The Holonomy Group

III. Metrisable Spaces*

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1. Description of the problems. This paper is a direct continuation of [1] and [2].† It is based on the results (and symbolism) of [2]. Whenever it is necessary, these results will be quoted in the text.

Let us denote by $L_{\omega\mu\lambda}{}^\nu$ either the curvature tensor $R_{\omega\mu\lambda}{}^\nu$ of a volume preserving L_n or the curvature tensor $K_{\omega\mu\lambda}{}^\nu$ of a Riemann space V_n . In the first case $L_{\omega\mu\lambda}{}^\nu$ must satisfy the identities

$$(1.1) \quad \begin{aligned} (a) \quad L_{(\omega\mu)\lambda}{}^\nu &= 0, & (d) \quad L_{[\omega\mu\lambda]}{}^\nu &= 0, \\ (c) \quad L_{\omega\mu\lambda}{}^\lambda &= 0, & (d) \quad \nabla_{[\xi} L_{\omega\mu]\lambda}{}^\nu &= 0. \end{aligned}$$

In the second case it must satisfy the identities (1.1a, b, c, d) and

$$(1.1) \quad \begin{aligned} (e) \quad L_{\omega\mu(\lambda\nu)} &= 0, \\ (f) \quad L_{\omega\mu\lambda\nu} &= L_{\lambda\nu\omega\mu}. \end{aligned}$$

Denote by $g_{\lambda\nu}$ a symmetric tensor of rank n and by $g^{\lambda\nu}$ its inverse,

$$(1.2) \quad g^{\lambda\nu} g_{\lambda\mu} = \delta_\mu^\nu,$$

and consider the following system of equations:

$$(1.3) \quad \begin{aligned} L_{\omega\mu\lambda}{}^\nu &= \partial_{[\mu} \{ g^{\nu\alpha} (\partial_{\omega]} g_{\alpha\lambda} + \partial_{|\lambda|} g_{\omega] \alpha} - \partial_{|\alpha|} g_{\omega] \lambda} \} + \frac{1}{2} g^{\nu\beta} (\partial_\alpha g_{\beta[\mu} \\ &\quad + \partial_{[\mu} g_{|\alpha\beta|} - \partial_\beta g_{\alpha[\mu}]) g^{\alpha\delta} (\partial_{\omega]} g_{\lambda\delta} + \partial_{|\lambda|} g_{\omega] \delta} - \partial_{|\delta|} g_{\omega] \lambda}). \end{aligned}$$

If $g_{\lambda\mu}$ is not known and $L_{\omega\mu\lambda}{}^\nu$ is given, (1.3) represents a system of differential equations for $g_{\lambda\mu}$. Then one of our problems may be formulated as follows:

1(a). One has to find necessary and sufficient conditions for (1.3) to admit at least one solution $g_{\lambda\nu}$.

1(b). If these conditions are satisfied, one has to find the explicit form of at least one solution $g_{\lambda\nu}$.

1(c). If one solution is given, one has to find the explicit form of all solutions.

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†These papers will be denoted by I and II.

The problem 1(a) has been solved for any holonomy group and for any n in terms of a rank of a matrix, and in the case of a complete holonomy group it is solved in terms of the tensors which lead to the generators of the group.

The problem 1(b) has been solved for a complete holonomy group and general (even) n and for some other holonomy groups in the case of a particular n .

The problem 1(c) has been solved for the case of general, degenerate or complete holonomy groups.

The problem 1(a), (b), (c) may be summarized roughly as follows: One has to find all metric tensors having the same curvature tensor. This leads to the following problem:

2(a). Given a symmetric connection one has to find the explicit form of at least one covariant constant quadratic symmetric tensor.

2(b). Given one solution of the problem 2(a), one has to find the explicit form of all solutions.

The problem 2(a) has been solved for the case of a complete holonomy group for the general (even) n and for some other groups in the case of a particular n . The problem 2(b) has been solved for the general and for the degenerate holonomy group.

The problem 2(a), (b) may be roughly summarized as follows: One has to find necessary and sufficient condition for which a given symmetric connection reduces to Christoffel symbols and one has to find all tensors which yield these Christoffel symbols.

Remark. All solutions of rank n of the second problem solve also the first problem. However, the class of solutions of the first problem is larger. It contains (in some special cases) solutions which are not covariant constant with respect to the connection, the curvature tensor of which is the given tensor $L_{\omega\mu\lambda}$. We shall see that this happens, for instance, in the case of a degenerate holonomy group.

It is worthwhile to stress that *both problems are solved by means of pure algebraic methods.*

2. Introductory notions. We shall first consider an L_n with special properties. We shall denote its curvature tensor $R_{\omega\mu\lambda}$ by $L_{\omega\mu\lambda}$:

$$(2.1) \quad L_{\omega\mu\lambda} = 2\{\partial_{[\mu}\Gamma_{\omega]\lambda} + \Gamma_{\alpha[\mu}\Gamma_{\omega]\lambda}^{\alpha}\},$$

and this will enable us later to identify it with the curvature tensor of a V_n .

We know that the system

$$(2.2a) \quad \nabla_{\mu}^* g_{\lambda\nu} = {}^*g_{\lambda\nu} \partial_{\mu} \ln \sigma$$

admits a solution ${}^*g_{\mu\lambda} = {}^*g_{\lambda\mu}$ if and only if its complete set of integrability conditions is satisfied:

$$(2.2b) \quad L_{\alpha\lambda}^{\alpha} {}^*g_{\alpha\nu} + L_{\alpha\nu}^{\alpha} {}^*g_{\lambda\alpha} = 0$$

(see Theorem II (4.11)). Hence if we put

$$(2.3) \quad g_{\lambda\mu} \stackrel{\text{def}}{=} *g_{\lambda\nu}\sigma^{-1},$$

we obtain

$$(2.4a) \quad \nabla_\mu g_{\lambda\nu} = \partial_\mu g_{\lambda\nu} - \Gamma_\lambda^\alpha{}_\mu g_{\alpha\nu} - \Gamma_\nu^\alpha{}_\mu g_{\lambda\alpha} = 0,$$

and (2.2b) is equivalent to

$$(2.4b) \quad L_{a\lambda}{}^\alpha g_{\alpha\nu} + L_{a\nu}{}^\alpha g_{\lambda\alpha} = 0$$

which is the complete set of integrability conditions of (2.4a).

Later on we shall need the following

Lemma (2.1). *Any solution $g_{\lambda\mu} = g_{\mu\lambda}$ of rank n of (2.4a) is also a solution of (1.3) where $L_{\omega\mu\lambda}$ is given by (2.1). A solution of (1.3) is not necessarily a solution of (2.4a).*

Proof. Assume that (2.4a) admits a solution $g_{\lambda\mu} = g_{\mu\lambda}$ of rank n and denote by $\{\lambda^\nu{}_\mu\}$ its Christoffel symbols. Then (2.4a) is equivalent to

$$(2.5) \quad \Gamma_\lambda^\nu{}_\mu = \{\lambda^\nu{}_\mu\}.$$

Substituting from (2.5) into (2.1), we obtain (1.3). Assume now that (1.3) admits a solution $g_{\lambda\mu} = g_{\mu\lambda}$ of rank n and denote by $\{\lambda^\nu{}_\mu\}$ its Christoffel symbols. Comparing (1.3) and (2.1) we obtain

$$(2.6) \quad \Gamma_\lambda^\nu{}_\mu = \{\lambda^\nu{}_\mu\} + Q_{\lambda\mu}{}^\nu$$

where the tensor $Q_{\lambda\mu}{}^\nu = Q_{\mu\lambda}{}^\nu$ is a solution of

$$(2.7) \quad {}^0\nabla_{[\mu} Q_{\omega]\lambda}{}^\nu = Q_{\lambda[\mu}{}^\alpha Q_{\omega]\alpha}{}^\nu$$

and ${}^0\nabla_\mu$ is the symbol of the covariant derivative with respect to $\{\lambda^\nu{}_\mu\}$ (see the equations I (3.5)). Hence if $Q_{\lambda\mu}{}^\nu \neq 0$ our solution of (1.3) is not a solution of (2.4a).

Remark. If (2.4a) admits a solution $g_{\lambda\mu} = g_{\mu\lambda}$ of rank n , the L_n under consideration reduces to a V_n with the metric tensor $g_{\lambda\mu}$. This justifies the following:

Definition (2.1). *An L_n for which (2.4a) admits a solution $g_{\lambda\mu} = g_{\mu\lambda}$ of rank n is termed a metrisable L_n .*

3. A solution of (1.3) in the case of the complete holonomy group. In the following three theorems, which will be proven simultaneously, we shall consider an L_n , the holonomy group of which is complete (for a complete group and a complete tensor see Section II (6)).

Theorem (3.1). *Consider an L_n ,*

$$n = 2m > 2,$$

the holonomy group of which is complete and has

$$r = \frac{1}{2}n(n-1) > 2 \dagger$$

parameters. A necessary and sufficient condition for this L_n to be metrisable is that the factors of proportionality of the vectors a^v , a_λ of the proper nonholonomic system can be chosen in such a way* that the nonholonomic components

$$L_{ab}{}^c \stackrel{\text{def}}{=} L_{a\lambda}{}^v a_\lambda^c a_v^b$$

of the tensors $L_{a\lambda}{}^{v**}$ satisfy the conditions

$$(3.1a) \quad L_{ax}{}^{v+\delta v} + L_{ay}{}^{x+\delta x} = 0, \quad *** \quad x \neq y,$$

$$\delta t \stackrel{\text{def}}{=} \begin{cases} +1 & \text{for } t \text{ odd} \\ -1 & \text{even} \end{cases}, \quad t = 1, \dots, n. \quad ****$$

This condition is not an identity for $x \neq y$, $a = 2, \dots, r$.*****

Theorem (3.2). If (3.1) are satisfied then the tensor of rank n

$$(3.2) \quad {}^*g_{\lambda\mu} = \sum_x \bar{a}_\lambda{}^x a_\mu{}^{x+\delta x} \dagger$$

has the following properties:

(a) It is invariant with respect to the change of factors of proportionality

$$(3.3) \quad \begin{aligned} (a) \quad & {}'a_x^v = \varphi_x a_x^v, \quad {}'\bar{a}_\lambda^x = \bar{\varphi}_\lambda^x \bar{a}_\lambda^x, \quad \varphi_x \bar{\varphi}_x = 1, \quad \S \\ (b) \quad & \varphi_x = {}^{x+\delta x} \bar{\varphi}_x. \end{aligned}$$

†Cf. The remark following the proof.

*See Theorems (3.4), (3.5) about the possibility of this choice.

**See Definition II (6.2).

***This equation shows that a factor of proportionality f common to all vectors a_b^v may be still chosen freely:

$${}_b' a^v = f {}_b a^v, \quad {}_\lambda' \bar{a}^b = f^{-1} {}_\lambda \bar{a}^b.$$

We remind the reader that the indices x, y, z, t from the range $1, \dots, n$ do not follow the usual summation convention.

****Hence $n = 2m \geq t + \delta t > 0$.

*****I.e., to every pair $x, y \neq x$ there is at least one $a = 2, \dots, r$ for which

$$(3.1b) \quad L_{ax}{}^{u+\delta u} \neq 0.$$

‡According to the definition of δx the right hand term of (3.2) is equal to

$$2 \sum_x \bar{a}_{(\lambda}{}^x a_{\mu)}{}^{x+\delta x}, \quad x = 1, 3, \dots, n-1.$$

§This condition secures the relation

$${}_b' a^\lambda {}_\lambda' \bar{a}^c = a^\lambda {}_\lambda \bar{a}^c = \delta_b^c.$$

(b) It satisfies the equations (2.2a) where the gradient $\partial_\mu \ln \sigma$ is defined by

$$(3.4) \quad \begin{aligned} (a) \quad n^* g \partial_\mu \ln \sigma &= \nabla_\mu^* g, \\ (b) \quad *g &\stackrel{\text{def}}{=} \text{Det} (*g_{\lambda\mu}). \end{aligned}$$

Theorem (3.3). If (3.1) are satisfied, then the tensor defined by

$$(3.5) \quad g_{\lambda\mu} \stackrel{\text{def}}{=} *g_{\lambda\mu} \sigma^{-1} = \sigma^{-1} \sum_x a_\lambda^{x+\delta x} a_\mu^x$$

(where σ is a solution of (3.4)) is a solution of (2.4a) and (1.3) (where $L_{\omega\mu\lambda}^v$ is defined by (2.1)). The L_n under consideration reduces to V_n with the metric tensor (3.5) and with the general holonomy group.

