A Uniqueness Theorem on Closed Convex Hypersurfaces in Euclidean Space

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The object of this note is to prove the following theorem:

Let Σ, Σ' be two closed, strictly convex, C^2 -differentiable hypersurfaces in a Euclidean space of dimension $n + 1$ (≥ 3). Let $f: \Sigma \rightarrow \Sigma'$ be a diffeomorphism such that Σ and Σ' have parallel outward normals at $p \in \Sigma$ and $p' = f(p)$ respectively. Denote by $P_l(p)$ (respectively $P_l(p')$) the l^{th} elementary symmetric function of the principal radii of curvature of Σ (resp. Σ') at p (resp. p'). If, for a fixed l , $2 \leq l \leq n$, we have

$$(1) \quad P_{l-1}(p) \leq P_{l-1}(p'), \quad P_l(p) \geq P_l(p')$$

for all points $p \in \Sigma$, then f is a translation.

The interest in this theorem lies in the fact that the conditions (1) involve only inequalities. The proof of the theorem depends on some integral formulas established in a previous paper of one of us² and on an algebraic inequality.

Let $(\lambda_{ik}), (\lambda'_{ik}), 1 \leq i, k \leq n$, be positive definite symmetric matrices. We define the polynomials $P_{rs}(\lambda, \lambda')$ by means of the equation

$$(2) \quad \det(\delta_{ik} + y\lambda_{ik} + y'\lambda'_{ik}) = \sum_{0 \leq r, s \leq n} \frac{n!}{r! s! (n - r - s)!} P_{rs}(\lambda, \lambda') y^r y'^s.$$

$P_{rs}(\lambda, \lambda')$ are thus homogeneous polynomials of degrees r, s in $\lambda_{ik}, \lambda'_{ik}$, respectively. It is easy to derive from the definition that

$$(3) \quad rP_{r-1,1}(\lambda, \lambda') = \sum_{i \leq k} \lambda'_{ik} \frac{\partial P_{r0}}{\partial \lambda_{ik}}.$$

Since $P_{r0}(\lambda, \lambda')$ is independent of λ'_{ik} , we shall write $P_{r0}(\lambda, \lambda') = P_r(\lambda)$; then $P_{0r}(\lambda, \lambda') = P_r(\lambda')$.

Lemma. Let $\alpha = (\alpha_{ik}), \lambda = (\lambda_{ik})$ be positive definite symmetric matrices such that, for a fixed l , $2 \leq l \leq n$,

$$(4) \quad P_{l-1}(\alpha) \leq P_{l-1}(\lambda), \quad P_l(\alpha) \geq P_l(\lambda).$$

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²S. S. CHERN, *Integral formulas for hypersurfaces in Euclidean space and their applications to uniqueness theorems*, this Journal, **8** (1959), pp. 947-955. This paper will be quoted as IFH.

Then

$$(5) \quad Q_i(\alpha, \lambda) \equiv P_i(\alpha) + P_i(\lambda) - 2P_{i-1,1}(\alpha, \lambda) \leq 0.$$

Proof. Let D be the convex domain of all positive definite symmetric matrices of order n . Let $\delta = (\delta_{ik}) \in D$ be the unit matrix. We have

$$\det(\delta_{ik} + \lambda_{ik}y) = \sum_{0 \leq r \leq n} \binom{n}{r} P_r(\lambda) y^r,$$

so that

$$\det(\delta_{ik} + \lambda_{ik}y + t \delta_{ik}y) = \sum_{0 \leq r \leq n} \binom{n}{r} P_r(\lambda + t \delta) y^r.$$

But the left-hand side can also be written

$$\det(\delta_{ik}(1 + ty) + \lambda_{ik}y) = \sum_{0 \leq r \leq n} \binom{n}{r} P_r(\lambda) y^r (1 + ty)^{n-r}.$$

Comparing the coefficients of y^l , we get

$$P_l(\lambda + t \delta) = P_l(\lambda) + lP_{l-1}(\lambda)t + \sum_{2 \leq i \leq l} c_i P_{l-i}(\lambda) t^i, \quad c_i > 0.$$

Similarly, comparing the coefficients of $y^{l-1}y'$ on both sides of the equation

$$\begin{aligned} \det(\delta_{ik} + y\alpha_{ik} + y' \delta_{ik}) &= \sum_{0 \leq r, s \leq n} \frac{n!}{r! s! (n-r-s)!} P_{rs}(\alpha, \delta) y^r y'^s \\ &= \det((1 + y') \delta_{ik} + y\alpha_{ik}) = \sum_{0 \leq r \leq n} \frac{n!}{r! (n-r)!} P_r(\alpha) y^r (1 + y')^{n-r}, \end{aligned}$$

we get

$$(6) \quad P_{l-1,1}(\alpha, \delta) = P_{l-1}(\alpha).$$

It follows that

$$(7) \quad P_{l-1,1}(\alpha, \lambda + t \delta) = P_{l-1,1}(\alpha, \lambda) + tP_{l-1}(\alpha),$$

and therefore that

$$Q_i(\alpha, \lambda + t \delta) = Q_i(\alpha, \lambda) + \{lP_{l-1}(\lambda) - 2P_{l-1}(\alpha)\}t + \sum_{2 \leq i \leq l} c_i P_{l-i}(\lambda) t^i, \quad c_i > 0.$$

Hence if $t > 0$ and $P_{l-1}(\lambda) \geq P_{l-1}(\alpha)$, then $Q_i(\alpha, \lambda + t\delta) > Q_i(\alpha, \lambda)$.

