Positive Real Matrices

LOUIS WEINBERG & PAUL SLEPIAN

Communicated by R. BELLMAN

1. Introduction. This paper is concerned with real rational functions of the complex variable $s \equiv \sigma + i\omega$ and symmetric matrices, each of whose elements is a real rational function. More specifically, we consider positive real (p.r.) functions and positive real (p.r.) matrices.

A rational function $K(s)$ is said to be p.r. if and only if:

- (1) $K(s)$ is real for s real.
- (2) $\operatorname{Re} [K(s)] \geq 0$ for $\operatorname{Re} [s] \geq 0$.

A symmetric n^{th} -order matrix of real rational functions $[a_{jk}(s)]$ is said to be a p.r. matrix if and only if its associated quadratic form

$$\sum_{j,k=1}^n a_{jk}(s)x_jx_k$$

is a p.r. function for all finite values of the real variables $x_1, x_2, \dots, x_n$.

The importance of the p.r. concept derives mainly from the following two facts:

- (1) A necessary and sufficient condition for a real rational function to be realizable as the driving-point impedance of a one-port network is that it be a p.r. function [1, 2].
- (2) A necessary and sufficient condition for a symmetric n^{th} -order matrix of real rational functions to be realizable as the open-circuit impedance matrix of an n -port network is that it be a p.r. matrix [3].

Some applications of the p.r. concept to networks are discussed in another paper [4] by the authors. The purpose of the present paper is to prove the previously given sets of necessary and sufficient conditions for a real rational function and a symmetric matrix of real rational functions to be p.r. The new theorems cited and proved are Theorems 1, 3, 5, 7, 9, and 10. It is believed that these new theorems are more suitable for practical testing procedures than the theorems previously given in the literature. For completeness the previously known theorems on p.r. functions and matrices are given with each new theorem.†

†An alternative characterization of *p.r.* matrices, which is equivalent to a synthesis procedure, is given by R. J. DUFFIN in his paper *Elementary Operations which Generate Network Matrices*, *Proc. Am. Math. Soc.*, 6, pp. 335–339, 1955.

2. Notation and Auxiliary Theorem. We first introduce some notation which will be consistently used throughout the paper.

If $[a_{jk}(s)]$ is a $t \times t$ matrix with determinant $A(s)$, if m is a positive integer not exceeding $t - 1$, and if $x_1, x_2, \dots, x_m$ is a sequence of m distinct positive integers, each not exceeding t , then by

$$A^{x_1 x_2 \dots x_m}(s)$$

we mean the determinant of the $(t - m) \times (t - m)$ matrix obtained by deleting rows $x_1, x_2, \dots, x_m$ and columns $x_1, x_2, \dots, x_m$ from the matrix $[a_{jk}(s)]$. If each element of the original matrix $[a_{jk}(s)]$ is a constant, we shall eliminate the (s) from the notation.

If $[a_{jk}(s)]$ is a matrix of rational functions and s_ν is a pole of the matrix,† we shall let ρ_ν be the rank of the matrix of residues at this pole. More precisely, for each j and k let $b_{jk}^{(\nu)}$ be the residue of $a_{jk}(s)$ at s_ν . Then ρ_ν is the rank of the matrix $[b_{jk}^{(\nu)}]$. If Q is any collection of poles of $[a_{jk}(s)]$, then by the residue rank of $[a_{jk}(s)]$ with respect to Q we shall mean the largest value of ρ_ν obtained for any pole in Q . We shall denote the residue rank with respect to Q by ρ , when the set Q is clearly specified.

The following matrix theorem is vital to the later discussion.

Theorem 1. *Let $[a_{jk}(s)]$ be a $t \times t$ matrix with determinant $A(s)$ and let α be a finite complex number such that:*

- (1) α is at most a simple pole of each element of $[a_{jk}(s)]$.
- (2) If $t = 1$, then $a_{11}(s)$ has a non-negative residue at α , while if $t > 1$ there exists a sequence of $t - 1$ distinct positive integers $x_1, x_2, \dots, x_{t-1}$, each not exceeding t , and a sequence of t functions $H_1(s), H_2(s), \dots, H_t(s)$, each of which has at most a simple pole at α with a positive residue, such that

$$A(s) = H_1(s)A^{x_1}(s),$$

$$A^{x_1 x_2 \dots x_{m-1}}(s) = H_m(s)A^{x_1 x_2 \dots x_m}(s)$$

for $m = 2, 3, \dots, t - 1$, and

$$A^{x_1 x_2 \dots x_{t-1}}(s) = H_t(s).$$

For each j and k let b_{jk} be the residue of $a_{jk}(s)$ at α and let $B = \det [b_{jk}]$. Then $B \geq 0$.