The proof of these theorems will be accomplished in four steps. In particular, the sufficiency of (3.1) will be proved by showing that the system

$$(3.6) \quad L_{ax}^a *g_{ay} + L_{ay}^a *g_{ax} = 0,$$

equivalent to (2.2b), admits a symmetric* solution of rank n if (3.1) are satisfied.

(1) The system (3.1a) may be split into two, namely

$$(3.7a) \quad L_{ax}^{y+\delta y} + L_{ay}^{x+\delta x} = 0, \quad x \neq y,$$

for

$$(3.8) \quad x + \delta x \neq y, \quad y + \delta y \neq x$$

and

$$(3.7b) \quad L_{ax}^x + L_{ax+\delta x}^{x+\delta x} = 0. \dagger$$

Consider first the case $\alpha = 1$. The tensor $L_{1\lambda}^v$ being complete, we have

$$(3.9) \quad \begin{aligned} (a) \quad L_{1x}^y &= \lambda \delta_x^y, \\ (b) \quad \lambda_x &\neq \lambda_y, \quad x \neq y. \end{aligned}$$

Hence (3.7a) is identically satisfied for $\alpha = 1$, while (3.7b) is for $\alpha = 1$ equivalent to

$$(3.9c) \quad \lambda_x + \lambda_{x+\delta x} = 0.$$

*We incorporated the symmetry of the unknown tensor by writing $*g_{ax}$ instead of $*g_{xa}$.

†(3.7a) is restricted by (3.8), while (3.7b) follows from (3.1) for $x + \delta x = y$, $y + \delta y = x$. Either one of these two conditions is a consequence of the remaining one. Hence we cannot have

$$x + \delta x \neq y, \quad y + \delta y = x$$

or

$$x + \delta x = y, \quad y + \delta y \neq x.$$

On the other hand, the equations (3.6) for $\alpha = 1$ are equivalent to

$$(3.10) \quad (\lambda_x + \lambda_y) * g_{xy} = 0,$$

so that according to (3.9b,c)

$$(3.11) \quad \begin{aligned} (a) \quad & *g_{xx+\delta x} \neq 0 \quad \text{undefined,} \\ (b) \quad & *g_{xy} = 0, \quad |x - y| \neq 1. \end{aligned}$$

(2) Substituting from (3.11) into (3.6) we obtain

$$(3.12) \quad \begin{aligned} (a) \quad & L_{ax}^{y+\delta y} * g_{y+\delta y, y} + L_{ay}^{x+\delta x} * g_{x+\delta x, x} = 0, \\ (b) \quad & L_{ax}^x * g_{x+\delta x, x} + L_{ax+\delta x}^{x+\delta x} * g_{x+\delta x, x} = 0. \end{aligned}$$

The equation (3.12a) reduces, by virtue of (3.7a) (and by virtue of the symmetry of $*g_{ab}$), to

$$(3.13) \quad L_{ax}^{y+\delta y} (*g_{y+\delta y, y} - *g_{x+\delta x, x}) = 0$$

for (3.8) while (3.12b) is identically satisfied because of (3.7b) (and because of the symmetry of $*g_{ab}$).

The conditions (3.1b) and (3.13) imply $*g_{y+\delta y, y} = *g_{x+\delta x, x}$. Therefore we assume, according to (3.11a),

$$(3.14) \quad *g_{x+\delta x, x} = 1. \quad \dagger$$

The nonholonomic components (3.11b) and (3.14) define a symmetric tensor of rank n . Its inverse has the components

$$(3.15) \quad *g^{xx+\delta x} = 1, \quad *g^{xy} = 0, \quad |x - y| \neq 1.$$

The tensor $*g_{ab}$ satisfies (3.6). Its holonomic components $*g_{\lambda\mu}$ are given by (3.2) while $*g^{\lambda\nu}$ is given by

$$(3.16) \quad *g^{\lambda\nu} = \sum_x a^\lambda_x a^\nu_{x+\delta x}.$$

(3) In (3.2) and (3.16) the factors of proportionality of a^ν_b , $\hat{a}_\lambda$ have been already chosen in such a way that (3.1a) holds. Moreover, from (3.14) and (3.11b) we see that

$$(3.17) \quad a^\lambda_x a^\mu_{x+\delta x} *g_{\lambda\mu} = 1, \quad a^\lambda_x a^\mu_y *g_{\lambda\mu} = 0, \quad |x - y| \neq 1.$$

* $*g_{ab}$ has to be of rank n .

† We could have put

$$*g_{x+\delta x, x} = f$$

where $f \neq 0$ is an arbitrary scalar. The assumption $f = 1$ is obviously no restriction of generality.

Hence we must have

$$(3.18) \quad a_x^\lambda * g_{\lambda\mu} = a_{x+\delta x}^\mu$$

to satisfy the basic equation relating a^ν and $\hat{a}_\lambda$:

$$a_b^\lambda \hat{a}_\lambda = \delta_b^c.$$

If $*g_{\lambda\mu}$ is required to be invariant with respect to (3.3a), then the equation (3.18) (or (3.2)) restricts the choice of the φ 's in (3.3a) as follows:

$$\varphi_x = \varphi_{x+\delta x}^{x+\delta x} = (\varphi_{x+\delta x})^{-1}, \quad \varphi_x = \varphi_{x+\delta x} = (\varphi_{x+\delta x}^{x+\delta x})^{-1}.$$

The tensor $*g_{\lambda\mu}$ satisfies (2.2b) and therefore it is a solution of (2.2a). Multiplying this equation by $*g^{\lambda\nu}$ we obtain (3.4). In order to prove that $\partial_\mu \ln \sigma$ in (3.4) is a gradient let us multiply (2.2b) by $*g^{\lambda\nu}$. The result

$$L_{a\lambda}^\lambda = 0$$

shows, in conjunction with

$$L_{\omega\mu\lambda}{}^\nu = M^\alpha{}_{\omega\mu} L_{a\lambda}{}^\nu \quad (\text{see II (1.4)}),$$

that

$$L_{\omega\mu\lambda}^\lambda = 0.$$

Hence L_n is volume preserving, *i.e.*

$$\partial_{[\mu} \Gamma_{\omega]}{}^\alpha{}_\alpha = 0,$$

and therefore $(\nabla_\mu *g)/*g$, and *a fortiori* also $\partial_\mu \ln \sigma$, defined by (3.4), is a gradient. Theorem (3.3) is then an immediate consequence of (2.3), (2.4a) and Lemma (2.1). Hence if (3.1) are satisfied, L_n is metrisable and Theorems (3.2), (3.3) hold.*

(4) Assume now that the L_n under consideration is metrisable, *i.e.*, that it reduces to a V_n with the metric tensor $g_{\lambda\nu}$, satisfying (2.4). The equation (2.4b) shows that

$$L_{a\lambda\nu} \stackrel{\text{def}}{=} L_{a\lambda}{}^\alpha g_{\alpha\nu}$$

is a bivector:

$$(3.19) \quad L_{a(\lambda\nu)} = 0.$$

This holds also, in particular, for the complete tensor $L_{1\lambda}{}^\nu$ so that (3.9a) is equivalent to

$$L_{1xy} = \lambda g_{xy}, \quad L_{1yx} = \lambda g_{xy}.$$

*The general holonomy group of a V_n has $r = \frac{1}{2} n(n-1)$ parameters.

These equations yield

$$(3.20) \quad (\lambda + \lambda)_{g_{xy}} = 0.$$

The eigenvalues λ of $L_{1\lambda}^{\nu}$ are solutions of the secular equations

$$(3.21) \quad \text{Det} ((\lambda g_{\lambda\nu} - L_{1\lambda\nu})) = 0.$$

Because $L_{1\lambda\nu}$ is a bivector we have

$$\mathfrak{G}^{\omega_1 \dots \omega_p \mu_1 \dots \mu_q} \mathfrak{G}^{\lambda_1 \dots \lambda_p \nu_1 \dots \nu_q} g_{\omega_1 \lambda_1} \dots g_{\omega_p \lambda_p} L_{1\mu_1 \nu_1} \dots L_{1\mu_q \nu_q} = 0, *$$

$$p + q = n, \quad p \text{ odd}.$$

Therefore (3.21) contains only even powers of λ , and consequently we may number its roots in such a way that (3.9c) holds. The conditions (3.9b, c) and (3.20) show that

$$(3.22) \quad \begin{aligned} (a) \quad & g_{xy} = 0, \quad |x - y| \neq 1, \\ (b) \quad & g_{xx+\delta x} \neq 0. ** \end{aligned}$$

Therefore we can choose the factors of proportionality of the vectors in such a way that

$$(3.22c) \quad g_{xx+\delta x} = 1.$$

Substituting from (3.22a, c) into

$$(3.22d) \quad L_{ax}^a g_{ay} + L_{ay}^a g_{ax} = 0$$

(equivalent to (2.2b)) and remembering the symmetry of g_{ab} , we obtain (3.7) equivalent to (3.1a). This proves the necessity of (3.1a).

Assume that (3.1b) does not hold and denote by ρ the rank of the $r \times n^2$ matrix $((L_{aa}^b))$. An easy inspection shows that for $r = \frac{1}{2}n(n-1)$, $n = 2m > 2$, we have $\rho < r$. Hence (3.1b) must hold.

Remark. The number r of parameters of the general holonomy group of a V_n is

$$r = \frac{1}{2}n(n-1).$$

Hence for a possibly metrisable L_n we must have

$$r \leq \frac{1}{2}n(n-1).$$

* $\mathfrak{G}$ is the indicator. Because $n = 2m$, q is odd whenever p is odd.

** g_{ab} has to be of rank n .

Therefore if $n = 2$ we have in this case $r = 1$. Then the conditions (3.1) are to be replaced by

$$\lambda_1 + \lambda_2 = 0. *$$

Otherwise the proof of Theorems (3.1)–(3.3) follows the previous pattern.

The case $n > 2$, $r < \frac{1}{2}n(n-2)$ requires a different method of proof.**

The question arises under what condition it is possible to choose the factors of proportionality of the vectors $a'_\nu, \dot{a}_\lambda$ so that (3.1) holds. The answer is given by the statements of the following two theorems which will be proved simultaneously.

Theorem (3.4). *The conditions (3.7b) are invariant under the transformations (3.3a).*

Theorem (3.5). *A necessary and sufficient condition for the existence of such proportionality factors for which (3.7a) holds is that there is a system of scalars $f \neq 0$, $z \neq t$, such that*

$$(3.23) \quad \left(f_{\nu+\delta\nu}^x a_\nu^{\lambda \nu+\delta\nu} + f_{x+\delta x}^y a_x^{\lambda x+\delta x} \right) L_{a\lambda}{}^\nu = 0, \quad x + \delta x \neq y, \quad y + \delta y \neq x.$$

If this condition is satisfied then the proportionality factors restricted by

$$(3.24a) \quad \varphi_{x+\delta x}^x \varphi_{\nu+\delta\nu}^y = \varphi_{\nu+\delta\nu}^x \varphi_{x+\delta x}^y f / f, \quad x + \delta x \neq y, \quad y + \delta y \neq x$$

lead to

$$(3.25) \quad 'L_{a\nu}{}^{\nu+\delta\nu} + 'L_{a\nu}{}^{x+\delta x} = 0, \quad x + \delta x \neq y, \quad y + \delta y \neq x, \\ 'L_{ab}{}^c \stackrel{\text{def}}{=} L_{a\lambda}{}^\nu 'a_\nu^\lambda 'a_\nu^c.$$

Proof. The invariance of (3.7b) follows immediately from (3.3). Assume that (3.23) holds. Substituting from (3.3) into (3.23) we obtain

$$(3.26) \quad \left(f_{\nu+\delta\nu}^x \varphi_{\nu+\delta\nu}^x 'a_\nu^\lambda 'a_\nu^{\nu+\delta\nu} + f_{x+\delta x}^y \varphi_{x+\delta x}^y 'a_x^\lambda 'a_x^{x+\delta x} \right) L_{a\lambda}{}^\nu = 0, \\ x + \delta x \neq y, \quad y + \delta y \neq x.$$

If (3.24a) holds then also

$$(3.24b) \quad f_{\nu+\delta\nu}^x \varphi_{\nu+\delta\nu}^x \varphi_{\nu+\delta\nu}^y = f_{x+\delta x}^y \varphi_{x+\delta x}^y \varphi_{x+\delta x}^x, \quad x + \delta x \neq y, \quad y + \delta y \neq x,$$

*Of course this condition is invariant with respect to (3.3).

