unsatisfactory because only the coefficients $a(0)$ and $a(r)$ have good geometric meaning. We have observed that if we use the increment $h = -1$ instead of $h = +1$, $a(0)$ and $a(r)$ remain unchanged, but all the intermediate

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coefficients $a(i)$ also acquire good geometric interpretations, namely as virtual arithmetic genera of well defined subvarieties of X .

On the grounds of this experience, we decided to study the expansion $\chi(X, F(x_1 D_1 + \cdots + x_n D_n)) = \sum a(i_1, \dots, i_n) [i_1, \dots, i_n]$, choosing all the increments $h_1, \dots, h_n$ equal to -1 , and this certainly paid off. The coefficient $a(1, \dots, 1)$ of this expansion turns out to be the correct *definition* of the virtual arithmetic genus $p_a(D_1, \dots, D_n, F)$ of the set of divisors $D_1, \dots, D_n$ (with values in F). This definition, together with the theory of expansion of polynomials in factorials, enables us in section 3 to establish the connections between virtual arithmetic genera and Euler-Poincaré characteristics, and the connections between the virtual arithmetic genera of sums and of intersections of divisors. In section 4 we show that if $n = r$, then $p_a(D_1, \dots, D_r, L)$ is the correct *definition* of the intersection number of the set of divisors $D_1, \dots, D_r$; in particular, if $D_1 = \cdots = D_r$, this intersection number is the correct *definition* of the virtual degree of D_1 . We observe that these definitions are not based on any previous local intersection theory. Effortlessly, classical formulas which interrelate virtual arithmetic genera, virtual degrees and intersection numbers of divisors on surfaces roll out as special cases of formulas which hold for arbitrary varieties. Examples 4.1 and 4.2 imply that we have enough information to compute rigorously the coefficients of the polynomial $\chi(X, F(x_1 D_1 + \cdots + x_n D_n))$ in many cases. The Hilbert polynomial is discussed in Example 4.4. Section 5 interrelates the virtual arithmetic genus of a divisor D of X with that of $-D$, and section 6 discusses canonical divisors. Here we find generalizations to arbitrary varieties of classical formulas which are concerned with intersection numbers of divisors with canonical divisors on surfaces. Finally, in section 7, we derive painlessly various forms of the Riemann-Roch theorem for curves, surfaces and higher-dimensional varieties.

The paper is completely formulated in terms of line classes, introduced in [3]. In this formulation, many theorems remain valid for reducible varieties and we obtain the further advantage that all definitions (as, for example, the definition of intersection number) are invariant under linear equivalence. The translation of the results into the language of divisors is immediate, as follows from [3]. For reasons set forth in the introduction of [3], a large part of the present paper should be valid for arbitrary complete varieties.

The necessary material on expansion of polynomials in factorials is collected in the first two sections. These sections consist in large part of extensions of results of [1] to polynomials in several variables. Since these extensions are immediate, we omitted proofs whenever the one variable case was treated in [1]. We changed some of the notations of [1], in order to avoid ambiguities and to facilitate printing. We found this book very clearly written and derived a great deal of help from it.

If we refer to, say, "Proposition 1.5" without further designation, we are referring to Proposition 5 of section 1 of the present paper. The support of the sheaf F is denoted by $\text{Sup}(F)$ and the dimension of this support by $\dim(\text{Sup}(F))$.

1. Finite differences. Z denotes the ring of rational integers and Q the field of rational numbers. The term "rational polynomial" designates a polynomial whose coefficients belong to Q .

Definition 1.1. Let $i \in Z$ and let h be a nonzero element of Q . The rational polynomial $C(x, i, h) \in Q[x]$ is defined as follows. If $i < 0$, $C(x, i, h) = 0$; if $i = 0$, $C(x, 0, h) = 1$; if $i > 0$, $C(x, i, h) = x(x - h)(x - 2h) \cdots (x - ih + h)/i!$.

We see that the degree of $C(x, i, h)$ is i for all $i \geq 0$ and that if $0 \leq i \leq m \in Z$, then $C(m, i, 1)$ is the usual binomial coefficient $\binom{m}{i}$. The polynomial $C(x, i, h)$, for $i \geq 0$, is denoted by $x^{(ih)}/i!$ in section 2.11 of [1].

We now study the polynomial ring $Q[x_1, \dots, x_n]$ in n variables $x_1, \dots, x_n$. We choose a fixed set of nonzero elements $h_1, \dots, h_n \in Q$, and denote the product of the n rational polynomials $C(x_1, i_1, h_1), \dots, C(x_n, i_n, h_n)$ by the symbol $[i_1, \dots, i_n]$. Hence $[i_1, \dots, i_n] \in Q[x_1, \dots, x_n]$ for all $i_1, \dots, i_n \in Z$ and $[i_1, \dots, i_n] = 0$ if one of the indices $i_i < 0$. The Q -vector space $Q[x_1, \dots, x_n]$ has the polynomials $[i_1, \dots, i_n]$, for $i_1, \dots, i_n \geq 0$, as a linearly independent Q -basis. Precisely, let $\alpha \in Q[x_1, \dots, x_n]$. There exist unique $a(i_1, \dots, i_n) \in Q$ such that $\alpha = \sum a(i_1, \dots, i_n)[i_1, \dots, i_n]$, where the summation is over all $i_1, \dots, i_n \geq 0$. In order to compute these coefficients $a(i_1, \dots, i_n)$ in terms of the given α , we denote by $\delta(x_i, m_i, h_i)(\alpha)$ the partial difference quotient of α with respect to x_i taken m_i times with increment h_i ; here $0 \leq m_i \in Z$ and $\delta(x_i, 0, h_i)(\alpha) = \alpha$. The operator $\delta(x_i, m_i, h_i)$ is designated by the symbol $\Delta_{x_i}^{m_i, h_i}$ in section 2.105 of [1]. The reason why we postulated that $h_i \neq 0$ was to insure that the operator $\delta(x_i, m_i, h_i)$ would be well defined. We denote by $\{m_1, \dots, m_n\}$ the operator which results from applying successively the n operators $\delta(x_1, m_1, h_1), \dots, \delta(x_n, m_n, h_n)$; the order of these operators is immaterial and $0 \leq m_1, \dots, m_n \in Z$.

Proposition 1.1. $\{m_1, \dots, m_n\}(\alpha) = \sum a(i_1, \dots, i_n)[i_1 - m_1, \dots, i_n - m_n]$ for all $0 \leq m_1, \dots, m_n \in Z$; the summation is over all $i_1, \dots, i_n \geq 0$. Consequently,

$$a(m_1, \dots, m_n) = (\{m_1, \dots, m_n\}(\alpha))(0, \dots, 0)$$

for all $0 \leq m_1, \dots, m_n \in Z$.