Proof. The following lemma must be established.

Lemma 1. *Let $[a_{jk}(s)]$ be an n^{th} -order matrix with determinant $A(s)$ and let α be a finite complex number which is at most a simple pole of each element of $[a_{jk}(s)]$. For each j and k let b_{jk} be the residue of $a_{jk}(s)$ at α and let $B = \det [b_{jk}]$. Then $\lim_{s \rightarrow \alpha} \{(s - \alpha)^n A(s)\} = B$.*

†By a pole of a matrix we mean a pole of some element of the matrix.

Proof of Lemma 1. The proof is by induction on n .

The result is trivial when $n = 1$.

Thus suppose $n > 1$ and the conclusion of the lemma is true for $n - 1$. Let $A_{jk}(s)$ be the cofactor of the element $a_{jk}(s)$ in $[a_{jk}(s)]$ for $j = 1, 2, \dots, n$ and $k = 1, 2, \dots, n$. Similarly, let B_{jk} be the cofactor of the element b_{jk} in $[b_{jk}]$ for $j = 1, 2, \dots, n$ and $k = 1, 2, \dots, n$. Then, using the inductive hypothesis and the Laplace expansion of a determinant,

$$\begin{aligned} \lim_{s \rightarrow \alpha} [(s - \alpha)^n A(s)] &= \lim_{s \rightarrow \alpha} \left[(s - \alpha)^n \sum_{k=1}^n a_{1k}(s) A_{1k}(s) \right] \\ &= \lim_{s \rightarrow \alpha} \left[\sum_{k=1}^n \{[(s - \alpha)a_{1k}(s)][(s - \alpha)^{n-1} A_{1k}(s)]\} \right] \\ &= \sum_{k=1}^n \left\{ \lim_{s \rightarrow \alpha} [(s - \alpha)a_{1k}(s)] \lim_{s \rightarrow \alpha} [(s - \alpha)^{n-1} A_{1k}(s)] \right\} \\ &= \sum_{k=1}^n \left\{ \left[\lim_{s \rightarrow \alpha} [(s - \alpha)a_{1k}(s)] \right] \left[\lim_{s \rightarrow \alpha} [(s - \alpha)^{n-1} A_{1k}(s)] \right] \right\} \\ &= \sum_{k=1}^n b_{1k} B_{1k} = B. \end{aligned}$$

Thus the lemma is proved.

Proof of Theorem 1. The proof is by induction on n .

When $n = 1$ the result is guaranteed by hypothesis (2).

Thus suppose $n > 1$ and suppose the theorem is true for $n - 1$. Then use of hypothesis (2) yields

$$\lim_{s \rightarrow \alpha} [(s - \alpha)^n A(s)] = \lim_{s \rightarrow \alpha} [(s - \alpha)^n H_1(s) A^{x_1}(s)].$$

Applying Lemma 1 to the left side gives

$$\lim_{s \rightarrow \alpha} [(s - \alpha)^n A(s)] = B.$$

Hypothesis (2) also guarantees that

$$\infty > \lim_{s \rightarrow \alpha} [(s - \alpha)H_1(s)] \geq 0.$$

Now applying Lemma 1 to the principal submatrix of $[a_{jk}(s)]$ corresponding to $A^{x_1}(s)$ and using the inductive hypothesis yield

$$\lim_{s \rightarrow \alpha} [(s - \alpha)^{n-1} A^{x_1}(s)] = B^{x_1} \geq 0.$$

Use of these facts then gives

$$\begin{aligned}
 B &= \lim_{s \rightarrow \alpha} [(s - \alpha)^n A(s)] = \lim_{s \rightarrow \alpha} [(s - \alpha)^n H_1(s) A^{x_1}(s)] \\
 &= \lim_{s \rightarrow \alpha} [\{(s - \alpha) H_1(s)\} \{(s - \alpha)^{n-1} A^{x_1}(s)\}] \\
 &= \{\lim_{s \rightarrow \alpha} [(s - \alpha) H_1(s)]\} \{\lim_{s \rightarrow \alpha} [(s - \alpha)^{n-1} A^{x_1}(s)]\} \\
 &= \{\lim_{s \rightarrow \alpha} [(s - \alpha) H_1(s)]\} B^{x_1} \geq 0.
 \end{aligned}$$

The proof is complete.