**This case is covered by Section 8. (Cf. also Section 9.)

and (3.25) results. Omitting the accents in (3.25) we obtain (3.7b). Assume that (3.25) holds and choose the proportionality factors according to (3.24a). Then (3.23) results.

4. All solutions of (1.3) in the case of a general holonomy group. In the previous section we found a solution of (1.3) assuming the completeness of the holonomy group. The holonomy group $G_r(R)$ of the metrisable L_n with the curvature tensor $R_{\omega\mu\lambda}{}^\nu = L_{\omega\mu\lambda}{}^\nu$ reduced to the holonomy group $G_r(K)$ of a V_n with the metric tensor $g_{\lambda\mu}$ and with the corresponding curvature tensor $K_{\omega\mu\lambda}{}^\nu = L_{\omega\mu\lambda}{}^\nu$. In this section we shall find *all* solutions of (1.3) for $L_{\omega\mu\lambda}{}^\nu = K_{\omega\mu\lambda}{}^\nu$ where $K_{\omega\mu\lambda}{}^\nu$ is the curvature tensor of a tensor $g_{\lambda\mu}$, *assumed to be known*. In other words, we shall find *all* tensors $g_{\lambda\mu}$ which lead to *the same* curvature tensor $K_{\omega\mu\lambda}{}^\nu$, provided that we know at least one of them. In this section we shall solve the above mentioned problem for a general holonomy group.

Let us denote by ${}^0g_{\lambda\mu}$ the metric tensor of a 0V_n and by ${}^0K_{\omega\mu\lambda}{}^\nu$ its curvature tensor. When we need to stress that an object belongs to 0V_n , we shall label it by "0" preceding the kernel letter.

We shall first prove a lemma concerning a perfect holonomy group.*

Lemma (4.1). *Let*

$$(4.1) \quad K_{\omega\mu\lambda\nu} = h^{a\epsilon} L_{a\omega\mu} L_{\epsilon\lambda\nu}, \quad {}^0K_{\omega\mu\lambda\nu} = {}^0h^{a\epsilon} {}^0L_{a\omega\mu} {}^0L_{\epsilon\lambda\nu}$$

be the curvature tensors of two Riemann spaces V_n and 0V_n . Let the holonomy groups of V_n and 0V_n have the same number r of parameters and let the holonomy group $G_r(K)$ of V_n be perfect. If

$$(4.2) \quad K_{\omega\mu\lambda\nu} = {}^0K_{\omega\mu\lambda\nu}$$

then the tensors $L_{a\lambda\nu}$ and ${}^0L_{a\lambda\nu}$ may be defined in such a way that

$$(4.3) \quad L_{a\lambda\nu} = {}^0L_{a\lambda\nu},$$

so that the holonomy groups $G_r(K)$ and ${}^0G_r({}^0K)$ of V_n and 0V_n are identical.

Proof. Put

$$(4.4) \quad A_a^i = {}^0L_{a\omega\mu}^{-1} L^{i\omega\mu}.$$

Then (4.1) and (4.2) yield

$$(4.5) \quad h^{a\epsilon} = A_a^i A_\epsilon^j {}^0h^{ij}.$$

Assume that $G_r(K)$ is perfect. Then the rank of $h^{a\epsilon}$ is r and (4.5) shows that the rank (of A_a^i as well as the rank) of ${}^0h^{a\epsilon}$ is r . This statement implies that ${}^0G_r({}^0K)$ is a perfect group, and (4.2) in conjunction with (4.5) yields

$$(4.6a) \quad {}^0h^{a\epsilon} (A_a^i A_\epsilon^j L_{i\omega\mu} L_{j\lambda\nu} - {}^0L_{a\omega\mu} {}^0L_{\epsilon\lambda\nu}) = 0$$

*For the definition and the properties of a perfect holonomy group see Section 8 in II.

equivalent to

$$(4.6b) \quad h^{0a*} A_i^b (A_a^i L_{g\omega\mu} - {}^0L_{a\omega\mu}) = 0,$$

and this equation leads by virtue of $\text{Det } ({}^0h^{a*}) \text{Det } (A_a^i) \neq 0$ to

$$(4.6c) \quad A_a^i L_{g\omega\mu} = {}^0L_{a\omega\mu}.$$

If we write $L_{g\omega\mu}$ instead of $A_a^i L_{g\omega\mu}$,* we obtain (4.3). Hence $G_r(K)$ is identical with ${}^0G_r({}^0K)$.**

Remark. The equation (4.6c) (equivalent to (4.3)) follows from (4.2) if and only if $G_r(K)$ is perfect. If this is not the case then (4.5) shows that the rank of ${}^0h^{a*}$ is not necessarily equal to r , so that instead of (4.6c) we obtain only

$${}^0h^{a*} A_i^b (A_a^i L_{g\omega\mu} - {}^0L_{a\omega\mu}) = 0$$

and, in general, this equation cannot be simplified to (4.6c).

Theorem (4.1). *If $n > 2$, $r = \frac{1}{2}n(n-1)$, then the metric tensors $g_{\lambda\mu}$ and ${}^0g_{\lambda\mu}$ of the Riemann spaces V_n and 0V_n mentioned in the Lemma (4.1) are related by*

$$(4.7) \quad g_{\lambda\mu} = {}^0g_{\lambda\mu} \text{ const.}$$

Only if (4.7) holds is the equation (4.2) satisfied.

Proof. It is obvious that (4.7) leads to (4.2). In order to prove the inverse statement, denote by u_a^v a set of mutually perpendicular unit vectors (with respect to ${}^0g_{\lambda\mu}$) and by u_λ the inverse set. From

$$(4.8a) \quad {}^0g^{ab} = {}^0g_{ab} \stackrel{\text{def}}{=} u_a^\lambda u_b^\mu {}^0g_{\lambda\mu} = \begin{cases} 0 & \text{for } a \neq b, \\ \epsilon_a & a = b, \end{cases} \quad \epsilon_a = \pm 1$$

we obtain

$$(4.8b) \quad u_\lambda^b = {}^0g^{ab} u_a^\mu {}^0g_{\lambda\mu}$$

equivalent to

$$(4.8c) \quad u_\lambda^x = \epsilon_x u^\mu {}^0g_{\lambda\mu} = \epsilon_x u_\lambda.$$

Hence

$$(4.9) \quad u^\lambda u_\lambda^b = \delta_a^b, \quad u_a^v u_\lambda^a = \delta_\lambda^v.$$

Therefore the set u^v, u_λ^b constitutes a nonholonomic frame and ${}^0g^{ab}, {}^0g_{ab}$ defined by (4.8a) are nonholonomic components (with respect to this frame) of ${}^0g^{\lambda\nu}$ and ${}^0g_{\lambda\mu}$.

*This is permissible because the rank of A_a^i is r . See Theorem II (4.1).

**See II (4.15) for the infinitesimal operators of the holonomy group.

According to Lemma (4.1) $G_r(K)$ is perfect and (4.3) may be assumed. Hence the complete set of integrability conditions of

$$(4.10) \quad \begin{aligned} (a) \quad {}^0\nabla_\mu {}^0g_{\lambda\nu} &= \partial_\mu {}^0g_{\lambda\nu} - {}^0\{\lambda^\alpha{}_\mu\} {}^0g_{\alpha\nu} - {}^0\{\lambda^\alpha{}_\mu\} {}^0g_{\lambda\alpha} = 0, \\ (b) \quad \nabla_\mu g_{\lambda\nu} &= \partial_\mu g_{\lambda\nu} - \{\lambda^\alpha{}_\mu\} g_{\alpha\nu} - \{\nu^\alpha{}_\mu\} g_{\lambda\alpha} = 0 \end{aligned}$$

acquires the form

$$(4.11a) \quad L_{a\lambda}{}^\alpha G_{\alpha\nu} + L_{a\nu}{}^\alpha G_{\lambda\alpha} = 0,$$

where either

$$(4.11b) \quad G_{\lambda\nu} \stackrel{\text{def}}{=} {}^0g_{\lambda\nu}$$

or

$$(4.11c) \quad G_{\lambda\nu} \stackrel{\text{def}}{=} g_{\lambda\nu}.$$

We shall deal with the system (4.11a, c). The nonholonomic components L_{axy} of the bivector $L_{a\lambda\nu}$ satisfy the relation

$$(4.12) \quad 2L_{a\lambda}{}^\nu = \sum_{x,y} L_{axy} (\overset{x}{u}_\lambda \overset{\nu}{u}_y \epsilon_y - \overset{\nu}{u}_\lambda \overset{x}{u}_x \epsilon_x),$$

so that (4.11a, c) is equivalent to

$$(4.13a) \quad \sum_{x,y} L_{axy} (g_{\nu(\nu} \overset{x}{u}_{\lambda)} \epsilon_\nu - g_{x(\nu} \overset{\nu}{u}_{\lambda)} \epsilon_x) = 0,$$

where

$$(4.13b) \quad g_{\iota\nu} \stackrel{\text{def}}{=} u_\iota^\mu g_{\mu\nu}.$$

Multiplying (4.13a) by $u^\lambda u^\nu$ we obtain

$$\sum_y L_{azy} g_{yz} \epsilon_y = 0,$$

and this equation is equivalent to

$$(4.14) \quad \sum_x L_{azz} g_{xz} \epsilon_x = 0, \quad x \neq z.$$

For a fixed z this system represents $r \geq n - 1^*$ homogeneous equations in $n - 1$ unknowns $g_{xz} = g_{zx}$, $x \neq z$. The matrix of this system is, by virtue of Theorem II (7.2), of rank $n - 1^\dagger$ and therefore (4.14) admits only the trivial solution

$$(4.15a) \quad g_{xz} = g_{zx} = 0, \quad z \neq x.$$

This result holds obviously for any $z = 1, \dots, n$. Multiplying (4.13a) by $u^\lambda u^\nu$, $z \neq t$, we obtain, by virtue of (4.15a),

* $n > 2$, $r = \frac{1}{2} n(n - 1)$.

$^\dagger G_r(K)$ is assumed to be general, $r = \frac{1}{2} n(n - 1)$.

$$(4.16) \quad L_{a_z t}(\epsilon_i g_{i t} - \epsilon_z g_{z z}) = 0, \quad z \neq t.$$

According to Theorem II (7.3) (for the general holonomy group) there is at least one a_0 in the range $1, \dots, r$ for which $L_{a_0 z t} \neq 0$ so that (4.16) yields

$$(4.15b) \quad \epsilon_i g_{i t} = \epsilon_z g_{z z}, \quad z \neq t.$$

Comparing (4.8a) and (4.15) we obtain

$$(4.17) \quad g_{\lambda\mu} = e^{2a} {}^0 g_{\lambda\mu}$$

where q is a function to be determined from (4.2). The Christoffel symbols involved in (4.1) are related by

$$(4.18) \quad \begin{aligned} (a) \quad \{\lambda^\nu_\mu\} &= {}^0\{\lambda^\nu_\mu\} + Q_{\lambda\mu}{}^\nu, \\ (b) \quad Q_{\lambda\mu}{}^\nu &= Q_{\mu\lambda}{}^\nu \stackrel{\text{def}}{=} \delta^\nu_\lambda q_\mu + \delta^\nu_\mu q_\lambda - q^\nu {}^0 g_{\lambda\mu}, \\ (c) \quad q_\mu &\stackrel{\text{def}}{=} \partial_\mu q, \quad q^\nu = q_\alpha {}^0 g^{\alpha\nu}. \end{aligned}$$

Therefore the curvature tensors are related by

$$(4.19) \quad K_{\omega\mu\lambda}{}^\nu = {}^0 K_{\omega\mu\lambda}{}^\nu + 2({}^0 \nabla_{[\mu} Q_{\omega]\lambda}{}^\nu + Q_{\alpha[\mu}{}^\nu Q_{\omega]\lambda}{}^\alpha).$$

Comparing (4.2) and (4.19a) we obtain (2.7), and this equation is equivalent to

$$(4.20a) \quad \delta_{[\omega}^\nu Q_{\mu]\lambda} - {}^0 g_{\lambda[\omega} Q_{\mu]}{}^\nu = 0$$

where

$$(4.20) \quad \begin{aligned} (b) \quad Q_{\mu\lambda} &\stackrel{\text{def}}{=} {}^0 \nabla_\mu q_\lambda - q_\mu q_\lambda + \frac{1}{2} {}^0 g_{\lambda\mu} q^\alpha q_\alpha, \\ (c) \quad Q_\mu{}^\nu &\stackrel{\text{def}}{=} Q_{\mu\alpha} {}^0 g^{\alpha\nu}. \end{aligned}$$