Proof. $(\{m_1, \dots, m_n\}(\alpha))(0, \dots, 0)$ designates the rational number obtained by substituting $x_1 = \dots = x_n = 0$ in the polynomial $\{m_1, \dots, m_n\}(\alpha)$. The proof for $n = 1$ is contained in section 2.12 of [1].

Proposition 1.2. For all $0 \leq m_1, \dots, m_n \in Z$,

$$\begin{aligned} \{m_1, \dots, m_n\}(\alpha) &= (1/h_1^{m_1} \cdots h_n^{m_n}) \sum (-1)^{i_1 + \cdots + i_n} \\ &\quad \cdot \binom{m_1}{i_1} \cdots \binom{m_n}{i_n} \alpha(x_1 + (m_1 - i_1)h_1, \dots, x_n + (m_n - i_n)h_n) \end{aligned}$$

where the summation is over $i_1, \dots, i_n$, each index i_i running from 0 to m_i .

Proof. This formula, which is also correct if α is not a polynomial, is the extension to n variables of formula 2 on page 33 of [1].

Proposition 1.3. If $h_1 = \dots = h_n = -1$ then, for all $0 \leq m_1, \dots, m_n \in \mathbb{Z}$,

$$a(m_1, \dots, m_n) = \sum (-1)^{i_1 + \dots + i_n} \binom{m_1}{i_1} \dots \binom{m_n}{i_n} \alpha(-i_1, \dots, -i_n)$$

where the summation is over $i_1, \dots, i_n$, each index i_i running from 0 to m_i .

Proof. If we substitute $x_1 = \dots = x_n = 0$ in the equality of Proposition 1.2, the left hand side becomes $a(m_1, \dots, m_n)$, according to Proposition 1.1. Since $\binom{m_i}{i_i} = \binom{m_i}{m_i - i_i}$ for $j = 1, \dots, n$ and

$$(-1)^{i_1 + \dots + i_n + m_1 + \dots + m_n} = (-1)^{(m_1 - i_1) + \dots + (m_n - i_n)},$$

we can use the new dummy indices $m_1 - i_1, \dots, m_n - i_n$ in the right hand side, which gives the required expression.

The last proposition expresses the coefficients $a(m_1, \dots, m_n)$ in terms of α . The next proposition tells us how α can be expressed in terms of these coefficients.

Proposition 1.4. If $h_1 = \dots = h_n = -1$ then, for all $0 \leq m_1, \dots, m_n \in \mathbb{Z}$,

$$\alpha(-m_1, \dots, -m_n) = \sum (-1)^{i_1 + \dots + i_n} \binom{m_1}{i_1} \dots \binom{m_n}{i_n} a(i_1, \dots, i_n)$$

where the summation is over $i_1, \dots, i_n$, each index i_i running from 0 to m_i .

Proof. If we extend Gregory's Theorem on page 33 of [1] to n variables, we obtain the formula which expresses $\alpha(x_1 + m_1 h_1, \dots, x_n + m_n h_n)$ in terms of the functions $\{i_1, \dots, i_n\}(\alpha)$. If we substitute $x_1 = \dots = x_n = 0$ and $h_1 = \dots = h_n = -1$ in this formula, and use from Proposition 1.1 that

$$(\{i_1, \dots, i_n\}(\alpha))(0, \dots, 0) = a(i_1, \dots, i_n),$$

we obtain Proposition 1.4.

We need the following observation concerning the case that $n = 1$. We assume that $\alpha \in \mathbb{Q}[x]$ and that the degree of α does not exceed r . We are going to compare the coefficients of the two expansions $\alpha = \sum a(i)C(x, i, -1)$ and $\alpha = \sum b(i)C(x, i, 1)$; the summation index i runs from 0 to r .

Proposition 1.5. $b(0) = a(0)$ and

$$b(i) = \sum \binom{j-1}{i-1} a(j)$$

for $1 \leq i \leq r$; the summation index j runs from i to r . In particular, $b(r) = a(r)$.

Proof. Since $b(0) = a(0) = \alpha(0)$ is obvious, we assume that $1 \leq i \leq r$. If we expand $C(x, i, -1) = \sum d(j)C(x, j, 1)$ where j runs from 1 to i , we conclude from Proposition 1.1 that

$$d(j) = (\delta(x, j, 1)(C(x, i, -1)))(0).$$

We see by direct computation that

$$\delta(x, j, 1)(C(x, i, -1)) = (x + j)(x + j + 1) \cdots (x + i - 1)/(i - j)!,$$

whence $d(j) = \binom{i-1}{j-1}$ for $1 \leq j \leq i$. If we substitute the resulting expansion for $C(x, i, -1)$ in $\alpha = \sum a(i)C(x, i, -1)$, we obtain Proposition 1.5.

2. The case $h_1 = \cdots = h_n = -1$. We continue the investigation of the base elements $[i_1, \cdots, i_n]$ of $Q[x_1, \cdots, x_n]$. We assume for the remainder of this paper that in the symbol $[i_1, \cdots, i_n]$ the increments $h_1, \cdots, h_n$ are always equal to -1 .

As in section 17.1 of [2], we introduce n new variables $u_1, \cdots, u_n$ and consider the ring $R = Q[u_1, \cdots, u_n, u_1^{-1}, \cdots, u_n^{-1}]$. An element $\rho \in R$ can be written uniquely as

$$\rho = \sum q(i_1, \cdots, i_n) u_1^{i_1} \cdots u_n^{i_n},$$

where $q(i_1, \cdots, i_n) \in Q$ and $i_1, \cdots, i_n \in Z$; of course, $i_1, \cdots, i_n$ may very well be negative. For any polynomial $\alpha \in Q[x_1, \cdots, x_n]$, we define the value $\rho(\alpha)$ of ρ at α by

$$\rho(\alpha) = \sum q(i_1, \cdots, i_n) \alpha(i_1, \cdots, i_n) \in Q.$$

Hence with each $\rho \in R$ we have associated a mapping from $Q[x_1, \cdots, x_n]$ into Q .

Proposition 2.1. *If $\alpha \in Q[x_1, \cdots, x_n]$ and $\alpha = \sum a(i_1, \cdots, i_n)[i_1, \cdots, i_n]$ then, for all $0 \leq m_1, \cdots, m_n \in Z$,*

$$a(m_1, \cdots, m_n) = ((1 - u_1^{-1})^{m_1} \cdots (1 - u_n^{-1})^{m_n})(\alpha).$$

Proof. Let $\rho = (1 - u_1^{-1})^{m_1} \cdots (1 - u_n^{-1})^{m_n}$. If we expand ρ by means of the binomial theorem in terms of $u_1^{i_1} \cdots u_n^{i_n}$, then $\rho(\alpha)$ becomes precisely equal to the sum which occurs in Proposition 1.3.