3. Positive Real Functions. The following known theorem [5] offers an alternative characterization of a p.r. function.

Theorem 2. *A real rational function $K(s)$ is p.r. if and only if:*

- (1) $K(s)$ has no poles in the right half-plane.
- (2) $\operatorname{Re} [K(i\omega)] \geq 0$ for each real ω .
- (3) If $K(s)$ has any imaginary axis[†] poles, each of these must be simple and the residue[‡] at each of these poles must be positive.

The following new* theorem [6, 7, 8] gives a simpler characterization of a p.r. function.

Theorem 3. *A real rational function $K(s) \equiv p(s)/q(s)$ is p.r. if and only if:*

- (1) The polynomial $p(s) + q(s)$ has no roots in the right half-plane or on the imaginary axis.
- (2) $\operatorname{Re} [K(i\omega)] \geq 0$ for each real ω .

[†]As is customary, we consider the closed complex plane; thus the point at infinity is on the imaginary axis and on both the positive and negative real axes.

[‡]We define the residue of a function at infinity in accordance with network theory usage. More precisely, the residue of a function $f(s)$ at infinity is the residue of $f(1/s)$ at $s = 0$. Observe that this is at variance with the usual mathematical terminology, but leads to simpler statements of the theorems.

*A word should, perhaps, be said about the sense in which this theorem is new. Both GUILLEMIN and TALBOT (in references [7] and [8], respectively) show that $K(s)$ is p.r. if and only if $|W(i\omega)| \leq 1$ for each real ω and $W(s)$ is analytic in the right half-plane and on the imaginary axis. TALBOT went on to recognize explicitly the superiority of these conditions in that they eliminated the necessity of testing for imaginary-axis poles. Neither, however, took the next step, though it appears to be an obvious one, of substituting the condition $\operatorname{Re} [K(i\omega)] \geq 0$ for $|W(i\omega)| \leq 1$.

Indeed, it was TALBOT who called the first-named author's attention to the fact that he had never seen the theorem in this form; in a private letter to the author, he expressed his debt to reference 6 for this theorem. A search of the synthesis literature confirms the fact that this test has not previously been used; many references even discuss at length how the test on the positive residues may be simplified, implying the condition on imaginary-axis poles as a necessary part of all known tests.

Proof of Theorem 3. Let

$$W(s) = \frac{1 - K(s)}{1 + K(s)} = \frac{q(s) - p(s)}{q(s) + p(s)}.$$

Then

$$\operatorname{Re} [K(s)] \geq 0 \text{ if and only if } |W(s)| \leq 1.$$

To establish the necessity note that if $K(s)$ is p.r., then $|W(s)| \leq 1$ when $\operatorname{Re} [s] \geq 0$; thus $p(s) + q(s)$ has no roots in the right half-plane or on the imaginary axis. In particular, for each real ω , $|W(i\omega)| \leq 1$ which implies $\operatorname{Re} [K(i\omega)] \geq 0$.

To establish the sufficiency observe that (1) permits the application of the maximum-modulus theorem to the restriction of $W(s)$ to the closure of the right half-plane. This theorem, in conjunction with (2), guarantees that $|W(s)| \leq 1$ when $\operatorname{Re} (s) \geq 0$. Thus $\operatorname{Re} [K(s)] \geq 0$ when $\operatorname{Re} [s] \geq 0$, and $K(s)$ is p.r.