The tangent hyperplane at α of the hypersurface $\{\lambda \mid P_l(\lambda) = P_l(\alpha)\}$ has the equation

$$\sum_{i \leq k} (\mu_{ik} - \alpha_{ik}) \frac{\partial P_l(\alpha)}{\partial \alpha_{ik}} = l(P_{l-1,1}(\alpha, \mu) - P_l(\alpha)) = 0.$$

Since the set $\{\lambda \mid P_l(\lambda) \geq P_l(\alpha)\}$ is strictly convex, we have $P_{l-1,1}(\alpha, \lambda) -$

$P_i(\alpha) > 0$ if $P_i(\lambda) > P_i(\alpha)$. (This also follows directly from GÄRDING's inequality.)

We now prove that $P_{i-1}(\alpha) \leq P_{i-1}(\lambda)$, $Q_i(\alpha, \lambda) \geq 0$ imply $P_{i-1,1}(\alpha, \lambda) \geq P_i(\alpha)$. Suppose the contrary be true. Then there exists $\lambda_0 \in D$ such that $P_{i-1}(\alpha) \leq P_{i-1}(\lambda_0)$, $Q_i(\alpha, \lambda_0) \geq 0$, $P_{i-1,1}(\alpha, \lambda_0) < P_i(\alpha)$. The ray $\lambda_0 + t\delta$, $t > 0$, does not belong to the hyperplane $P_{i-1,1}(\alpha, \lambda) = P_{i-1,1}(\alpha, \lambda_0)$, and hence meets the hyperplane $P_{i-1,1}(\alpha, \lambda) = P_i(\alpha)$ in a point $\lambda_0 + t_0\delta$, $t_0 > 0$. Then we have $P_{i-1,1}(\alpha, \lambda_0 + t_0\delta) = P_i(\alpha)$. From the result of the last paragraph this implies that $P_i(\lambda_0 + t_0\delta) \leq P_i(\alpha)$. It follows that

$$Q_i(\alpha, \lambda_0 + t_0\delta) \leq 2\{P_i(\alpha) - P_{i-1,1}(\alpha, \lambda_0 + t_0\delta)\} = 0.$$

On the other hand, we have, since $P_{i-1}(\alpha) \leq P_{i-1}(\lambda_0)$,

$$Q_i(\alpha, \lambda_0 + t_0\delta) > Q_i(\alpha, \lambda_0) \geq 0.$$

But this is a contradiction, and our statement is proved.

Suppose therefore that $P_{i-1}(\alpha) \leq P_{i-1}(\lambda)$, $Q_i(\alpha, \lambda) > 0$. Then

$$P_i(\lambda) - P_{i-1,1}(\alpha, \lambda) > -P_i(\alpha) + P_{i-1,1}(\alpha, \lambda) \geq 0$$

or

$$P_i(\alpha) \leq P_{i-1,1}(\alpha, \lambda) < P_i(\lambda).$$

This obviously implies the statement of the lemma.

To prove the theorem we utilize the integral formulas established in IFH. Suppose the first hypersurface to be defined by the mapping $x: S_0 \rightarrow \Sigma$, where S_0 is the unit hypersphere in the Euclidean space and $x(\xi)$, $\xi \in S_0$, is the point of Σ whose outward unit normal vector is ξ . Similarly, the second hypersurface is defined by a mapping $x': S_0 \rightarrow \Sigma'$. From their second fundamental forms we construct the mixed scalar invariants P_{rs} , $0 \leq r, s \leq n$ (cf. IFH). Then we have the following integral formula:

$$(8) \quad \int_{S_0} (P_{rs} - h'P_{r,s-1}) dV = 0, \quad 1 \leq s \leq n,$$

where dV is the volume element of S_0 and $h'(\xi)$ is the support function of Σ' . By a proper choice of the origin we can always suppose $h'(\xi) > 0$. From (8) we derive the formula

$$(9) \quad \int_{S_0} \{(P_{i0} + P_{0i} - 2P_{i-1,1}) + (P_{0i} - P_{i0}) + 2h'(P_{i-1,0} - P_{0,i-1})\} dV = 0.$$

The hypotheses of the theorem give $P_{i-1,0} \leq P_{0,i-1}$, $P_{0i} \leq P_{i0}$. From these and the lemma it follows that $P_{0i} + P_{i0} - 2P_{i-1,1} \leq 0$. Hence the integrand in (9) is ≤ 0 , and must vanish identically since it is continuous. In particular, we have $P_{0i} = P_{i0}$ for all $p \in \Sigma$. Our theorem then follows from the theorem of ALEXANDROFF-FENCHEL-JESSEN.

Corollary. *Let Σ be a closed strictly convex hypersurface in Euclidean space. If there is a constant c such that*

$$(10) \quad P_{i-1}^{1/(i-1)}(p) \geq c \geq P_i^{1/i}(p), \quad p \in \Sigma,$$

then Σ is a hypersphere.

Remark. The theorem can obviously be extended to a pair of convex hypersurfaces with boundaries.

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