Let $\beta(x_2, \cdots, x_n) \in Q[x_2, \cdots, x_n]$ be a polynomial in which x_1 does not occur. Let $\alpha = \beta(x_1 + x_2, \cdots, x_n)$ be the polynomial of $Q[x_1, \cdots, x_n]$ which is obtained from β by means of the substitution $x_2 \rightarrow x_1 + x_2$. We are going to compare the coefficients of the two expansions $\alpha = \sum a(i_1, \cdots, i_n)[i_1, \cdots, i_n]$ and $\beta = \sum b(i_2, \cdots, i_n)[i_2, \cdots, i_n]$.

Proposition 2.2. *$a(m_1, m_2, \cdots, m_n) = b(m_1 + m_2, \cdots, m_n)$ for all $0 \leq m_1, \cdots, m_n \in Z$.*

Proof. For any $\rho \in R$, $\rho(\alpha) = \rho'(\beta)$ where ρ' is obtained from ρ by means of the substitution $u_1 = u_2$. Consequently,

$$\begin{aligned} a(m_1, \dots, m_n) &= ((1 - u_1^{-1})^{m_1} \cdots (1 - u_n^{-1})^{m_n})(\alpha) \\ &= ((1 - u_2^{-1})^{m_1 + m_2} \cdots (1 - u_n^{-1})^{m_n})(\beta) = b(m_1 + m_2, \dots, m_n). \end{aligned}$$

We need one more proposition of the same kind as 2.2. We consider a polynomial $\beta(x_0, x_1, \dots, x_n) \in Q[x_0, x_1, \dots, x_n]$ in $n + 1$ variables $x_0, x_1, \dots, x_n$. Let $\alpha = \beta(sx_1, tx_1, \dots, x_n)$ be the polynomial of $Q[x_1, \dots, x_n]$ which is obtained from β by means of the substitution $x_0 \rightarrow sx_1$ and $x_1 \rightarrow tx_1$ for some fixed $s, t \in Q$. Again, $\alpha = \sum a(i_1, \dots, i_n)[i_1, \dots, i_n]$ and $\beta = \sum b(i_1, \dots, i_n)[i_1, \dots, i_n]$.

Proposition 2.3. *If the degree of β does not exceed n , then*

$$a(1, \dots, 1) = sb(1, 0, 1, \dots, 1) + tb(0, 1, \dots, 1).$$

Here all the indices which are not filled in are meant to be equal to 1.

Proof. $a(1, \dots, 1)$ is the coefficient of the monomial $[1, \dots, 1] = x_1 \cdots x_n$ in the expansion $\alpha = \sum a(i_1, \dots, i_n)[i_1, \dots, i_n]$. We assert that $a(1, \dots, 1)$ is also the coefficient of the monomial $x_1 \cdots x_n$ in the usual expansion $\alpha = \sum q(i_1, \dots, i_n)x_1^{i_1} \cdots x_n^{i_n}$. Namely, since the degree of the polynomial $[i_1, \dots, i_n]$ is equal to $i_1 + \cdots + i_n$ and since the degree of α cannot exceed n , only the terms $a(i_1, \dots, i_n)[i_1, \dots, i_n]$ with $i_1 + \cdots + i_n = n$ can possibly contain the monomial $x_1 \cdots x_n$. Furthermore, if $i_1 + \cdots + i_n = n$, then $[i_1, \dots, i_n]$ contains the monomial $x_1 \cdots x_n$ only if $i_1 = \cdots = i_n = 1$, which proves the italicized assertion. We now consider the effect of the substitution in question on the expansion $\beta = \sum b(i_0, \dots, i_n)[i_0, \dots, i_n]$. We denote by $[i_0, \dots, i_n]'$ the polynomial of $Q[x_1, \dots, x_n]$ which arises from $[i_0, \dots, i_n]$ by means of this substitution, and hence $\alpha = \sum b(i_0, \dots, i_n)[i_0, \dots, i_n]'$. The same degree arguments as above show that in this expansion only those terms for which $i_0 + \cdots + i_n = n$ can contain the monomial $x_1 \cdots x_n$. Furthermore, if $i_0 + \cdots + i_n = n$, only the two polynomials $[1, 0, 1, \dots, 1]'$ and $[0, 1, \dots, 1]'$ involve all the variables $x_1, \dots, x_n$. Since

$$\begin{aligned} b(1, 0, 1, \dots, 1)[1, 0, 1, \dots, 1]' + b(0, 1, \dots, 1)[0, 1, \dots, 1]' \\ = (sb(1, 0, 1, \dots, 1) + tb(0, 1, \dots, 1))x_1 \cdots x_n, \end{aligned}$$

the coefficient of the monomial $x_1 \cdots x_n$ in the expansion $\alpha = \sum q(i_1, \dots, i_n)x_1^{i_1} \cdots x_n^{i_n}$ is also equal to $sb(1, 0, 1, \dots, 1) + tb(0, 1, \dots, 1)$, and we are done.

3. Arithmetic genera. *We assume for the remainder of this paper that X is a projective variety, defined over an arbitrary ground field. X may be reducible, and we denote the sheaf of local rings of X by L . Let F be an algebraic coherent sheaf of X and $f_1, \dots, f_n$ a set of line classes of X . It is not necessary that $f_1, \dots, f_n$ are mutually distinct. The rational polynomial $\chi(X, f_1^{e_1} \cdots f_n^{e_n}(F))$,*

discussed in Theorem 9.1 of [3], is denoted by α . Hence $\alpha \in Q[x_1, \dots, x_n]$, and we are going to study the expansion $\alpha = \sum a(i_1, \dots, i_n)[i_1, \dots, i_n]$.

Definition 3.1. The coefficient $a(1, \dots, 1)$ of the monomial $[1, \dots, 1] = x_1 \dots x_n$ is called the virtual arithmetic genus $p_a(f_1, \dots, f_n, F)$ of the sequence of line classes $f_1, \dots, f_n$ with values in F .

In the notation of Proposition 2.1,

$$p_a(f_1, \dots, f_n, F) = ((1 - u_1^{-1}) \dots (1 - u_n^{-1}))(\alpha).$$

This shows that $p_a(f_1, \dots, f_n, F) \in \mathbb{Z}$ and that $p_a(f_1, \dots, f_n, L)$ is precisely the “virtual arithmetic genus of the virtual subvariety $(f_1, \dots, f_n)$ ” defined on page 133 of [2].