The equation (4.20a) yields

$$(4.21) \quad (n-2)Q_{\mu\lambda} = -{}^0 g_{\lambda\mu} Q_\alpha{}^\alpha.$$

Because $n > 2$ we may substitute $Q_{\mu\lambda}$ from this equation into (4.20a). In doing so we obtain easily $Q_\alpha{}^\alpha = 0$, so that (4.21) reduce to

$$Q_{\mu\lambda} = 0,$$

equivalent to

$$(4.22) \quad {}^0 \nabla_\mu q_\lambda = q_\mu q_\lambda - \frac{1}{2} q^\alpha q_\alpha {}^0 g_{\lambda\mu}.$$

An easy inspection shows that the first integrability conditions of (4.22) are

$$(4.23a) \quad {}^0 K_{\omega\mu\lambda}{}^\nu q_\nu = 0.$$

Because ${}^0 G_r({}^0 K)$ is identical with $G_r(K)$ and this latter group is perfect, the equation (4.23) is equivalent, by virtue of (4.3), to

$$L_{a\lambda}{}^\nu q^\lambda = 0$$

or in nonholonomic form

$$(4.23b) \quad 0 = \sum_x L_{axx} q^x, \quad z \neq x.$$

For a fixed z the system (4.23b) consists of $r \geq n - 1^*$ linear homogeneous equations for $n - 1$ unknowns q^x , $x \neq z$. Its matrix $((L_x))$ is, according to Theorem II (7.2), of rank $n - 1$, so that (4.23b) admits only a trivial solution

$$q^x = 0, \quad x \neq z.$$

This result holds obviously for all $z = 1, \dots, n$, so that we have $q^r = 0$. This equation is equivalent to

$$(4.24) \quad q = c, \quad c = \text{const.}$$

This value reduces (4.23a) to an identity and (4.17) to (4.7).

Theorem (4.1) could be expressed in a slightly different form.

Corollary (4.1). *Let the holonomy group $G_r(K)$ of a V_n , $n > 2$, be general and perfect. Put $L_{\omega\mu\lambda}{}^r = K_{\omega\mu\lambda}{}^r$ in (1.3). If ${}^0g_{\lambda\mu} = {}^0g_{\mu\lambda}$ is a solution (of rank n) of (1.3), then the most general solution of this system is $g_{\lambda\mu}$ given by (4.7).*

The next theorem deals with solutions of (4.10a).

Theorem (4.2). *Let 0V_n be a Riemann space with the general holonomy group. The most general solution $h_{\lambda\mu} = h_{\mu\lambda}$ of*

$$(4.25) \quad {}^0\nabla_\mu h_{\lambda\nu} = 0$$

which is of rank $< n$ is

$$(4.26) \quad h_{\lambda\nu} = 0.$$

If ${}^0g_{\lambda\nu} = h_{\lambda\nu}$ is a solution of rank n of (4.25), then the most general solution $g_{\lambda\mu} = g_{\mu\lambda}$ of rank n is given by (4.7).

Proof. Any solution of (4.25) must satisfy the condition

$$(4.27) \quad {}^0L_{a\lambda}{}^\alpha h_{\alpha\nu} + {}^0L_{a\nu}{}^\alpha h_{\lambda\alpha} = 0.$$

If the metric tensor ${}^0g_{\lambda\nu}$ of 0V_n is referred to the nonholonomic frame $u^r, \underset{b}{u}_\lambda$, so that (4.8a) holds, and if the holonomy group ${}^0G_r({}^0K)$ is general, then, using the first part of the proof of Theorem (4.1), we obtain instead of (4.15) the analogous conditions

$$(4.28) \quad h_{xz} = 0, \quad x \neq z, \quad h_{zz}\epsilon_z = h_{t\epsilon_t}, \quad z \neq t.$$

Hence if the rank of h_{ab} is $< n$, then we obtain at once (4.26) and for $h_{\lambda\nu}$ of rank n we obtain

$$(4.29a) \quad h_{\lambda\mu} = e^{2\alpha} {}^0g_{\lambda\mu}.$$

* $n > 2$, $r = \frac{1}{2} n(n - 1)$.

Substituting from this equation into (4.10a) we have

$$(4.29b) \quad q = \text{const.}$$

The equations (4.29) lead to

$$h_{\lambda\mu} = {}^0g_{\lambda\mu} \text{ const.}$$

or, in a slightly different notation, to (4.7).

Remark. Observe that in this theorem we did not require that ${}^0G_r({}^0K)$ be perfect and did not restrict n by the requirement $n > 2$.

5. All solutions of (1.3) in the case of a degenerate holonomy group. In this section we shall consider the same problem as in the previous section *assuming that the holonomy group $G_r(K)$ of V_n is degenerate.**

Let us denote by $u^{\nu}, u_{\lambda}^{\nu}$ the nonholonomic frame consisting again of n mutually orthogonal vectors u^{ν} (with respect to the metric tensor $g_{\lambda\mu}$ of V_n) and of the inverse vectors

$$\tilde{u}_{\lambda} = \epsilon_x u^{\nu} g_{\nu\lambda} = \epsilon_x \tilde{u}_{\lambda}.$$

The first q vectors $u^{\nu**}$ will be chosen in the local E_q described by $L_a^{\lambda\nu}$ and the remaining u^{ν} in the local $E_{n-q} \perp E_q$. Hence we have also

$$(5.1) \quad (a) \quad \tilde{u}_{\lambda}^1 = \epsilon_{x_1} u_{\lambda}^1, \quad (b) \quad \tilde{u}_{\lambda}^2 = \epsilon_{x_2} u_{\lambda}^2.$$

In the following two lemmas, which will be proved simultaneously, we shall describe some properties of the vectors (5.1).

Lemma (5.1a). *The vectors u^{ν} are V_q -forming.*

Lemma (5.1b). *There is a set of $n - q$ scalars u such that the vectors*

$$(5.2) \quad u_{\lambda}^{a_2} \stackrel{\text{def}}{=} \partial_{\lambda} u^{a_2}$$

have the following properties:

- (1) *They are mutually perpendicular unit vectors.*
- (2) *The vectors*

$$(5.3) \quad u^{\nu} \stackrel{\text{def}}{=} \epsilon_{x_2} u^{x_2\nu}$$

*See Definition II (6.4).

**The range of

$$\begin{cases} a_1, b_1, \dots, x_1, y_1, z_1, t_1 \\ a_2, b_2, \dots, x_2, y_2, z_2, t_2 \end{cases} \text{ is } \begin{cases} 1, \dots, q \\ q+1, \dots, n \end{cases}.$$

The indices $x_1, \dots, t_2$ do not follow the summation convention.

are V_{n-q} -forming, $V_{n-q} \perp V_q$.

(3) They are covariant constant:

$$(5.4) \quad (a) \quad \nabla_{\mu}^{a_2} u_{\lambda} = 0, \quad (b) \quad \nabla_{\mu} u_{a_2}^{\nu} = 0.$$

Proof. The complete set of integrability conditions of

$$(5.5a) \quad \nabla_{\mu} v^{\nu} = v^{\nu} \partial_{\mu} \sigma$$

is

$$(5.5b) \quad L_{a\lambda}^{\nu} v^{\lambda} = 0. *$$

Hence any solution of (5.5a) (and (5.5b)) must be perpendicular to the local E_q 's spanned by the bivectors $L_a^{\lambda\nu}$, i.e., perpendicular to u^{ν} . Denote by v^{ν} any such vector which is not a null vector.** Hence the corresponding unit vector

$$w^{\nu} \stackrel{\text{def}}{=} v^{\nu} / \sqrt{|v^{\alpha} v_{\alpha}|}$$

must satisfy, by virtue of (5.5a), the condition

$$(5.6) \quad (a) \quad \nabla_{\mu} w_{\nu} = 0, \quad (b) \quad \nabla_{\mu} w^{\nu} = 0.$$

Denote by w^{ν} $n - q$ linearly independent unit vectors, solutions of (5.5b) and (5.6a). Then $g_{\lambda\mu} w^{\lambda} w^{\mu}$ is constant. Hence there is a set of constants $C_{a_2}^{b_2}$ such that the vectors

$$(5.7) \quad u_{a_2}^{\nu} \stackrel{\text{def}}{=} C_{a_2}^{b_2} w_{b_2}^{\nu}, \quad u_{\nu}^{x_2} \stackrel{\text{def}}{=} \epsilon_{x_2} u_{\nu}$$

are mutually perpendicular vectors satisfying the conditions (5.4) and

$$(5.8) \quad (a) \quad u_{a_1}^{\nu} u_{\nu}^{b_2} = 0, \quad (b) \quad u_{\lambda}^{a_1} u_{b_2}^{\lambda} = 0.$$

The equation (5.4a) leads at once to (5.2). Moreover,

$$u^{\lambda} u^{\mu} \nabla_{\mu} u_{\lambda}^{a_2} = 0,$$

so that the vectors u^{ν} are V_q -forming. Using (5.8b) we obtain, by virtue of (5.4b),

$$2 u^{\lambda} u^{\mu} \nabla_{\mu} u_{\lambda}^{a_2} = - u^{\mu} u_{\lambda}^{a_2} \nabla_{\mu} u^{\lambda} + u^{\mu} u_{\lambda}^{a_2} \nabla_{\mu} u^{\lambda} = 0.$$

Hence the $n - q$ vectors u^{ν} are V_{n-q} -forming and obviously every V_q is perpendicular to every V_{n-q} at the common point of V_q and V_{n-q} .

The results of Lemma (5.1) will be used in the following two theorems.

*See Theorem II (4.ii).

**The E_q described by $L_a^{\lambda\nu}$ is not tangent to the local cone $p^{\lambda} p^{\mu} g_{\lambda\mu} = 0$. Hence we may choose $n - q$ linearly independent vectors $v^{\nu} \perp E_q$ which are not null vectors.

Theorem (5.1). Let 0V_n , $n > 2$, be a Riemann space* with a degenerate perfect holonomy group ${}^0G_r({}^0K)$. Let V_n be another Riemann space with the metric tensor

$$(5.9) \quad g_{\lambda\mu} = e^{2\alpha} {}^0g_{\lambda\mu}.$$

A necessary and sufficient condition for

$$(5.10) \quad K_{\omega\mu\lambda}{}^{\nu} = {}^0K_{\omega\mu\lambda}{}^{\nu}$$

is that there exists a scalar q such that the gradient

$$(5.11) \quad q_{\mu} = \partial_{\mu}q$$

satisfies the condition

$$(5.12) \quad (a) \quad q_{\lambda}q_{\mu}{}^0g^{\lambda\mu} = 0, \quad (b) \quad {}^0\nabla_{\mu}q_{\lambda} = q_{\mu}q_{\lambda}^{**}$$

so that q is a harmonic function:

$$(5.13) \quad {}^0\Delta_2(q) \stackrel{\text{def}}{=} {}^0g^{\lambda\mu} {}^0\nabla_{\lambda}q_{\mu} = 0.$$

Proof. Starting with (5.9) we obtain (4.18) and (4.19). Assuming that (5.10) holds we obtain (4.20)–(4.22). The first set of integrability conditions of (4.22) reduces to

$$(5.14a) \quad {}^0L_{\alpha\lambda}{}^{\nu}q^{\lambda} = 0.$$

If we denote by ${}^0u_{\alpha}{}^{\nu}$, ${}^0u_{\lambda}{}^{\nu}$ the vectors $u_{\alpha}{}^{\nu}$, $u_{\lambda}{}^{\nu}$ of the previous lemmas we see from (5.14a) that q^{ν} must be a linear combination of ${}^0u_{\alpha}{}^{\nu}$:

$$(5.14b) \quad q^{\nu} = \omega_{\alpha}{}^{\nu} {}^0u_{\alpha}{}^{\nu}.$$

The second set of integrability conditions of (4.22) reduces to

$$(5.15) \quad L_{\alpha\mu}{}^{\nu}q^{\alpha}q_{\alpha} = 0$$

and yields (5.12a). Substituting this value into (4.22) we obtain (5.12b) and (5.13).

Assume now that (5.12) holds. Then (4.20b) reduces to $Q_{\mu\lambda} = 0$, so that (4.20a) is identically satisfied and therefore (5.10) holds.