4. Positive Real Matrices. The following known theorem characterizing p.r. matrices is generally given [6].

Theorem 4. *An n^{th} -order symmetric matrix of real rational functions $[a_{ik}(s)]$ is p.r. if and only if:*

- (1) *Each element of the matrix has no poles in the right half-plane and no multiple-order poles on the imaginary axis.*
- (2) *If for each j and k and real ω , $a_{jk}(i\omega)$ is separated into its rectangular components:*

$$a_{jk}(i\omega) = r_{jk}(\omega) + ix_{jk}(\omega),$$

the matrix of the real parts $[r_{jk}(\omega)]$ is positive semidefinite.†‡

- (3) *If ξ_j is a pole of the matrix on the imaginary axis, and if $b_{jk}^{(r)}$ is the residue of $a_{jk}(s)$ at this pole for each j and k , then the matrix of residues $[b_{jk}^{(r)}]$ must be positive semidefinite.*

The following new theorem gives conditions equivalent to those of Theorem 4.

Theorem 5. *An n^{th} -order symmetric matrix of real rational functions $[a_{ik}(s)]$, whose residue rank with respect to the set of its finite imaginary-axis poles is ρ , is p.r. if and only if:*

- (1) *Same as (1) of Theorem 4.*
- (2) *Same as (2) of Theorem 4.*

†By a positive semidefinite matrix we mean a real symmetric matrix whose associated quadratic form is non-negative for all values of the real variables; thus, in our terminology, a positive semidefinite matrix may be positive definite.

‡In Appendix A a theorem is cited and proved giving conditions sufficient for (2) to be true.

- (3) If infinity is a pole of $[a_{ik}(s)]$, the matrix of residues at infinity is positive semidefinite.
- (4) Let $A(s)$ be any principal minor of $[a_{ik}(s)]$ of order $t \leq \rho$. If $t = 1$, $A(s)$ is p.r.; if $t > 1$ there exists a sequence of $t - 1$ distinct positive integers $x_1, x_2, \dots, x_{t-1}$, each of which does not exceed t , and a sequence of $t - 1$ p.r. functions $H_1(s), H_2(s), \dots, H_{t-1}(s)$ such that

$$A(s) = H_1(s)A^{x_1}(s)$$

and

$$A^{x_1 x_2 \dots x_{m-1}}(s) = H_m(s)A^{x_1 x_2 \dots x_m}(s)$$

for $m = 2, 3, \dots, t - 1$.†

Proof of Theorem 5. The necessity of (1), (2), and (3) of Theorem 5 is implied by Theorem 4. To establish (4) of Theorem 5 as a necessary condition let $A(s)$ be any principal minor of $[a_{ik}(s)]$ of order $t \leq n$. If $t = 1$, then $A(s)$ is a diagonal element of the matrix and is p.r. [6]. Thus suppose $t > 1$; we shall construct sequences $x_1, x_2, \dots, x_{t-1}$ and $H_1(s), H_2(s), \dots, H_{t-1}(s)$ meeting the requirements of condition (4).

If $A(s) \equiv 0$ let $x_1 = 1$ and let $H_1(s) \equiv 0$.

Suppose now that $A(s) \not\equiv 0$. Observe that the matrix corresponding to the minor $A(s)$ is p.r. If the inverse of this matrix is denoted by $[y_{ik}(s)]$, then it is known [6] that $[y_{ik}(s)]$ is a p.r. matrix.

We assert that there exists a positive integer $u \leq t$ such that $A^u(s) \not\equiv 0$. For if not, this implies that for $u = 1, 2, \dots, t$

$$y_{uu}(s) = \frac{A^u(s)}{A(s)} \equiv 0.$$

Now by Theorem 4 it follows that each principal submatrix of $[y_{ik}(s)]$ of order 1 is p.r.; thus the last equation implies that each element of $[y_{ik}(s)]$ is identically 0, and hence $\det [y_{ik}(s)] \equiv 0$. But this is impossible since $\det [y_{ik}(s)] = 1/A(s) \not\equiv 0$. Thus our assertion is proved.

Let x_1 be such a positive integer not exceeding t with $A^{x_1}(s) \not\equiv 0$ and let

$$H_1(s) \equiv \frac{A(s)}{A^{x_1}(s)}.$$

Since $H_1(s)$ is the reciprocal of a p.r. function, it is p.r. [7].