The word “sequence” in Definition 3.1 is in need of explanation. It is obvious that $p_a(f_1, \dots, f_n, F)$ does not depend on the order of $f_1, \dots, f_n$. However, it does depend on how often each f_i occurs in the sequence $f_1, \dots, f_n$. Hence, instead of saying “sequence” in Definition 3.1, we could have said “set with multiplicities.” In general, if $g_1, \dots, g_s$ are line classes of X , we denote by $p_a(g_1^{[t_1]}, \dots, g_s^{[t_s]}, F)$ the virtual arithmetic genus with values in F of the sequence $g_1, \dots, g_1, \dots, g_s, \dots, g_s$, where each g_i occurs t_i times. Clearly, if $g_1 = g_2$, then

$$p_a(g_1^{[t_1]}, g_1^{[t_2]}, \dots, g_s^{[t_s]}, F) = p_a(g_1^{[t_1+t_2]}, \dots, g_s^{[t_s]}, F)$$

and similarly if $g_i = g_j$ for any $1 \leq i, j \leq s$. We define

$$p_a(g_1^{[0]}, \dots, g_s^{[t_s]}, F) = p_a(g_2^{[t_2]}, \dots, g_s^{[t_s]}, F)$$

and similarly if $t_i = 0$ for some $1 \leq j \leq s$. Finally, we define

$$p_a(g_1^{[0]}, \dots, g_s^{[0]}, F) = \chi(X, F)$$

and hence the symbol $p_a(g_1^{[t_1]}, \dots, g_s^{[t_s]}, F)$ is well defined for all $0 \leq t_1, \dots, t_s \in \mathbb{Z}$. We now return to the given set of line classes $f_1, \dots, f_n$ and the expansion $\alpha = \sum a(i_1, \dots, i_n)[i_1, \dots, i_n]$.

Theorem 3.1. $a(m_1, \dots, m_n) = p_a(f_1^{[m_1]}, \dots, f_n^{[m_n]}, F)$ for all $0 \leq m_1, \dots, m_n \in \mathbb{Z}$.

Proof. Case 1. $m_1, \dots, m_n \geq 1$. Let α' be the rational polynomial in the $m_1 + \dots + m_n$ variables $x_{11}, \dots, x_{1m_1}, \dots, x_{n1}, \dots, x_{nm_n}$ which is defined by

$$\alpha' = \chi(X, f_1^{x_{11}} \dots f_1^{x_{1m_1}} \dots f_n^{x_{n1}} \dots f_n^{x_{nm_n}}(F)).$$

where we write momentarily $m(1)$ for m_1 and $m(n)$ for m_n . We consider the expansion $\alpha' = \sum a'(i_{11}, \dots, i_{nm_n})[i_{11}, \dots, i_{nm_n}]$ and we have to show that $a'(1, \dots, 1) = a(m_1, \dots, m_n)$. Accordingly we observe that α' is obtained from α by means of the substitution $x_i \rightarrow x_{i1} + \dots + x_{im_i}$ for $j = 1, \dots, n$. Repeated application of Proposition 2.2 now gives the required result immediately.

Case 2. $m_1 = \cdots = m_n = 0$. It is trivial that $a(0, \cdots, 0) = \chi(X, F)$, and we have defined that $p_a(f_1^{[0]}, \cdots, f_n^{[0]}, F) = \chi(X, F)$.

Case 3. $m_i = 0$ for some $1 \leq j \leq n$, but $m_1 + \cdots + m_n > 0$. Since $p_a(f_1^{[m_1]}, \cdots, f_n^{[m_n]}, F)$ is independent of the order of $f_1, \cdots, f_n$, we may assume that $m_1 = \cdots = m_s = 0$ and $m_{s+1}, \cdots, m_n \geq 1$ for some $1 \leq s < n$. According to our definitions,

$$p_a(f_1^{[0]}, \cdots, f_s^{[0]}, \cdots, f_n^{[m_n]}, F) = p_a(f_{s+1}^{[m_{s+1}]}, \cdots, f_n^{[m_n]}, F).$$

We conclude from Case 1 that $p_a(f_{s+1}^{[m_{s+1}]}, \cdots, f_n^{[m_n]}, F) = b(m_{s+1}, \cdots, m_n)$, where

$$\chi(X, f_{s+1}^{x_{s+1}} \cdots f_n^{x_n}(F)) = \sum b(i_{s+1}, \cdots, i_n) [i_{s+1}, \cdots, i_n].$$

This last polynomial is obtained from α by means of the substitution $x_1 = \cdots = x_s = 0$, whence $b(m_{s+1}, \cdots, m_n) = a(0, \cdots, 0, m_{s+1}, \cdots, m_n)$.

Corollary 3.1. $\chi(X, f_1 \cdots f_n(F)) = \sum p_a(f_1^{[i_1]}, \cdots, f_n^{[i_n]}, F)$, where the summation is over all $i_1, \cdots, i_n \geq 0$. This sum is finite because

$$p_a(f_1^{[m_1]}, \cdots, f_n^{[m_n]}, F) = 0 \quad \text{if} \quad m_1 + \cdots + m_n > \dim(\text{Sup}(F)).$$

Proof. Theorem 3.1 states that the rational polynomial $\chi(X, f_1^{x_1} \cdots f_n^{x_n}(F))$ is equal to the polynomial $\sum p_a(f_1^{[i_1]}, \cdots, f_n^{[i_n]}, F) [i_1, \cdots, i_n]$, where the summation is over all $i_1, \cdots, i_n \geq 0$. We know from Theorem 9.1 of [3] that the degree of the first polynomial does not exceed $\dim(\text{Sup}(F))$, whence $p_a(f_1^{[m_1]}, \cdots, f_n^{[m_n]}, F) = 0$ for all $m_1 + \cdots + m_n > \dim(\text{Sup}(F))$. If we substitute $x_1 = \cdots = x_n = 1$ in both polynomials in question, we obtain the remainder of Corollary 3.1.

If we apply Corollary 3.1 in the case that $n = 1$ and $F = L$, we obtain $\chi(X, f_1(L)) = \sum p_a(f_1^{[i_1]}, L)$, where the summation is over all $i_1 \geq 0$. This is precisely formula (14) on page 151 of [2]; the same formula also occurs in [6].

So far, we have been concerned with expressing Euler-Poincaré characteristics in terms of virtual arithmetic genera. The following corollary tells us how to express virtual arithmetic genera in terms of Euler-Poincaré characteristics.

Corollary 3.2. For all $0 \leq m_1, \cdots, m_n \in \mathbb{Z}$,

$$p_a(f_1^{[m_1]}, \cdots, f_n^{[m_n]}, F) = \sum (-1)^{i_1 + \cdots + i_n} \binom{m_1}{i_1} \cdots \binom{m_n}{i_n} \chi(X, f_1^{-i_1} \cdots f_n^{-i_n}(F)),$$

where the summation is over $i_1, \cdots, i_n$, each index i_i running from 0 to m_i .