In the next theorem we shall use equation (5.2), writing ${}^{0\alpha}u$, ${}^{0\alpha}u_{\lambda}$ instead of ${}^{\alpha}u$, ${}^{\alpha}u_{\lambda}$.

Theorem (5.2). Let $C_{\alpha\alpha} \neq 0$ be a set of arbitrary constants satisfying the relation

$$(5.16) \quad \sum_{x_{\alpha}} \epsilon_{x_{\alpha}} (C_{x_{\alpha}})^2 = 0, \quad (\epsilon_{x_{\alpha}} \stackrel{\text{def}}{=} {}^0g_{\lambda\mu} {}^0u^{\lambda} {}^0u^{\mu}).$$

*The meaning of "0" preceding a kernel letter is the same as in the previous section.

**For a solution q of these equations see Theorem (5.2).

Then

$$(5.17) \quad q \stackrel{\text{def}}{=} -\ln C_{a_2} {}^{0a_2}u$$

is a solution of (5.12).

Proof. We have from (5.17)

$$C_{a_2} {}^{0a_2}u = \bar{e}^a$$

so that, according to (5.2),

$$-q_\lambda \stackrel{\text{def}}{=} -\partial_\lambda q = C_{a_2} {}^{0a_2}u_\lambda e^a$$

and, by virtue of (5.4a),

$$-{}^0\nabla_\mu q_\lambda = q_\mu e^a C_{a_2} {}^{0a_2}u_\lambda = -q_\mu q_\lambda.$$

Moreover, according to (5.16),

$$q_\lambda q_\mu {}^0g^{\lambda\mu} = e^{2a} C_{a_2} C_{b_2} {}^{0a_2}u_\lambda {}^{0b_2}u_\mu {}^0g^{\lambda\mu} = e^{2a} \sum_{x_2} \epsilon_{x_2} (C_{x_2})^2 = 0.$$

Remark. According to Theorem (4.1), the equality of curvature tensors could be achieved in the case of a perfect and general group only by a constant stretching of metric. If the group is degenerate and perfect it is possible to achieve the equality of curvature tensors by means of a certain conformal transformation which is not a constant stretching. In Section 7 we shall find all possible tensors with the same curvature tensor for the case of a degenerate holonomy group, and in Section 8 we shall deal with a similar problem for an arbitrary holonomy group.

6. Illustration. Several of the previous results may be applied even if the holonomy group does not display all the characteristic features of a degenerate group. We shall discuss here one such case when $n = 4$. Denote by z the function

$$z \stackrel{\text{def}}{=} x^{\text{III}} - x^{\text{IV}}$$

and by $A, B, C > 0, D, E$ five functions of z . Define the metric tensor $g_{\lambda\mu}$ by

$$(6.1) \quad ((g_{\lambda\mu})) \stackrel{\text{def}}{=} \begin{bmatrix} -A & -D & 0 & 0 \\ -D & -B & 0 & 0 \\ 0 & 0 & E - C & -E \\ 0 & 0 & -E & E + C \end{bmatrix},$$

assuming that $(dx^{\text{I}})^2 g_{\text{II}} + 2 dx^{\text{I}} dx^{\text{II}} g_{\text{I II}} + (dx^{\text{II}})^2 g_{\text{II II}}$ is negative definite. The curvature tensor $K_{\omega\mu\lambda\nu}$ of $g_{\lambda\mu}$ is given by the following table of its components $K_{\omega\mu\lambda\nu}$:

(6.2)

$\begin{matrix} \lambda \nu \\ \omega \mu \end{matrix}$	I II	III IV	II III	I IV	III I	II IV
I II	0	0	0	0	0	0
III IV	0	0	0	0	0	0
II III	0	0	v	$-w$	$-w$	$-v$
I IV	0	0	$-w$	u	u	w
III I	0	0	$-w$	u	u	w
II IV	0	0	$-v$	w	w	v

where u, v, w are explicitly given functions of A, B, C, D and their first two derivatives with respect to z^* and $wv - w^2$ is not identically equal to zero. We shall assume

$$(6.3) \quad wv - w^2 \neq 0.$$

The table (6.2) shows that the $\lambda \nu$ rank of $K_{\omega \mu \lambda \nu}$ is 2. If we put

$$(6.4) \quad \begin{aligned} & \text{I II} \quad \text{III IV} \quad \text{II III} \quad \text{I IV} \quad \text{III I} \quad \text{II IV} \\ (a) \quad L_{1 \lambda \nu} & \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}, \\ (b) \quad L_{2 \lambda \nu} & \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \\ (c) \quad ((h^{a*})) & = \begin{bmatrix} v & -w \\ -w & u \end{bmatrix}, \end{aligned}$$

we have

$$(6.5) \quad K_{\omega \mu \lambda \nu} = h^{a*} L_{a \omega \mu} L_{\epsilon \lambda \nu}, \quad a = 1, 2.$$

From (6.3) and (6.4c) we see that

$$\text{Det}((h^{a*})) \neq 0.$$

Therefore** the holonomy group $G_r(K)$ is perfect and $r = 2$. The equations (5.5b), equivalent to

$$(6.6a) \quad L_{a \lambda \nu} v^\lambda = 0,$$

*The formulas (6.1) (6.2) are taken from [3].

**See Theorem II (8.3).

admit only one solution (up to an arbitrary factor of proportionality), namely

$$(6.6b) \quad v''(0, 0, 1, 1).$$

Therefore the bivectors $L_a{}^{\lambda\nu}$ span an E_3 given by the vector $v_\lambda = v''g_{\nu\lambda}$, *i.e.*

$$(6.6c) \quad v_\lambda(0, 0, -C, C).$$

This E_3 is isotropic because

$$(6.7) \quad v^\lambda v_\lambda = 0.$$

Our holonomy group displays one characteristic feature of a degenerate group, namely, the $L_a{}^{\lambda\nu}$ span an E_q , $q = 3$. However, it is not a degenerate group, for this E_q is isotropic and $r = 2 \neq \frac{1}{2}q(q-1)$.

From the manner in which we obtained v_λ it is clear that this is the only field defining parallel three-directions. As a matter of fact, an easy inspection based on (6.1) shows that

$$(6.8) \quad \nabla_\mu v_\lambda = 0.$$

Hence v_λ is the only vector field (up to an arbitrary constant factor of proportionality) which is covariant constant.* Hence if γ is an arbitrary constant, every tensor $h_{\lambda\mu}$ of the pencil

$$(6.9) \quad h_{\lambda\mu} = \alpha g_{\lambda\mu} + \gamma v_\lambda v_\mu \quad (\alpha \neq 0 \text{ const.})$$

is of rank 4 and covariant constant:

$$(6.10) \quad \nabla_\mu h_{\lambda\nu} = 0,$$

and therefore yields the same Christoffel symbols and the same curvature tensor (6.2) as $g_{\lambda\mu}$.

There is still another tensor which has the same curvature tensor as $g_{\lambda\mu}$. In fact, Theorem (5.1) is based on the fact that there is a gradient (5.11) for which (5.14a) holds. Therefore it is to be expected that (6.6a) will lead to a similar result. Put

$$(6.11a) \quad v \stackrel{\text{def}}{=} - \int C \, dz.$$

Then v_λ , defined by (6.6c), is of the form

$$(6.11b) \quad v_\lambda = \partial_\lambda v.$$

If we put

$$(6.12) \quad q \stackrel{\text{def}}{=} -\ln v, \quad v = e^{-q},$$

*TAKENO [3] obtained this result by means of some differential equations, without the help of the holonomy group.

we obtain, by virtue of (6.8),

$$(6.13) \quad \begin{aligned} (a) \quad q_\lambda &= \partial_\lambda q = -e^a v_\lambda, \\ (b) \quad \nabla_\mu q_\lambda &= -v_\lambda q_\mu e^a = q_\lambda q_\mu, \end{aligned}$$

and, according to (6.7) and (6.13a),

$$(6.13c) \quad q_\lambda q^\lambda = 0.$$

Hence Theorem (5.1) could be applied: *The tensor*

$$G_{\lambda\mu} \stackrel{\text{def}}{=} e^{2a} g_{\lambda\mu} = g_{\lambda\mu} v^{-2} = \frac{g_{\lambda\mu}}{\left(\int C dz\right)^2}$$

has the same curvature tensor (6.2) as $g_{\lambda\mu}$. Of course, $G_{\lambda\mu}$ is not covariant constant with respect to the connection of Christoffel symbols of $g_{\lambda\mu}$.

7. Continuation of Section 5. In this section we shall pursue our investigation of the V_n ($n > 2$) with a degenerate perfect holonomy group $G_r(K)$. Put

$$(7.1a) \quad T_\lambda{}^\alpha \stackrel{\text{def}}{=} u_\lambda^{a_1} u^\alpha_{a_1}, \quad N_\lambda{}^\alpha \stackrel{\text{def}}{=} u_\lambda^{a_2} u^\alpha_{a_2}.$$

$T_\lambda{}^\alpha$ ($N_\lambda{}^\alpha$) is the unit tensor of the V_a 's (V_{n-a} 's) mentioned in Lemmas (5.1) and

$$(7.1b) \quad g'_{\lambda\mu} = T_\lambda{}^\alpha g_{\alpha\mu} = T_\mu{}^\alpha g_{\alpha\lambda} \quad (g''_{\lambda\mu} = N_\lambda{}^\alpha g_{\alpha\mu} = N_\mu{}^\alpha g_{\alpha\lambda})$$

is their metric tensor. This follows at once from the obvious formulas

$$(7.2) \quad \begin{aligned} (a) \quad \delta_\lambda{}^\alpha &= T_\lambda{}^\alpha + N_\lambda{}^\alpha, \\ (b) \quad T_\lambda{}^\alpha N_\alpha{}^\beta &= T_\alpha{}^\beta N_\lambda{}^\alpha = 0, \\ (c) \quad u^\lambda_{a_1} N_\lambda{}^\alpha &= u^\lambda_{a_2} T_\lambda{}^\alpha = 0, \\ (d) \quad u^\alpha_{a_1} N_\lambda{}^\alpha &= u^\alpha_{a_2} T_\lambda{}^\alpha = 0. \end{aligned}$$

In the next lemma we denote by $K'_{\omega\mu\lambda}{}^\alpha$ ($K''_{\omega\mu\lambda}{}^\alpha$) the curvature tensor of the V_a 's (of the V_{n-a} 's).

Lemma (7.1). *We have*

$$(7.3) \quad \begin{aligned} (a) \quad g_{\lambda\mu} &= g'_{\lambda\mu} + g''_{\lambda\mu}, \\ (b) \quad K'_{\omega\mu\lambda}{}^\alpha &= K_{\omega\mu\lambda}{}^\alpha, \\ (c) \quad K''_{\omega\mu\lambda}{}^\alpha &= 0, \end{aligned}$$

so that the V_{n-a} 's reduce to E_{n-a} 's.

Proof. (7.3a) follows immediately from

$$g_{\lambda\mu} = \delta_\lambda{}^\alpha \delta_\mu{}^\beta g_{\alpha\beta}$$

and from (7.1), (7.2). In order to prove (7.3b, c) let us introduce first the second fundamental tensors $\overset{a_2}{h}_{\lambda\mu}$ ($\overset{a_1}{h}_{\lambda\mu}$) of the V_a 's (V_{n-a} 's):

$$\overset{a_2}{h}_{\lambda\mu} \stackrel{\text{def}}{=} T_{\lambda}^{\alpha} T_{\mu}^{\beta} \nabla_{\alpha} \overset{a_2}{u}_{\beta} \quad (\overset{a_1}{h}_{\lambda\mu} \stackrel{\text{def}}{=} N_{\lambda}^{\alpha} N_{\mu}^{\beta} \nabla_{\alpha} \overset{a_1}{u}_{\beta}).$$

Using (5.4a) we obtain at once

$$(7.4a) \quad \overset{a_2}{h}_{\lambda\mu} = 0$$

and, by virtue of (7.2c, d) and (5.4b),

$$(7.4b) \quad \overset{a_1}{h}_{\lambda\mu} = -\overset{a_1}{u}_{\beta} N_{\lambda}^{\alpha} \nabla_{\alpha} N_{\mu}^{\beta} = -\overset{a_2}{u}_{\beta} N_{\lambda}^{\alpha} (\nabla_{\alpha} \overset{a_2}{u}^{\beta}) \overset{a_1}{u}_{\mu}^{\alpha} = 0.$$

Hence Gauss' equations reduce here to

$$(7.5) \quad \begin{aligned} (a) \quad K'_{\omega\mu\lambda\nu} &= T_{\omega}^{\alpha} T_{\mu}^{\beta} T_{\lambda}^{\gamma} T_{\nu}^{\delta} K_{\alpha\beta\gamma\delta}, \\ (b) \quad K''_{\omega\mu\lambda\nu} &= N_{\omega}^{\alpha} N_{\mu}^{\beta} N_{\lambda}^{\gamma} N_{\nu}^{\delta} K_{\alpha\beta\gamma\delta}. \end{aligned}$$

The group under consideration is degenerate and the $L_a^{\lambda\nu}$ span the space of the vectors $\overset{a_1}{u}^{\nu}$ perpendicular to $\overset{a_2}{u}^{\nu}$. Therefore

$$(7.6a) \quad L_{a\alpha\lambda} N_{\mu}^{\alpha} = -L_{a\lambda\alpha} N_{\mu}^{\alpha} = 0$$

so that

$$(7.6b) \quad L_{a\alpha\lambda} T_{\mu}^{\alpha} = L_{a\mu\lambda} = L_{a\alpha\beta} T_{\mu}^{\alpha} T_{\lambda}^{\beta}.$$

The equations (7.5), (7.6) and

$$K_{\omega\mu\lambda\nu} = h^{a*} L_{a\omega\mu} L_{a\lambda\nu}$$

lead at once to (7.3b, c).