From the matrix corresponding to $A(s)$ delete row x_1 and column x_1 to obtain a symmetric matrix of order $t - 1$. Noting that this new matrix of order $t - 1$

†The following fact may facilitate the use of this condition for testing purposes:

Suppose that $[a_{ik}(s)]$ is a p.r. matrix and let $A(s)$ be any principal minor of $[a_{ik}(s)]$ of order t with $1 < t \leq \rho$. Let $x_1, x_2, \dots, x_{t-1}$ be any sequence of distinct positive integers, each of which does not exceed t . Thus there exists a sequence of $t - 1$ p.r. functions $H_1(s), H_2(s), \dots, H_{t-1}(s)$ satisfying the conclusions cited in condition (4).

is p.r. [6], apply the preceding analysis to it to obtain x_2 and $H_2(s)$. In a similar manner we obtain the entire sequences $x_1, x_2, \dots, x_{t-1}$ and $H_1(s), H_2(s), \dots, H_{t-1}(s)$. This completes the necessity proof.

To establish sufficiency all we must show is that condition (3) of Theorem 4 is implied by the four conditions of Theorem 5. In view of (3) of Theorem 5 we need only consider finite poles in establishing (3) of Theorem 4. Thus let ω_v be real such that $i\omega_v$ is a pole of $[a_{jk}(s)]$, and for each j and k let $b_{jk}^{(v)}$ be the residue of $a_{jk}(s)$ at this pole. Let $\rho(v)$ be the rank of $[b_{jk}^{(v)}]$; then note that $\rho(v) > 0$. We shall now show that the matrix $[b_{jk}^{(v)}]$ is positive semidefinite.

In Appendix B it is shown that each element of the matrix $[b_{jk}^{(v)}]$ is real.

Let $[\beta_{jk}]$ be a $\rho(v) \times \rho(v)$ principal submatrix of $[b_{jk}^{(v)}]$ with a nonzero determinant. Let B be the determinant of this principal submatrix $[\beta_{jk}]$ and let $A(s)$ be the principal minor of $[a_{jk}(s)]$ corresponding to those values of j and k for which elements appear in $[\beta_{jk}]$. The statement of condition (4) of Theorem 5, as applied to $A(s)$, permits the application of Theorem 1 to infer that $B \geq 0$, and hence $B > 0$.

There exists a positive integer $x_1 \leq \rho(v)$ such that $B^{x_1} \neq 0$, for otherwise $B = 0$. Then applying condition (4) of Theorem 5 to $A^{x_1}(s)$, we can invoke Theorem 1 to infer that $B^{x_1} > 0$. In this way we establish a sequence of $\rho(v) - 1$ distinct positive integers $x_1, x_2, \dots, x_{\rho(v)-1}$, each not exceeding $\rho(v)$, such that

$$\begin{aligned} B &> 0 \\ B^{x_1} &> 0 \\ B^{x_1 x_2} &> 0 \\ &\dots\dots\dots \\ B^{x_1 x_2 \dots x_{\rho(v)-1}} &> 0. \end{aligned}$$

This completes the proof of Theorem 5.

5. Special Types of Positive Real Matrices. Two other types of real rational functions of importance† in network theory are here designated for convenience as F_1 and F_2 functions; these are defined below.

A nonconstant‡ real rational function is an F_1 function if and only if the following conditions are satisfied:

- (1) Poles and zeros are simple and occur only on the imaginary axis.
- (2) Poles and zeros alternate on the imaginary axis.
- (3) The constant multiplier is positive.*

†Every LC driving-point impedance is an F_1 function and conversely; every RC driving-point impedance is an F_2 function and conversely. Thus each F_1 or F_2 function is p.r.

‡The function identically zero is defined to be an F_1 function, while any other constant function is not.

*The constant multiplier of a rational function $p(s)/q(s)$ is the coefficient of the term of highest degree in the numerator divided by the coefficient of the term of highest degree in the denominator.

A real rational function is an F_2 function if and only if it is identically zero or the following conditions are satisfied:

- (1) All poles and zeros are simple and lie on the nonpositive real axis.
- (2) Poles and zeros alternate on the nonpositive real axis.
- (3) The critical frequency (zero or pole) of smallest magnitude is a pole.
- (4) The constant multiplier is positive.