Proof. This is an immediate corollary of Theorem 3.1 and Proposition 1.3.

If we apply Corollary 3.2 in the case that $n = m_1 = 1$ and $F = L$, we obtain

$$p_a(f_1, L) = \chi(X, L) - \chi(X, f_1^{-1}(L)).$$

If X is irreducible and normal, we know from [3] that this is precisely formula (2) on page 134 of [4].

We are now in a position to derive the important formula which interrelates the two symbols $p_a(f_1^{[m_1]}, \dots, f_n^{[m_n]}, F)$ and $p_a(f_1^{m_1} \dots f_n^{m_n}, F)$. We observe that the latter symbol denotes the virtual arithmetic genus (with values in F) of the product of the line classes $f_1^{m_1}, \dots, f_n^{m_n}$.

Theorem 3.2. *For all $0 \leq m_1, \dots, m_n \in \mathbb{Z}$,*

$$p_a(f_1^{m_1} \dots f_n^{m_n}, F) = \sum (-1)^{i_1 + \dots + i_n + 1} \binom{m_1}{i_1} \dots \binom{m_n}{i_n} p_a(f_1^{[i_1]}, \dots, f_n^{[i_n]}, F),$$

where the summation is over $i_1, \dots, i_n$, each index i , running from 0 to m_i and $i_1 + \dots + i_n \geq 1$.

Proof. It follows from Theorem 3.1 and Proposition 1.4 that the above sum is equal to $\chi(X, F) - \chi(X, f_1^{-m_1} \dots f_n^{-m_n}(F))$. This difference is precisely $p_a(f_1^{m_1} \dots f_n^{m_n}, F)$, as we observed immediately after Corollary 3.2.

If we put $m_1 = \dots = m_n = 1$ in Theorem 3.2, we obtain the following corollary which we will use in the section on intersection theory.

Corollary 3.3. $p_a(f_1 \dots f_n, F) = \sum (-1)^{s+1} p_a(f_{i_1}, \dots, f_{i_s}, F)$, where the summation is over all sequences $1 \leq i_1 < \dots < i_s \leq n$ and where $s = 1, \dots, n$.

If $n = 2$, we conclude from Corollary 3.3 that

$$p_a(f_1 f_2, F) = p_a(f_1, F) + p_a(f_2, F) - p_a(f_1, f_2, F).$$

It is not difficult to see how equation (40) on page 581 of [5] follows from this result.

4. Intersection numbers. We continue our assumptions about X, L, F and $f_1, \dots, f_n$. We repeat that X may be reducible and that the line classes $f_1, \dots, f_n$ are not necessarily mutually distinct. We recall from Corollary 3.1 that $p_a(f_1, \dots, f_n, F) = 0$ if $n > \dim(\text{Sup}(F))$.

Definition 4.1. Let $n = \dim(\text{Sup}(F))$. The intersection number $i(f_1, \dots, f_n, F)$ of the line classes $f_1, \dots, f_n$ with values in F is the virtual arithmetic genus $p_a(f_1, \dots, f_n, F)$.

We know from section 3 that $i(f_1, \dots, f_n, F)$ depends only on how often each f_i occurs in the sequence $f_1, \dots, f_n$, but is independent of the order of $f_1, \dots, f_n$. Bearing this in mind, the following lemma expresses the fact that the symbol $i(f_1, \dots, f_n, F)$ is linear in each factor.

Lemma 4.1. Let $n = \dim(\text{Sup}(F))$. Let $f_1 = g_0^s g_1^t$, where g_0 and g_1 are line classes of X and where $s, t \in \mathbb{Z}$. Then

$$i(f_1, \dots, f_n, F) = si(g_0, f_2, \dots, f_n, F) + ti(g_1, f_2, \dots, f_n, F).$$

Proof. Consider the rational polynomial $\beta = \chi(X, g_0^{x_0} g_1^{x_1} f_2^{x_2} \dots f_n^{x_n}(F))$ and

denote, as in Proposition 2.3, the expansion of β by $\beta = \sum b(i_0, \dots, i_n)$ $[i_0, \dots, i_n]$. If we substitute $x_0 = 0$ in β we obtain the polynomial equality

$$\chi(X, g_1^{x_1} f_2^{x_2} \cdots f_n^{x_n}(F)) = \sum b(0, i_1, \dots, i_n) [i_1, \dots, i_n].$$

We conclude that $i(g_1, f_2, \dots, f_n, F) = b(0, 1, \dots, 1)$, where the indices in $b(0, 1, \dots, 1)$ which are indicated by dots are all equal to one. We show in the same way that $i(g_0, f_2, \dots, f_n, F) = b(1, 0, 1, \dots, 1)$. Now consider the rational polynomial $\alpha = \chi(X, (g_0^s g_1^t)^{x_1} f_2^{x_2} \cdots f_n^{x_n}(F))$ which is obtained from β by means of the substitution $x_0 \rightarrow sx_1$ and $x_1 \rightarrow tx_1$. If $\alpha = \sum a(i_1, \dots, i_n)$ $[i_1, \dots, i_n]$ denotes the expansion of α , we see that $i(g_0^s g_1^t, f_2, \dots, f_n, F) = a(1, \dots, 1)$, where again the indices of $a(1, \dots, 1)$ which are indicated by dots are all equal to one. We conclude from Theorem 9.1 of [3] that *the degree of β is at most equal to n* , and Lemma 4.1 now follows immediately from Proposition 2.3.

Let r be the dimension of X . If $n = r$, we call $i(f_1, \dots, f_r, L)$ *the intersection number of the line classes $f_1, \dots, f_r$* , and denote it by $i(f_1, \dots, f_r)$. If $f_1 = \dots = f_r$, we call $i(f_1, \dots, f_r)$ *the virtual degree of f_1* , and denote it by $d(f_1)$. If X is irreducible and p is the line class of X discussed in section 8 of [3], we call $d(p)$ *the degree of X* . This terminology is justified by what happens if X is irreducible and nonsingular. In that case, if D_i is a divisor of the divisor class $||f_i||$ (see section 5 of [3] for the divisor class $||f||$ associated with a line class f), $i(f_1, \dots, f_r)$ is the intersection number of $D_1, \dots, D_r$ as defined in section 10 of [8]. In particular, if $r = 1$, the virtual degree of f_1 is the usual degree of the divisor D_1 .

We return to the case that X is an arbitrary projective variety of dimension r and f a line class of X . We call $p_a(f, L)$ *the virtual arithmetic genus of f* and denote it by $p_a(f)$. If X is irreducible, we call $(-1)^r + (-1)^{r-1} p_a(f)$ the classical virtual arithmetic genus of f and denote it by $p'_a(f)$; see equation (3) on page 134 of [4] for the justification of this terminology. As an application, let us assume that in the last equation of the previous section $F = L$ and $r = 2$. This equation then states that

$$p'_a(f_1 f_2) = p'_a(f_1) + p'_a(f_2) + i(f_1, f_2) - 1,$$

which is equation (2) on page 86 of [7].