In order to formulate the next theorem let us label all objects we considered in this section by "0" so that, in particular, we have instead of (7.3) the equations

$$(7.7) \quad \begin{aligned} (a) \quad {}^0g_{\lambda\mu} &= {}^0g_{\lambda\mu}' + {}^0g''_{\lambda\mu}, \\ (b) \quad {}^0K'_{\omega\mu\lambda\nu} &= {}^0K_{\omega\mu\lambda\nu}, \\ (c) \quad {}^0K''_{\omega\mu\lambda\nu} &= 0. \end{aligned}$$

Moreover, denote by $s_{a_1b_1} = s_{b_1a_1}$ an arbitrary set of constants not all equal to zero and put

$$(7.8) \quad s_{\lambda\mu} \stackrel{\text{def}}{=} s_{a_2b_2} {}^{0a_2}u_{\lambda} {}^{0b_2}u_{\mu}.$$

Theorem (7.1). *Let 0V_n be a Riemann space with the metric tensor ${}^0g_{\lambda\mu}$ and curvature tensor ${}^0K_{\omega\mu\lambda\nu}$ and with the degenerate holonomy group ${}^0G_r({}^0K)$. The tensor*

$$(7.9a) \quad g_{\lambda\mu} = c {}^0g_{\lambda\mu} + s_{\lambda\mu} \quad (c = \text{const.} \neq 0)$$

has the following properties:

(a) It is of rank R where $R = q, q + 1, \dots, n$ according to the choice of $s_{a,b}$.

(b) It is the most general solution of rank R of

$$(7.10a) \quad {}^0\nabla_\mu g_{\lambda\nu} = 0.$$

(c) If $R = n$, then it has the curvature tensor ${}^0K_{\omega\mu\lambda}$ satisfying (7.7b).

The most general symmetric solution of

$$(7.10b) \quad {}^0\nabla_\mu h_{\lambda\nu} = 0, \quad h_{\lambda\alpha} T_\mu^\alpha = 0$$

of rank $R'' \leq n - q$ is

$$(7.9b) \quad h_{\lambda\mu} = s_{\lambda\mu}$$

where the constants $s_{a,b}$ are chosen in such a way that the rank of $s_{\lambda\mu}$ is R'' .

Proof. The nonholonomic components g_{ab} of $g_{\lambda\mu}$ with respect to the frame ${}^0u_\lambda, {}^0u^\nu$ are, according to (7.9a),

$$(7.11) \quad ((g_{ab})) \stackrel{\text{def}}{=} \begin{pmatrix} \epsilon_1 c & & & & & \\ & \ddots & & & & \\ & & \epsilon_q c & & & \\ & & & \epsilon_{q+1} c + s_{q+1\ q+1} & \cdots & s_{q+1\ n} \\ & 0 & & \vdots & & \vdots \\ & & s_{n\ q+1} & \cdots & \epsilon_n c + s_{nn} & \end{pmatrix}.$$

Because the rank of g_{ab} is the same as the rank of $g_{\lambda\mu}$ the statement (a) follows immediately from (7.11).

The complete set of integrability conditions of (7.10a) reduce to

$$(7.12) \quad {}^0L_{a\lambda}{}^\alpha g_{\alpha\nu} + {}^0L_{a\nu}{}^\alpha g_{\alpha\lambda} = 0$$

for any tensor $g_{\lambda\mu}$ (not necessarily given by (7.9a)). Using the nonholonomic frame ${}^0u_\alpha, {}^0u^\alpha$ derived from ${}^0g_{\lambda\mu}$, we may use instead of (7.12) an equation similar to (4.13a), namely

$$(7.13a) \quad \sum_{x,y} {}^0L_{axy} (g_{y(\nu)} {}^0u_\lambda^x \epsilon_y - g_{x(\nu)} {}^0u_\lambda^y \epsilon_x) = 0$$

where

$$(7.13b) \quad g_{\nu t} = g_{t\nu} \stackrel{\text{def}}{=} {}^0u^\alpha g_{\alpha\nu}.$$

Some of the components ${}^0L_{axy} = -{}^0L_{ayx}$ vanish in our case because of (7.6a) (labeled by "0"):

$$(7.13c) \quad {}^0L_{axy_2} = {}^0L_{ax_2y} = 0.$$

Hence if we multiply (4.13a) by $\begin{pmatrix} {}^0u^\lambda & {}^0u^\nu \\ z_1 & z_1 \\ {}^0u^\lambda & {}^0u^\nu \\ z_1 & t_2 \end{pmatrix}$

we obtain

$$(7.14) \quad \begin{aligned} (a) \quad & \sum_{x_1} {}^0L_{ax_1x_1}\epsilon_{x_1}g_{x_1z_1} = 0, \quad z_1 \neq x_1, \\ (b) \quad & \sum_{x_1} {}^0L_{ax_1x_1}\epsilon_{x_1}g_{x_1t_2} = 0, \quad z_1 \neq x_1. \end{aligned}$$

If we keep $\begin{cases} z_1 \\ z_1, t_2 \end{cases}$ fixed, the system (7.14) $\begin{cases} (a) \\ (b) \end{cases}$ consists of $r = \frac{1}{2}q(q-1)$ linear homogeneous equations for $q-1$ unknowns $\begin{cases} g_{x_1z_1} \\ g_{x_1t_2} \end{cases}$, $x_1 \neq z_1$. By virtue of Theorem II (7.2) (for a degenerate holonomy group), the matrix $((L_x))$ of this system is of rank $q-1$ and therefore this system admits only the trivial solution

$$(7.15) \quad \begin{aligned} (a) \quad & g_{x_1z_1} = 0 \\ (b) \quad & g_{x_1t_2} = 0 \end{aligned} \quad x_1 = 1, 2, \dots, z_1-1, z_1+1, \dots, q.$$

This result holds obviously for any $\begin{cases} z_1 \\ z_1, t_2 \end{cases}$. Multiplying (7.13a) by $\begin{smallmatrix} {}^0u^\lambda & {}^0u^\nu \\ z_1 & t_1 \end{smallmatrix}$ ($z_1 \neq t_1$) we obtain, by virtue of (7.15),

$$(7.16) \quad L_{ax_1t_1}(\epsilon_{t_1}g_{t_1t_1} - \epsilon_{x_1}g_{x_1z_1}) = 0, \quad z_1 \neq t_1.$$

Using now Theorem II (7.3) for the degenerate group, we obtain

$$(7.15c) \quad g_{t_1t_1}\epsilon_{t_1} = g_{x_1z_1}\epsilon_{x_1} = e^{2q}, \quad z_1 \neq t_1,$$

where g is an arbitrary function. Multiplying (7.13a) by $\begin{smallmatrix} {}^0u^\lambda & {}^0u^\nu \\ z_2 & t_2 \end{smallmatrix}$ we obtain

the identity $0 = 0$. Hence the complete set (7.13) of integrability conditions does not yield $g_{x_2y_2}$ and therefore we may put

$$(7.15d) \quad g_{x_2y_2} = e^{2q}\delta_{x_2y_2}\epsilon_{x_2} + s_{x_2y_2}$$

where $s_{x_2y_2}$ are arbitrary functions. Comparing (7.15) with

$$(7.17) \quad {}^0g_{ab} = \begin{cases} 0 & \text{for } a \neq b, \\ \epsilon_a & a = b \end{cases},$$

we see that

$$(7.18) \quad \begin{aligned} g_{\lambda\mu} &= e^{2q} {}^0g_{\lambda\mu} + s_{\lambda\mu}, \\ S_{\lambda\mu} &\stackrel{\text{def}}{=} \sum_{x_2, y_2} S_{x_2y_2} \begin{smallmatrix} {}^0x_2 & {}^0y_2 \\ u_\lambda & u_\mu \end{smallmatrix}. \end{aligned}$$

Substituting from (7.18) into (7.10a) we obtain, by virtue of (5.4a),

$${}^0\nabla_\mu g_{\lambda\nu} = 2e^{2q} \{ (\partial_\mu q) \sum_{x_1} \epsilon_{x_1} \begin{smallmatrix} x_1 & x_1 \\ u_\lambda & u_\nu \end{smallmatrix} + \sum_{x_2, y_2} [(\partial_\mu q) \epsilon_{x_2} \delta_{x_2y_2} + \frac{1}{2}e^{-2q} \partial_\mu s_{x_2y_2}] \begin{smallmatrix} {}^0x_2 & {}^0y_2 \\ u_\lambda & u_\nu \end{smallmatrix} \} = 0.$$

This equation can be satisfied only by

$$q = \text{const.}, \quad s_{x_2 y_2} = \text{const.}$$

In this way we obtain (7.9a). The remaining part of the theorem is obvious.

We are now able to generalize the results of Theorems (5.1) and (5.2).

Theorem (7.2). Let 0V_n ($n > 2$) be a Riemann space with the metric tensor ${}^0g_{\lambda\mu}$ and a curvature tensor ${}^0K_{\omega\mu\lambda}$. Let its holonomy group be degenerate and perfect. Denote by V_n another Riemann space with the metric tensor $G_{\lambda\mu}$ and the curvature tensor $K_{\omega\mu\lambda}$. Sufficient conditions for (5.10) are the relations

$$(7.19) \quad G_{\lambda\mu} = e^{2q} g_{\lambda\mu}$$

where $g_{\lambda\mu}$ is any tensor of rank n described in Theorem (7.1) and the function q satisfies the conditions (5.11)–(5.13) with ${}^0\nabla_\mu$ and ${}^0\Delta$ referring to $g_{\lambda\mu}$.

Theorem (7.3). Denote by $\overset{a}{u}_\lambda$, u^ν a nonholonomic frame of unit vectors derived in the usual way from the tensor $g_{\lambda\mu}$ mentioned in the previous theorem. Put

$$\epsilon_{x_2} \stackrel{\text{def}}{=} g_{\lambda\mu} \overset{a}{u}^\lambda u^\mu = \pm 1$$

and let $C_{a_2} \neq 0$ be a set of arbitrary constants satisfying the relation

$$(7.20) \quad \sum_{x_2} \epsilon_{x_2} (C_{x_2})^2 = 0.$$

Then (5.2) holds and

$$(7.21) \quad q = -\ln \sum_{x_2} C_{x_2} \overset{x_2}{u}$$

is a solution of (5.12) (where ${}^{**}{}^0\nabla$ refers to $g_{\lambda\mu}$).

Proof. According to Lemma (7.1) the V_{n-q} reduces to an E_{n-q} . Hence replacing ${}^0g'_{\lambda\mu}$ by

$$\begin{aligned} s_{\lambda\mu} &= S_{a_2 b_2} {}^{0a_2}{u}_\lambda {}^{0b_2}{u}_\mu, \\ S_{x_2 y_2} &= c \epsilon_{x_2} \delta_{x_2 y_2} + s_{x_2 y_2}, \\ \text{Det } ((S_{x_2 y_2})) &\neq 0 \end{aligned}$$

means only a redefining of the Euclidean metric of the E_{n-q} under consideration. On the other hand, $c {}^0g'_{\lambda\mu}$ has the same curvature tensor ${}^0K'_{\omega\mu\lambda}$ as ${}^0g'_{\lambda\mu}$. Therefore we may apply the method of the proof of Theorem (5.1) and prove in this way the sufficiency of the conditions mentioned in Theorem (7.2).