These definitions are then extended to described an F_1 matrix and an F_2 matrix. More precisely, the following definition applies for $r = 1, 2$.

A symmetric n^{th} -order matrix of real rational functions $[a_{ik}(s)]$ is said to be an F_r matrix if and only if the associated quadratic form

$$\sum_{i,k=1}^n a_{ik}(s)x_i x_k$$

is an F_r function for all finite values of the real variables $x_1, x_2, \dots, x_n$.

The importance of the F_1 and F_2 matrices for the synthesis of LC and RC n -ports, respectively, is discussed in reference [4].

The following known theorem [9] characterizes F_1 matrices.

Theorem 6. *An n^{th} -order symmetric matrix of real rational functions $[a_{ik}(s)]$ is an F_1 matrix if and only if:*

- (1) *For $j = 1, 2, \dots, n$, $a_{ji}(s)$ is an F_1 function.*
- (2) *For $j = 1, 2, \dots, n$, $k = 1, 2, \dots, n$, and $j < k$, each pole of $a_{jk}(s)$ is a simple pole on the imaginary axis.*
- (3) *At each pole the matrix of residues is positive semidefinite.*

The following new theorem gives conditions equivalent to those of Theorem 6.

Theorem 7. *An n^{th} -order symmetric matrix of real rational functions $[a_{ik}(s)]$, whose residue rank with respect to the set of its finite poles is ρ , is an F_1 matrix if and only if:*

- (1) *Same as (1) of Theorem 6.*
- (2) *For $j = 1, 2, \dots, n$, $k = 1, 2, \dots, n$, and $j < k$, each function $a_{jk}(s)$ is an odd rational function whose denominator divides the denominator of $a_{ji}(s)$.*
- (3) *If infinity is a pole of $[a_{ik}(s)]$, the matrix of residues at infinity is positive semidefinite.*
- (4) *If $A(s)$ is any principal minor of $[a_{ik}(s)]$ of order $t \leq \rho$ with $t > 1$, there exists a sequence of $t - 1$ distinct positive integers $x_1, x_2, \dots, x_{t-1}$, each of which does not exceed t , and a sequence of $t - 1$ F_1 functions $H_1(s), H_2(s), \dots, H_{t-1}(s)$ such that*

$$A(s) = H_1(s)A^{x_1}(s)$$

and

$$A^{x_1 x_2 \cdots x_{m-1}}(s) = H_m(s) A^{x_1 x_2 \cdots x_m}(s)$$

for $m = 2, 3, \dots, t - 1$.†

Proof of Theorem 7. To establish the necessity of condition (2) consider positive integers $j \leq n$ and $k \leq n$ with $j < k$. Since the matrix

$$\begin{bmatrix} a_{jj}(s) & a_{jk}(s) \\ a_{jk}(s) & a_{kk}(s) \end{bmatrix}$$

is a principal submatrix of an F_1 matrix, and thus an F_1 matrix, it is easy to show [9] that $a_{jk}(s)$ is an odd rational function and that each of its poles is a pole of $a_{jj}(s)$.

To establish the necessity of condition (4) we proceed as in the necessity proof of Theorem 5. Let $A(s)$ be a principal minor of $[a_{jk}(s)]$ of order $t \leq n$. If $A(s) \equiv 0$ let $x_1 = 1$ and let $H_1(s) \equiv 0$. If $A(s) \not\equiv 0$, there exists a positive integer $x_1 \leq t$ such that $A^{x_1}(s) \equiv 0$. Let $H_1(s) \equiv A(s)/A^{x_1}(s)$; observe that $1/H_1(s)$ is an F_1 function and thus $H_1(s)$ is an F_1 function. A similar procedure is applied to the matrix of order $t - 1$ corresponding to the determinant $A^{x_1}(s)$. In this way the entire sequence $x_1, x_2, \dots, x_{t-1}$ and $H_1(s), H_2(s), \dots, H_{t-1}(s)$ can be established, completing the necessity proof.