For the purpose of examples, the following proposition is convenient. We assume that X is irreducible and normal and that the divisor class $||f||$, associated with the line class f of X , contains a nonnegative divisor D which has no multiple components. We denote the variety of D by V and the sheaf of local rings of this closed subvariety of X by L' ; see section 6 of [3] for varieties of divisors.

Proposition 4.1. $p_a(f) = \chi(V, L')$.

Proof. Since $D \geq 0$, f is a regular line class of X and we have an exact sequence

$$0 \rightarrow f^{-1}(L) \rightarrow L \rightarrow L'' \rightarrow 0,$$

where L'' is the extension by zero of L' ; this is contained in Proposition 6.4 of [3]. Hence

$$\chi(X, L'') = \chi(X, L) - \chi(X, f^{-1}(L)).$$

The right hand side of this equality is $p_a(f)$, as we observed after Corollary 3.2. The left hand side is equal to $\chi(V, L')$, as follows from Proposition 8 of section 26 of [9].

We finish this section with some examples.

Example 4.1. Let X be a nonsingular quadric imbedded in 3-dimensional complex projective space, and Y a straight line on X . Let f be the line class of X which corresponds to the unique prime divisor D of X which is centered at Y . We claim that $d(f) = 0$. Accordingly, we select another straight line Y' on X such that $Y \cap Y' \neq \emptyset$, and we denote by g the line class of X which corresponds to Y' . We know from Lemma 4.1 that $i(f, fg) = d(f) + i(f, g)$. Now $i(f, fg)$ is the intersection multiplicity of the point of intersection of Y and any plane section of X , whence $i(f, fg) = 1$. In the same way, $i(f, g)$ is the intersection multiplicity of the point of intersection of Y and Y' , whence $i(f, g) = 1$. This proves the italicized claim. Since $r = 2$, we now conclude from Theorem 9.1 of [3] and Theorem 3.1 of the present paper that $\chi(X, f^x(L)) = \chi(X, L) + p_a(f)x$ for all $x \in \mathbb{Z}$. We know from Proposition 5 of section 78 of [9] that $\chi(X, L) = 1$. Proposition 4.1 of the present paper tells us that $p_a(f) = \chi(Y, L')$, where L' is the sheaf of local rings of Y . Again we conclude from Proposition 5 of section 78 of [9] that $\chi(Y, L') = 1$. Hence $\chi(X, f^x(L)) = 1 + x$ for all $x \in \mathbb{Z}$. This throws a better light on Example 10.1 of [3], since it ties up the deficiency of the degree of the polynomial $\chi(X, f^x(L))$ with intersection theory. Precisely, we see that *if X is an arbitrary projective variety and f any line class of X , the polynomial $\chi(X, f^x(L))$ has its maximal degree r if and only if $d(f) \neq 0$.*

Example 4.2. Let X be a nonsingular, cubic surface imbedded in 3-dimensional complex projective space. Let Y be one of the 27 straight lines on X and f the line class of X which corresponds to the unique prime divisor D of X which is centered at Y . We claim that $d(f) = -1$. In order to prove this, select a plane P which passes through Y , and denote by Y' the conic on X which is such that $Y' \cup Y = X \cap P$. If g is the line class of X which corresponds to Y' , we conclude from Lemma 4.1 that $i(f, fg) = d(f) + i(f, g)$. Now $i(f, fg)$ is the intersection multiplicity of the point of intersection of Y with any plane section of X , whence $i(f, fg) = 1$. Similarly, $i(f, g)$ is the sum of the intersection multiplicities of the two points of intersection of Y and Y' , whence $i(f, g) = 2$. The italicized claim has now been proved. We conclude, in the same way as in the previous example, that

$$\chi(X, f^x(L)) = \chi(X, L) + p_a(f)x - \frac{1}{2}x(x + 1)$$

and that $\chi(X, L) = p_a(f) = 1$. Consequently,

$$\chi(X, f^x(L)) = 1 + \frac{1}{2}x(1 - x)$$

for all $x \in \mathbb{Z}$. It is interesting to go a little further and to conclude from the duality theorem that $h^2(X, f^x(L)) = 0$ for large x . Furthermore, according to Lemma 4.1, $d(f^x) = x^2 d(f) = -x^2$. Hence if $x \neq 0$, $d(f^x) < 0$, from which we conclude that $h^0(X, f^x(L)) = 1$ for all $x \geq 0$. This shows that if x is large, $h^1(X, f^x(L)) = \frac{1}{2}x(x-1)$. Here then we have an example of a line class f with the property that $h^1(X, f^x(L))$ does not become constant for large x . It is interesting to observe that $h^1(X, f^x(L))$ is a polynomial for large x .

The computations of $d(f)$, based on the linearity of the symbol i , are of course classical in algebraic geometry. See for example Chapter 1 of [7].

Example 4.3. Let X be any 2-dimensional projective variety whatsoever and f_1 and f_2 two line classes of X . We conclude from Lemma 4.1 that

$$d(f_1 f_2) = d(f_1) + d(f_2) + 2i(f_1, f_2),$$

which is precisely equation (2) on page 87 of [7].

Example 4.4. Let X be an irreducible, projective variety and p the line class of X , discussed in section 8 of [3]. The polynomial

$$\chi(X, p^x(L)) = \sum p_a(p^{(i)}, L) x(x+1) \cdots (x+i-1)/i!,$$

where the summation index i runs from 0 to r , is the Hilbert polynomial of X . The degree of this polynomial is precisely r because $d(p) = \text{degree}(X) \neq 0$. We know from Proposition 4.1 that if X is normal, the coefficient $p_a(p, L)$ of this expansion is the arithmetic genus (in the sense of section 80 of [9]) of a hyperplane section of X , where the hyperplane must be chosen in such a way that the intersection divisor has no multiple components on X . *For arbitrary $1 \leq i \leq r$, $p_a(p^{(i)}, L)$ is the arithmetic genus (in the sense of section 80 of [9]) of the variety $X \cap P^{n-i}$, where P^{n-i} is a linear subspace of dimension $n-i$ of the ambient, n -dimensional projective space and where P^{n-i} has been chosen in such a way that the intersection components of $X \cap P^{n-i}$ on X are all simple.* This italicized statement can be proved by developing Proposition 4.1 further. *It gives a geometric characterization of all the coefficients of the Hilbert polynomial.* For example, let X be a normal surface, imbedded in 3-dimensional complex, projective space. Then

$$\chi(X, p^x(L)) = \chi(X, L) + p_a(p)x + \frac{1}{2}d(p)x(x+1).$$

It follows from Proposition 5 of section 78 of [9] that $\chi(X, L) = 1 + \frac{1}{6}(d-1)(d-2)(d-3)$, where $d = d(p)$ denotes the degree of X . That same proposition, applied to a plane section of X , gives that $p_a(p) = 1 - \frac{1}{2}(d-1)(d-2)$. If $d = 2$, we find that $\chi(X, p^x(L)) = 1 + 2x + x^2$, a fact we made use of in Example 10.2 of [3]. If X is again an arbitrary irreducible, projective variety, the classical expansion of the Hilbert polynomial is $\chi(X, p^x(L)) = \sum b(i) \binom{x}{i}$, where the summation index i runs from 0 to r , and where $\binom{x}{i}$ is the usual binomial coefficient. We know from Proposition 1.5 that $b(0) = a(0) =$