The proof of Theorem (7.3) follows the pattern of the proof of Theorem (5.2).

Remark I. The significance of Theorem (7.2) lies in the fact that the tensors ${}^0g_{\lambda\mu}$ and $G_{\lambda\mu}$ have the same curvature tensor without being proportional.

Remark II. In previous sections we succeeded in finding the explicit form of tensors which have the same curvature tensor (see equation (1.3)). This

*This remark (inserted only for convenience sake) is not necessary because of (7.10a).

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was due to the fact that we confined ourselves to particular holonomy groups (complete, general perfect, degenerate perfect). In the next sections we shall derive some results concerning any kind of holonomy group. Due to this general point of view we will not be able to find the explicit form of solutions of (1.3) for general n . However, the method displayed in the subsequent sections leads for particular n to the explicit form of solutions of (1.3).

8. Solutions of (1.3). Our starting point will be again the system (1.3), where $L_{\omega\mu\lambda}$ stands for the curvature tensor $R_{\omega\mu\lambda}$ of a volume preserving L_n . Hence the conditions (1.1a-d) must be satisfied.

Denote by X_i^j nonholonomic components of a tensor X_λ^ν , with respect to an arbitrary nonholonomic frame v^ν, v_λ . In subsequent considerations we shall consider a matrix $[X]$ of these components which is defined as follows: It is the matrix of the homogeneous linear system

$$(8.1) \quad \begin{aligned} (a) \quad & Y_{ki}^{ij} g_{ij} = 0, \\ (b) \quad & Y_{kl}^{ij} \stackrel{\text{def}}{=} 4X_{(k}^{(i} \delta_{l)}^{j)} \end{aligned}$$

for the unknowns $g_{ij} = g_{ji}$. For the convenience of the reader we display the matrix $[X]$ in the case $n = 4^*, \dagger$

$kl \backslash ij$	12	13	14	23	24	34	11	22	33	44
12	$X_1^1 + X_2^2$	X_2^3	X_2^4	X_1^3	X_1^4	0	X_2^1	X_1^2	0	0
13	X_3^2	$X_1^1 + X_3^3$	X_3^4	X_1^2	0	X_1^4	X_3^1	0	X_1^3	0
14	X_4^2	X_4^3	$X_1^1 + X_4^4$	0	X_1^2	X_1^3	X_4^1	0	0	X_1^4
23	X_3^1	X_2^1	0	$X_2^2 + X_3^3$	X_3^4	X_2^4	0	X_3^2	X_2^3	0
24	X_4^1	0	X_2^1	X_4^3	$X_2^2 + X_4^4$	X_2^3	0	X_4^2	0	X_2^4
34	0	X_4^1	X_3^1	X_4^2	X_3^2	$X_3^3 + X_4^4$	0	0	X_4^3	X_3^4
11	X_1^2	X_1^3	X_1^4	0	0	0	X_1^1	0	0	0
22	X_2^1	0	0	X_2^3	X_2^4	0	0	X_2^2	0	0
33	0	X_3^1	0	X_3^2	0	X_3^4	0	0	X_3^3	0
44	0	0	X_4^1	0	X_4^2	X_4^3	0	0	0	X_4^4

*We multiplied each row where $k = l$ by $\frac{1}{2}$.

†It is not difficult to derive the rule for the construction of this matrix without the help of (8.1a, b).

Lemma (8.1). $[X]$ is the matrix of the system

$$(8.1c) \quad X_i^i g_{ik} + X_k^i g_{il} = 0$$

for $\frac{1}{2}n(n+1)$ unknown components of a tensor $g_{ij} = g_{ji}$. The rank of $[X]$ is independent of the choice of the nonholonomic frame.

Proof. The first statement follows from (8.1a, b). The system (8.1c) admits p linearly independent non-trivial solutions if and only if the rank of $[X]$ is $\frac{1}{2}n(n+1) - p$. The equation (8.1c) is clearly independent of the choice of the nonholonomic frame and therefore the same must be true for the numbers p , $\frac{1}{2}n(n+1) - p$ which are arithmetic invariants of the system (8.1c).

In subsequent considerations we denote by $[L]_a$ the matrix $[X]$ built up from the elements $X_i^i = L_{a,i}^i$ and by $[L]$ the matrix

$$(8.2) \quad [L] \stackrel{\text{def}}{=} \begin{pmatrix} [L]_1 \\ [L]_2 \\ \vdots \\ [L]_r \end{pmatrix}.$$

Theorem (8.1a). A sufficient condition for the L_n under consideration not to be metrisable* is that the rank of $[L]$ is not less than $\frac{1}{2}n(n+1)$.

Proof. If our L_n is metrisable then the system (2.4a) admits at least one solution. The integrability conditions of (2.4a) reduce in the nonholonomic form to

$$(8.3) \quad L_{a,i}^i g_{ik} + L_{a,k}^i g_{il} = 0$$

and the matrix of this system is $[L]$. The proof follows from these facts.

Theorem (8.1b). The condition introduced in Theorem (8.1a) is not necessary.

Proof. It is sufficient to exhibit an L_n , the matrix $[L]$ of which is of rank less than $\frac{1}{2}n(n+1)$, while the corresponding system (8.3) admits only solutions of rank less than n . Consider the case $n = 3$, $r = 2$ and assume

$$(8.4)^\dagger \quad ((L_{1a}^b)) \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad ((L_{2a}^b)) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The tensors $L_{a,a}^b$ are linearly independent and satisfy the condition

$$(8.5) \quad L_{ab}^b = 0.$$

*I.e., for (1.3) not to admit any solution $g_{\lambda\mu} = g_{\mu\lambda}$ of rank n .

†The index $\begin{Bmatrix} b \\ a \end{Bmatrix}$ labels the $\begin{Bmatrix} \text{columns} \\ \text{rows} \end{Bmatrix}$.

The matrix $[L]$ built up from (8.4) is of rank $5 < \frac{1}{2}n(n+1)$, so that (8.3) admits only one solution (up to a multiplicative factor), namely

$$((g_{ab})) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \xi \\ 0 & \xi & 0 \end{pmatrix}$$

with arbitrary ξ . This solution is of rank $2 < n$.

Corollary (8.1). *A sufficient condition for the L_n under consideration not to be metrisable is that one of the matrices $[L]_a$ is of rank $\frac{1}{2}n(n+1)$.*

The proof, based on Theorem (8.1a), is obvious.

Remark. If the holonomy group of our L_n is complete and the eigenvalues $\lambda_x \neq 0$ of its complete tensor $L_{1\lambda}^x$ satisfy the condition

$$|\lambda_x| \neq |\lambda_y|, \quad x \neq y, *$$

then $[L]_1$ is obviously of rank $\frac{1}{2}n(n+1)$, and therefore the L_n under consideration is not metrisable.

The following two theorems will be proved simultaneously.

Theorem (8.2a). *A necessary condition for the L_n under consideration to be metrisable is that the rank of $[L]$ is $R = \frac{1}{2}n(n+1) - p$. If this condition is satisfied then there exists a set of linearly independent solutions ${}^s g_{ab}$ ($s = 1, \dots, p$) of (8.3) and a set of p functions ${}^s q$ such that every tensor*

$$(8.6) \quad {}^s g_{\lambda\mu} = e^{2q} {}^s g_{ab} v_\lambda^a v_\mu^b \quad (\text{no sum for } s)$$

is a solution of (2.4a), i.e.,

$$(8.7) \quad \nabla_\mu {}^s g_{\lambda\nu} = 0.$$

Theorem (8.2b). *If there are $\frac{1}{2}n(n+1)$ linearly independent tensors ${}^s g_{\lambda\mu}$, $s = 1, \dots, \frac{1}{2}n(n+1)$, satisfying (8.7) (and (8.3)), the L_n under consideration is flat.*

Proof. The conditions (8.3) represent also a complete set of integrability conditions of

$$(8.8) \quad \nabla_\mu g_{\lambda\nu} = g_{\lambda\nu} \partial_\mu \sigma.$$

Hence if L_n is metrisable there is at least one tensor ${}^s g_{\lambda\nu}$ satisfying (8.3) and

$$\nabla_\mu {}^s g_{\lambda\nu} = {}^s g_{\lambda\nu} \partial_\mu \sigma \quad (\text{no sum}).$$

*This is a stronger condition than $\lambda_x \neq \lambda_y$, $x \neq y$, required for the complete tensor $L_{1\lambda}^x$.

Hence the tensor

$$(8.9a) \quad \begin{aligned} {}^s g_{\lambda\nu} &= e^{2\dot{s}} \dot{g}_{\lambda\nu} \quad (\text{no sum}), \\ 2\dot{q} &= -\sigma + \text{const.} \end{aligned}$$

satisfies

$$(8.9b) \quad \nabla_\mu {}^s g_{\lambda\nu} = 0$$

and (8.3). Therefore $[L]$ must be of rank less than $\frac{1}{2}n(n+1)$. In particular, if $s = 1, \dots, p$ and the tensors ${}^s g_{\lambda\nu}$ are linearly independent the rank of $[L]$ is $\frac{1}{2}n(n+1) - p$. If $p = \frac{1}{2}n(n+1)$ (i.e., if there are $\frac{1}{2}n(n+1)$ linearly independent tensors satisfying (8.9b)), then the rank of $[L]$ is zero and consequently $L_{\omega\mu\lambda}{}^\nu = 0$.

Theorem (8.3). *If the condition introduced in Theorem (8.2a) is satisfied, then a necessary and sufficient condition for the L_n under consideration to be metrisable is that there is a set of constants ${}^s c$, $s = 1, \dots, p$, such that the tensor*

$$(8.10) \quad g_{\lambda\mu} = \sum_{s=1}^p {}^s c {}^s g_{\lambda\mu}$$

is of rank n .

If this condition is satisfied, the L_n under consideration reduces to a V_n with the metric tensor (8.10) which satisfies (2.4a) and (1.3).

Proof. Assume that the conditions of Theorem (8.2a) are satisfied. If the L_n under consideration is metrisable its metric tensor $g_{\lambda\nu}$ must satisfy (2.4a), and therefore it must be of the form (8.10) where the functions ${}^s c$ have to be determined by means of (2.4a) and (8.7). Because the tensors ${}^s g_{\lambda\nu}$ are linearly independent, the equations (2.4a) and (8.7) yield at once ${}^s c = \text{const.}$ If (8.10) is satisfied with constant ${}^s c$'s, then we see from (8.7) that (2.4a) (and therefore also (1.3)) is satisfied; the L_n under consideration is metrisable.

Remark. Theorems (8.2a) and (8.3) present a simplified version of Theorem I (8.4).