To establish sufficiency note that because of (3) of Theorem 7 we need only consider finite poles in establishing (3) of Theorem 6. Now observe that by condition (1) of Theorem 7 for $j = 1, 2, \dots, n$ each finite pole of $a_{jj}(s)$ is a simple pole on the imaginary axis with positive residue. Thus for $j < k$ condition (2) guarantees that any finite pole of $a_{jk}(s)$ is a simple pole on the imaginary axis; since $a_{jk}(s)$ is an odd real rational function its residue is real at such a pole. Thus at any finite pole of any element of $[a_{jk}(s)]$ each element of the residue matrix is real.

With this established the remainder of the sufficiency proof is very similar to that of Theorem 5 and is omitted.

The following known [6] theorem characterizes F_2 matrices.

Theorem 8. *An n^{th} -order symmetric matrix of real rational functions $[a_{jk}(s)]$ is an F_2 matrix if and only if:*

- (1) *For $j = 1, 2, \dots, n$, $a_{jj}(s)$ is an F_2 function.*
- (2) *For $j = 1, 2, \dots, n$, $k = 1, 2, \dots, n$, and $j < k$ each pole of $a_{jk}(s)$ is simple and lies on the finite nonpositive real axis.*
- (3) *At each pole the matrix of residues is positive semidefinite.*

†The following fact may facilitate the use of this theorem for testing purposes:

Suppose that $[a_{jk}(s)]$ is an F_1 matrix and let $A(s)$ be any principal minor of $[a_{jk}(s)]$ of order t with $1 < t \leq \rho$. Let $x_1, x_2, \dots, x_{t-1}$ be any sequence of distinct positive integers, each of which does not exceed t . Thus there exists a sequence of $t - 1$ F_1 functions $H_1(s), H_2(s), \dots, H_{t-1}(s)$ satisfying the conclusions cited in condition (4).

- (4) For each j and k let $c_{jk} = \lim_{s \rightarrow \infty} a_{jk}(s)$. Then the matrix $[c_{jk}]$ must be positive semidefinite.

The following new theorem gives conditions equivalent to those of Theorem 8.

Theorem 9. An n^{th} -order symmetric matrix of real rational functions $[a_{jk}(s)]$, whose residue rank with respect to the set of all its poles is ρ , is an F_2 matrix if and only if:

- (1) Same as (1) of Theorem 8.
- (2) For $j = 1, 2, \dots, n, k = 1, 2, \dots, n$, and $j < k$, $a_{jk}(s)$ has no pole at infinity and its denominator divides the denominator of $a_{ji}(s)$.
- (3) If $A(s)$ is any principal minor of $[a_{jk}(s)]$ of order $t \leq \rho$ with $t > 1$, there exists a sequence of $t - 1$ distinct positive integers $x_1, x_2, \dots, x_{t-1}$, each of which does not exceed t , and a sequence of $t - 1$ F_2 functions $H_1(s), H_2(s), \dots, H_{t-1}(s)$ such that

$$A(s) = H_1(s) A^{x_1}(s)$$

and

$$A^{x_1 x_2 \dots x_{m-1}}(s) = H_m(s) A^{x_1 x_2 \dots x_m}(s)$$

for $m = 2, 3, \dots, t - 1$.†

- (4) Same as (4) of Theorem 8.

The proof of this theorem is analogous to the proof of Theorem 7 and is omitted.

APPENDIX A

In this appendix we shall cite a theorem which may be applied to check whether the matrix $[r_{jk}(\omega)]$ of condition (2), Theorem 5, is positive semidefinite for each real ω .

We must first specify the fractional representation to be used for each element of the matrix $[r_{jk}(\omega)]$. For each j and k write $r_{jk}(\omega)$ as a quotient of polynomials,

$$r_{jk}(\omega) = \frac{v_{jk}(\omega)}{w_{jk}(\omega)},$$

reduced to lowest terms. Next let $T(\omega)$ be the least common multiple of all the reduced denominator polynomials $w_{jk}(\omega)$ for $j = 1, 2, \dots, n$ and $k = 1, 2, \dots, n$, and suppose that $T(\omega) \geq 0$ for each real ω .* Then for each j and k write

$$r_{jk}(\omega) = \frac{u_{jk}(\omega)}{T(\omega)}.$$

†An observation analogous to the footnotes associated with (4) of Th. 5 and (4) of Th. 7 can also be made here.