$\chi(X, L)$ and that $b(i) = \sum \binom{j-1}{i-1} p_a(p^{(i)}, L)$ for $1 \leq i \leq r$; here the summation index j runs from i to r . *This then explains completely the geometric meaning of all the coefficients of the classical expansions of the Hilbert polynomial.* For example, if $r = 2$, then $b(0) = \chi(X, L)$, $b(2) = \text{degree}(X)$ and

$$b(1) = p_a(p) + d(p) = 1 - p'_a(p) + \text{degree}(X).$$

This formula for $b(1)$ can, for instance, be found on page 87 of [10].

5. Relations between $p_a(f, F)$ and $p_a(f^{-1}, F)$. We assume again that X is an arbitrary, not necessarily irreducible, projective variety with sheaf of local rings L . We select an algebraic, coherent sheaf F of X and a line class f of X and denote $\dim(\text{Sup}(F))$ by s . For the sake of brevity, we restrict ourselves in the remainder of this paper to just one line class, although many of the propositions can be extended to several line classes. The relations which occur in Theorem 5.1 are convenient for the derivations of various forms of the Riemann-Roch theorem.

Theorem 5.1. $p_a((f^{-1})^{(i)}, F) = (-1)^i \sum \binom{j-1}{i-1} p_a(f^{(j)}, F)$ for $i = 1, \dots, s$.

The summation index j runs from i to s . (We may of course replace f by f^{-1} in this equality.)

Proof. We know from Theorem 9.1 of [3] and from Theorem 3.1 of the present paper that

$$\chi(X, (f^{-1})^x(F)) = \sum p_a((f^{-1})^{(i)}, F)[i],$$

where $[i] = x(x+1) \cdots (x+i-1)/i!$ and where the summation index i runs from 0 to s . If we replace x by $-x$, we conclude that

$$\chi(X, f^x(F)) = \sum (-1)^i p_a((f^{-1})^{(i)}, F) \binom{x}{i}.$$

Proposition 1.5 enables us to compare this expansion of $\chi(X, f^x(F))$ in terms of binomial coefficients with the expansion $\chi(X, f^x(F)) = \sum p_a(f^{(i)}, F)[i]$. Theorem 5.1 is an immediate consequence of this comparison.

If $F = L$ and $\dim(X) = 2$, the equality of Theorem 5.1 for $i = 1$ states that $p_a(f^{-1}) = -p_a(f) - d(f)$, which is equivalent to $p'_a(f^{-1}) + p'_a(f) = d(f) + 2$. (The second sentence of the above proof explains the meaning of the coefficients of the expansion in terms of the classical binomial coefficients.)

6. The canonical line class. We denote $\chi(X, L)$ by $p_a(X)$ and call it the (virtual) arithmetic genus of X . If X is irreducible and of dimension r , we call $(-1)^{r+1} + (-1)^r p_a(X)$ the classical (virtual) arithmetic genus of X and denote it by $p'_a(X)$; see section 80 of [9] for the justification of this terminology.

We assume for the remainder of this section that X is an irreducible, nonsingular,

projective variety of dimension r . We call the line class of X whose associated divisor class contains the canonical divisors of X *the canonical line class of X* and denote it by c . We know from the duality theorem that if f is any line class of X , then $H^q(X, f(L))$ is isomorphic to the dual of $H^{r-q}(X, cf^{-1}(L))$ for all $0 \leq q \leq r$, whence $\chi(X, f(L)) = (-1)^r \chi(X, cf^{-1}(L))$.

Theorem 6.1. *Let f be a line class of X and $0 \leq i \leq r$. Then*

$$p_a((f^{-1})^{[i]}) = (-1)^r \sum p_a(f^{[i]}, c^{[i]})$$

where the summation index j runs from 0 to $r - i$.

Proof. If we substitute $x \rightarrow -x$ and $y = 1$ in the expansion

$$\chi(X, f^x c^y(L)) = \sum p_a(f^{[i]}, c^{[j]})[i, j],$$

we obtain

$$\chi(X, cf^{-x}(L)) = \sum (-1)^i p_a(f^{[i]}, c^{[i]}) \binom{x}{i}.$$

The coefficient of $\binom{x}{i}$ in this sum is equal to $(-1)^i \sum p_a(f^{[i]}, c^{[j]})$, where the summation index j runs from 0 to $r - i$. According to the duality theorem, $\chi(X, cf^{-x}(L)) = (-1)^r \chi(X, f^x(L))$, and Theorem 6.1 now follows easily from Proposition 1.5 and Theorem 5.1.

For $i = r$, Theorem 6.1 states that $p_a((f^{-1})^{[r]}) = (-1)^r p_a(f^{[r]})$, which also follows from Lemma 4.1. For $i = 0$, we obtain that $p_a(X) = (-1)^r \sum p_a(c^{[j]})$, where the summation index j runs from 0 to r . If r is odd, this means that $2(p'_a(X) - 1) = \sum p_a(c^{[j]})$, where the summation index j runs from 1 to r . For example, if $r = 1$, this gives us the well known fact that the usual degree of the canonical divisor, i.e. $p_a(c)$, is equal to $2(p'_a(X) - 1)$. If r is even, our formula states that $\sum p_a(c^{[j]}) = 0$, where the summation index j runs from 1 to r . For instance, if $r = 2$, we obtain that $1 - p'_a(c) + d(c) = 0$, which is equation (2) on page 100 of [7]. Continuing the case that $r = 2$, we obtain from Theorem 6.1 for $i = 1$ that $p_a(f^{-1}) = p_a(f) + i(f, c)$. If we combine this result with the equation $p_a(f^{-1}) = -p_a(f) - d(f)$ of the previous section, we obtain that $i(f, c) = 2p'_a(f) - 2 - d(f)$, which is equation (α) on page 92 of [7]. In general, important relationships between $p_a(f^{[i]})$ and $p_a(f^{[i]}, c^{[j]})$ can be obtained by combining Theorems 5.1 and 6.1.