9. Illustration. Assume $n = 3$, $r = 1$ so that

$$(9.1) \quad \begin{aligned} (a) \quad (L_{\omega\mu\lambda}{}^\nu) R_{\omega\mu\lambda}{}^\nu &= M_{\omega\mu}^1 L_{1\lambda}{}^\nu, \\ (b) \quad L_{1\lambda}{}^\lambda &= 0. \end{aligned}$$

Take as nonholonomic frame the eigenvectors a_b^ν , $\dot{a}_\lambda^\nu$ of $L_{1\lambda}{}^\nu$. Hence

$$(9.2) \quad \begin{aligned} (a) \quad \lambda_{xx} a_x^\nu &= L_{1\lambda}{}^\nu a_x^\lambda, & (b) \quad \lambda_{\dot{x}\mu}^{\dot{x}} &= L_{1\mu}{}^\nu \dot{a}_\nu^{\dot{x}}, \\ (c) \quad L_{1x}{}^b &= \lambda \delta_x^b. \end{aligned}$$

Because our L_3 is assumed to be volume preserving we must have, by virtue of (9.1b) and (9.2c),

$$(9.2d) \quad \lambda_1 + \lambda_2 + \lambda_3 = 0.$$

The eigenvectors are defined up to transformations

$$(9.3) \quad 'a^v_x = \varphi^v_x a^v_x, \quad 'a^x_\lambda = \varphi^x_\lambda a^x_\lambda, \quad \varphi^x_x \varphi^v_v = 1$$

so that

$$(9.4) \quad \begin{aligned} 'a^\lambda_b 'a^c_\lambda &= a^\lambda_b a^c_\lambda = \delta^c_b, \\ 'a^v_b 'a^b_\lambda &= a^v_b a^b_\lambda = \delta^v_\lambda. \end{aligned}$$

Using (9.2c) we obtain

$$(9.5a) \quad [L] = \begin{pmatrix} \lambda_1 + \lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 + \lambda_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 + \lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{pmatrix}.$$

Hence if $\lambda_1 \lambda_2 \lambda_3 \neq 0$ the rank of $[L]$ is 6 and our L_3 is not metrisable. Let us assume

$$(9.6) \quad (a) \quad \lambda_3 = 0, \quad (b) \quad \lambda_1 \lambda_2 \neq 0,$$

so that $[L]$ reduces to

$$(9.5b) \quad [L] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The rank of this matrix is 4 and therefore there are two linearly independent solutions $\dot{g}_{ab}$ ($s = 1, 2$) of (8.3). These solutions are

$$(9.7) \quad \begin{aligned} (a) \quad \dot{g}^1_{ab} &= \delta^3_a \delta^3_b, \\ (b) \quad \dot{g}^2_{ab} &= 2\delta^1_{(a} \delta^2_{b)}. \end{aligned}$$

$g_{ab}^1 (g_{ab}^2)$ is of rank 1 (2). The holonomic components of these tensors are

$$(9.8) \quad (a) \quad g_{\lambda\mu}^1 = a_{\lambda}^3 a_{\mu}^3, \quad (b) \quad g_{\lambda\mu}^2 = 2a_{(\lambda}^1 a_{\mu)}^2.$$

Let us consider first $g_{\lambda\mu}^1$. The conditions (9.3), (9.6a) and (9.2a, b) lead to

$$(9.9) \quad (a) \quad L_{1\lambda}^r \cdot {}^r a_{\lambda}^3 = 0, \quad (b) \quad L_{1\lambda}^r \cdot {}^r a_{\lambda}^3 = 0.$$

The system (9.9) $\begin{cases} (a) \\ (b) \end{cases}$ is the complete set of integrability conditions of

$$(9.10) \quad \begin{aligned} (a) \quad & \nabla_{\mu} {}^r a_{\lambda}^3 = -{}^r a_{\lambda}^3 \partial_{\mu} \ln a, * \\ (b) \quad & \nabla_{\mu} {}^r a_{\lambda}^r = {}^r a_{\lambda}^r \partial_{\mu} \ln a. \end{aligned}$$

Hence if we put

$$\varphi = a \text{ const.} = {}^3\varphi^{-1}$$

we obtain

$$(9.11) \quad (a) \quad \nabla_{\mu} a_{\lambda}^3 = 0, \quad (b) \quad \nabla_{\mu} a_{\lambda}^r = 0$$

so that, by virtue of (9.8a),

$$(9.12) \quad \nabla_{\mu} g_{\lambda\nu}^1 = 0.$$

On the other hand we have, by virtue of (9.6b),

$$(9.13) \quad L_{1(\lambda}^a g_{\mu)\alpha}^2 = (\lambda_{\lambda}^1 + \lambda_{\mu}^2) a_{(\lambda}^1 a_{\mu)}^2 = 0.$$

Hence we put

$$g_{\lambda\nu} \stackrel{\text{def}}{=} \sum_{u=1}^2 g_{\lambda\nu}^u \varphi_u$$

the equation (2.4a) is satisfied if and only if

$$(9.14) \quad \sum_{u=1}^2 (\varphi_u \nabla_{\mu} g_{\lambda\nu}^u + g_{\lambda\nu}^u \partial_{\mu} \varphi_u) = 0.$$

The integrability conditions of this system for the unknowns φ are satisfied because of (9.13) and (9.12). If we denote by φ^u two sets of solutions of (9.14)

with $\text{Det} ((\varphi^u)) \neq 0$ we obtain

$$\begin{aligned} {}^s g_{\lambda\nu} &= \sum_{u=1}^2 g_{\lambda\nu}^u \varphi_u^s, \quad \text{Det} (({}^s g_{\lambda\nu})) \neq 0, \\ \nabla_{\mu} {}^s g_{\lambda\nu} &= 0, \quad s = 1, 2. \end{aligned}$$

*The factor $-\partial_{\mu} \ln a$ in (9.10a) comes from (9.4) and (9.10b).

Hence there are always constants ${}^s c$ such that

$$(9.15) \quad g_{\lambda\nu} = \sum_{s=1}^n {}^s c {}^s g_{\lambda\nu}$$

is of rank 3 and satisfies (2.4a) and (1.3). Our L_3 reduces for any pair of constants ${}^1 c, {}^2 c$ to a V_3 with the metric tensor (9.15). Therefore (9.1a) acquires the form

$$(L_{\omega\mu\lambda}{}^\nu =) K_{\omega\mu\lambda}{}^\nu = h^{11} L_{1\omega\mu} L_{1\lambda}{}^\nu$$

and consequently

$$(9.16) \quad \begin{aligned} \nabla_\xi K_{\omega\mu\lambda}{}^\nu &= h_\xi L_{1\omega\mu} L_{1\lambda}{}^\nu = k_\xi K_{\omega\mu\lambda}{}^\nu, \\ h_\xi &\stackrel{\text{def}}{=} D_\xi h^{11} = k_\xi h^{11}. \end{aligned}$$

Hence every one of the V_3 is recurrent.

10. Appendix. Covariant constant tensor fields. The method used in section 8 could be generalized for any arbitrary tensor. Denote by $t_{\lambda_1 \dots \lambda_w}$ a general tensor. The complete set of integrability conditions of

$$(10.1) \quad \nabla_\mu t_{\lambda_1 \dots \lambda_w} = 0$$

is

$$(10.2) \quad L_{a\lambda_1 \dots \lambda_w}{}^{\nu_1 \dots \nu_w} t_{\nu_1 \dots \nu_w} = 0,$$

where for $w > 1^*$

$$(10.3) \quad \begin{aligned} L_{a\lambda_1 \dots \lambda_w}{}^{\nu_1 \dots \nu_w} &\stackrel{\text{def}}{=} L_{a\lambda_1}{}^{\alpha_1} \delta_{\alpha_1}{}^{\nu_1} \delta_{\lambda_2}{}^{\nu_2} \dots \delta_{\lambda_w}{}^{\nu_w} + L_{a\lambda_2}{}^{\alpha_2} \delta_{\lambda_1}{}^{\nu_1} \delta_{\alpha_2}{}^{\nu_2} \delta_{\lambda_3}{}^{\nu_3} \dots \delta_{\lambda_w}{}^{\nu_w} \\ &\quad + L_{a\lambda_w}{}^{\alpha_w} \delta_{\lambda_1}{}^{\nu_1} \delta_{\lambda_2}{}^{\nu_2} \dots \delta_{\lambda_{w-1}}{}^{\nu_{w-1}} \delta_{\alpha_w}{}^{\nu_w}. \end{aligned}$$

Then $[L]_a$ is the matrix of $L_{a\lambda_1 \dots \lambda_w}{}^{\nu_1 \dots \nu_w}$ and $[L]$ is built up from these matrices according to (8.2). Then results similar to those of Section 8 may be obtained for $t_{\lambda_1 \dots \lambda_w}$.

One could simplify this method in the following way: If the $\nu_1 \dots \nu_w$ -rank of the set of r tensors $L_{a\lambda_1 \dots \lambda_w}{}^{\nu_1 \dots \nu_w}$ is s , then there is a set of s linearly independent tensors $V^{\nu_1 \dots \nu_w}{}_\dagger$ such that

$$(10.4a) \quad L_{a\lambda_1 \dots \lambda_w}{}^{\nu_1 \dots \nu_w} = L_{a\lambda_1 \dots \lambda_w}{}^A V^{\nu_1 \dots \nu_w}{}_A,$$

where

$$(10.4b) \quad L_{a\lambda_1 \dots \lambda_w}{}^A \stackrel{\text{def}}{=} L_{a\lambda_1 \dots \lambda_w}{}^{\nu_1 \dots \nu_w} V_{\nu_1 \dots \nu_w}{}^A$$

*If $w = 1$ then we have

$$L_{a\lambda}{}^\nu t_\nu = 0.$$

We shall deal with this case again later.

†Capital Latin indices have the range $\bar{1}, \dots, \bar{s}$.

and $\overset{A}{V}_{\nu_1 \dots \nu_w}$ is a system of inverse tensors

$$\overset{A}{V}_{\nu_1 \dots \nu_w} \overset{A}{V}^{\nu_1 \dots \nu_w} \stackrel{\text{def}}{=} \delta_B^A.$$

Then (10.2) is equivalent to

$$(10.5) \quad \overset{A}{V}^{\nu_1 \dots \nu_w} t_{\nu_1 \dots \nu_w} = 0.$$

If $s = n^w - 1$, then any tensor $t_{\nu_1 \dots \nu_w}$, a solution of (10.5) multiplied by a convenient factor, is covariant constant.

Similar considerations hold for contravariant or mixed tensors.

If $w = 1$ we have

$$L_{a\lambda}{}^\nu = \overset{A}{L}_{a\lambda} \overset{A}{V}^\nu.$$

Denote by $V^{\nu_1 \dots \nu_s} = V^{[\nu_1 \dots \nu_s]}$ the s -vector of the vectors $\overset{A}{V}^\nu$ and assume $s \leq n - 1$. Its s -direction is independent of the possible choice of the vectors $\overset{A}{V}^\nu$. Denote by $u_{s+1}, \dots, u_n$ an arbitrary (even) permutation of $s+1, \dots, n$. There are a set of factors $f, \dots, f$ and a set of linearly independent vectors $\overset{A}{V}^\nu, \dots, \overset{A}{V}^\nu$ not incident with $V^{\nu_1 \dots \nu_s}$ such that *each of the linearly independent vectors*

$$(10.6) \quad \begin{aligned} t_\lambda &\stackrel{\text{def}}{=} \frac{f}{\mathfrak{v}} \overset{u_n}{e}_{\lambda_1 \dots \lambda_{n-1} \lambda} V^{\lambda_1 \dots \lambda_s} \overset{u_{s+1}}{V}^{\lambda_{s+1}} \dots \overset{u_{n-1}}{V}^{\lambda_{n-1}}, \\ \mathfrak{v} &\stackrel{\text{def}}{=} e_{\lambda_1 \dots \lambda_n} V^{\lambda_1 \dots \lambda_s} \overset{s+1}{V}^{\lambda_{s+1}} \dots \overset{n}{V}^{\lambda_n} \end{aligned}$$

is covariant constant. This follows from the fact that

$$\overset{u_n}{V}^\lambda t_\lambda = \frac{f}{\mathfrak{v}} e_{\lambda_1 \dots \lambda_{n-1} \lambda} V^{\lambda_1 \dots \lambda_s} \overset{u_{s+1}}{V}^{\lambda_{s+1}} \dots \overset{u_{s+1}}{V}^{\lambda_{n-1}} \overset{A}{V}^\lambda = 0$$

and consequently

$$L_{a\mu}{}^\lambda t_\lambda = \overset{A}{L}_{a\mu} \overset{u_n}{V}^\lambda t_\lambda = 0. *$$

*In fact, if $*V^\nu, \dots, *V^\nu$ is another choice of the $n - s$ linearly independent vectors (such that $\overset{s+1}{V}^\nu, \dots, \overset{n}{*V}^\nu$ are linearly independent) and if $*t_\lambda$ is the corresponding vector, defined by an equation similar to (10.6), then

$$\overset{u_n}{V}^\lambda *t_\lambda = 0.$$

Consequently the set

$$\sum_a \left[*t_\lambda \overset{a}{\partial}_\mu \overset{p}{\varphi} + \overset{p}{\varphi} \overset{a}{\nabla}_\mu *t_\lambda \right] = 0, \quad p, q = s+1, \dots, n,$$

is completely integrable and $\overset{p}{t}_\lambda = \sum_a \overset{p}{\varphi} \overset{a}{*t}_\lambda$ is covariant constant.

If $s = n$ there are no covariant vectors which are covariant constant.

Similar considerations hold for contravariant vectors as the following illustration shows. In the case dealt with in Section 6, we have

$$L_{a\lambda\nu} = L_{a\lambda}^A V_\nu, \quad A = \bar{1}, \bar{2}, \bar{3},$$

with

$$\bar{1} V_\lambda = (1 \ 0 \ 0 \ 0), \quad \bar{2} V_\lambda = (0 \ 1 \ 0 \ 0), \quad \bar{3} V_\lambda = (0 \ 0 \ 1 \ -1),$$

so that

$$V_\lambda^A v^\lambda = 0$$

leads at once to (6.6b).

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