We observe that since $\chi(X, c^x(L)) = \sum p_a(c^{[j]})[j]$, where the summation index j runs from 0 to r , the arithmetic genera $p_a(X), p_a(c), \dots, p_a(c^{[r]})$ are $r + 1$ birational invariants of X . These invariants are not independent of one another, but satisfy the following linear equations.

Theorem 6.2. $p_a(X) = (-1)^r \sum p_a(c^{[i]})$, where the summation index j runs from 0 to r . Furthermore, if $3 \leq r$ and $2 \leq i \leq r - 1$, then

$$\sum \left((-1)^{r+i} \binom{j-1}{i-1} - 1 \right) p_a(c^{(i)}) = 0,$$

where the summation index j runs from i to r .

Proof. The first equation is a corollary of Theorem 6.1, as we observed earlier. If we use the canonical line class c as the f in Theorem 6.1, we obtain $p_a((c^{-1})^{(i)}) = (-1)^r \sum p_a(c^{(i)})$, where the summation index j runs from i to r . We conclude from Theorem 5.1 that

$$p_a((c^{-1})^{(i)}) = (-1)^i \sum \binom{j-1}{i-1} p_a(c^{(i)}),$$

where the summation index j runs from i to r . Theorem 6.2 follows immediately from these two equations.

Formally, we could have let i run from 1 to r in Theorem 6.2. However, if $i = r$, we get $0 = 0$ and if $i = 1$, we get $0 = 0$ if r is odd and we get back the first equation of Theorem 6.2 if r is even. For $r = 1, 2$ the implications of the first equation of Theorem 6.2 were discussed after Theorem 6.1. For $r = 1$ the implication is that $p_a(c)$ is determined by $p_a(X)$ and, for $r = 2$, that $p_a(c^{(2)})$ is determined by "the linear genus of the surface X ," i.e. by $p_a(c)$. If $r = 3$, the two equations of Theorem 6.2 are linearly independent, and $p_a(c^{(2)})$ and $p_a(c^{(3)})$ are completely determined by $p_a(X)$ and $p_a(c)$. If $r = 4$, the system of three equations of Theorem 6.2 has rank 2, and $p_a(c^{(3)})$ and $p_a(c^{(4)})$ are determined by $p_a(c)$ and $p_a(c^{(2)})$. Depending on other relations between these invariants which we did not observe, it seems that for $r = 4$ the $p_a(c^{(2)})$ is a new invariant which is independent of $p_a(X)$ and $p_a(c)$. It should not be hard to give a discussion of the system of equations of Theorem 6.2 for general r .

7. Riemann-Roch theorems. We first assume only that X is an irreducible, normal, projective variety of dimension r with sheaf of local rings L , and that the groundfield over which X is defined is algebraically closed in the function field of X . We select a line class f of X and a divisor $D \in |f|$. We know that

$$\chi(X, f^x(L)) = 1 + \dim |xD| + (-1)^r h^r(X, f^x(L)) + \sum (-1)^q h^q(X, f^x(L))$$

where the summation index q runs from 1 to $r - 1$. We denote this last sum by $-s(x)$ and obtain the formula

$$(7.1) \quad \dim |xD| + (-1)^r h^r(X, f^x(L)) = (-1)^r p'_a(X) + s(x) + \sum p_a(f^{(i)})[i]$$

where the summation index i runs from 1 to r . This formula holds for all $x \in \mathbb{Z}$ and we may always replace $p_a(f)$ by $1 + (-1)^{r-1} p'_a(f)$ and $p_a(f^{(r)})$ by $d(f)$. In particular, if $x = 1$, we obtain the formula

$$(7.2) \quad \begin{aligned} \dim |D| + (-1)^r h^r(X, f(L)) \\ = (-1)^r p'_a(X) + s(1) + 1 + (-1)^{r-1} p'_a(f) + d(f) + \sum p_a(f^{(i)}) \end{aligned}$$

where the summation index i runs from 2 to $r - 1$ and where $r \geq 2$.

We now consider some special cases.

Case 7.1. f is the line class p , discussed in section 8 of [3]. Let P^{n-i} have the same meaning as in Example 4.4. Formulas 7.1 and 7.2 can now be improved by using that

$$p_a(p^{(i)}) = p_a(X \cap P^{n-i}) = 1 + (-1)^{r-i} p'_a(X \cap P^{n-i})$$

for $i = 1, \dots, r$. Furthermore, if x is large enough, $h^r(X, p^x(L)) = s(x) = 0$, as proved in [9]. Hence, for large x , we have the formula

$$\dim |xD| = (-1)^r p'_a(X) + \sum (1 + (-1)^{r-i} p'_a(X \cap P^{n-i})) [i],$$

where the summation index i runs from 1 to r .

Case 7.2. f is arbitrary, but X is nonsingular. We denote the canonical line class of X by c and select a divisor $K \in ||c||$. We know that

$$h^r(X, f^x(L)) = h^0(X, cf^{-x}(L)) = 1 + \dim |K - xD|$$

which, if $xD \geq 0$, is the so called "index of specialty" $j(xD)$ of the divisor xD . Hence formulas 7.1 and 7.2 can now be improved by replacing $h^r(X, f^x(L))$ and $h^r(X, f(L))$ by, respectively, $j(xD)$ and $j(D)$. We conclude, furthermore, that if $D \geq 0$, then $j(xD) = 0$ for large x . The real difficulty now lies with the sum $s(x)$ about which practically nothing is known, even for large x .

Case 7.3. X is nonsingular and $r = 1$. Formula 7.1 now becomes $\dim |xD| - j(xD) = -p'_a(X) + d(f)x$, whence

$$\dim |D| - j(D) = -p'_a(X) + d(f).$$

Since $d(f)$ is now the usual degree of D , the last formula is the classical Riemann-Roch formula for curves.

Case 7.4. X is nonsingular and $r = 2$. Formula 7.1 now becomes

$$\dim |xD| + j(xD) = p'_a(X) + s(x) + (1 - p'_a(f))x + \frac{1}{2}d(f)x(x+1).$$

Here, $s(x)$ is the so called "superabundance" of the divisor xD . Formula 7.2 becomes

$$\dim |D| + j(D) = p'_a(X) + s(1) + 1 - p'_a(f) + d(f),$$

which is the classical Riemann-Roch theorem for surfaces; see, for example, page 106 of [7]. If we replace $d(f)$ by $p'_a(f) + p'_a(f^{-1}) - 2$ (see the last formula of section 5), we get the Riemann-Roch formula which is given on page 137 of [4]. (The -2 is missing in [4], which must have been a misprint.)

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