*Such a representation for $r_{jk}(\omega)$ can always be obtained if $[a_{jk}(s)]$ is a p.r. matrix; in fact, $T(\omega) > 0$ for all real ω . But Th. 10 is stated here to conform with its statement in reference 4.

Let $m \leq n$ be the rank of $[u_{jk}(\omega)]$; we may suppose $m > 0$.[‡] Let $U(\omega) \equiv 0$ be an m^{th} -order principal minor of $[u_{jk}(\omega)]$. Then the following theorem is true.

Theorem 10. *The matrix $[r_{jk}(\omega)]$ is positive semidefinite for each real ω if each element of the matrix is finite for each real ω and the following condition is satisfied: If $m = 1$ then $U(\omega) \geq 0$ for each real ω , while if $m > 1$ then $U(\omega) \geq 0$ for each real ω and, in addition, there exists a sequence of $m - 1$ distinct positive integers $x_1, x_2, \dots, x_{m-1}$, each not exceeding m , such that*

$$\begin{aligned} U^{x_1}(\omega) &\geq 0 \quad \text{for each real } \omega \quad \text{and} \quad U^{x_1}(\omega) \equiv 0, \\ U^{x_1 x_2}(\omega) &\geq 0 \quad \text{for each real } \omega \quad \text{and} \quad U^{x_1 x_2}(\omega) \equiv 0, \\ &\dots\dots\dots \\ U^{x_1 x_2 \dots x_{m-1}}(\omega) &\geq 0 \quad \text{for each real } \omega \quad \text{and} \quad U^{x_1 x_2 \dots x_{m-1}}(\omega) \equiv 0.* \end{aligned}$$

Proof of Theorem 10. Suppose that $[r_{jk}(\omega)]$ is not positive semidefinite for each real ω . Thus there exists a principal minor of $[u_{jk}(\omega)]$, say $V(\omega)$, and a real number ω_1 such that

$$V(\omega_1) < 0.$$

Hence, by the continuity of $V(\omega)$, we can establish an open interval I of real numbers with $\omega_1 \in I$ such that for all $\omega \in I$, $V(\omega) < 0$. But there exists $\omega' \in I$ such that if $m = 1$, $U(\omega') > 0$, while if $m > 1$,

$$\begin{aligned} U(\omega') &> 0 \\ U^{x_1}(\omega') &> 0 \\ U^{x_1 x_2}(\omega') &> 0 \\ &\dots\dots\dots \\ U^{x_1 x_2 \dots x_{m-1}}(\omega') &> 0. \end{aligned}$$

But this implies that the matrix of real numbers $[u_{jk}(\omega')]$ is positive semidefinite of rank m . Hence $V(\omega') \geq 0$, which contradicts the fact that $\omega' \in I$. The proof is complete.

[‡]If $m = 0$ then each element of $[u_{jk}(\omega)]$ is identically 0 for each real ω , and thus $r_{jk}(\omega)$ is positive semidefinite for each real ω .

*For testing purposes the following observation will be useful:

If $[r_{jk}(\omega)]$ is positive semidefinite for each ω , $U(\omega) \equiv 0$ is an m^{th} -order principal minor of $[u_{jk}(\omega)]$ with $m > 1$, and $x_1, x_2, \dots, x_{m-1}$ is any sequence of $m - 1$ distinct positive integers, each not exceeding m , then

$$\begin{aligned} U^{x_1}(\omega) &\geq 0 \quad \text{for each real } \omega \quad \text{and} \quad U^{x_1}(\omega) \equiv 0, \\ U^{x_1 x_2}(\omega) &\geq 0 \quad \text{for each real } \omega \quad \text{and} \quad U^{x_1 x_2}(\omega) \equiv 0, \\ &\dots\dots\dots \\ U^{x_1 x_2 \dots x_{m-1}}(\omega) &\geq 0 \quad \text{for each real } \omega \quad \text{and} \quad U^{x_1 x_2 \dots x_{m-1}}(\omega) \equiv 0. \end{aligned}$$

APPENDIX B

In this appendix we show that if $[a_{jk}(s)]$ is an n^{th} -order symmetric matrix of real rational functions satisfying (1), (2), and (3) of Theorem 5, ω_* is real, and $i\omega_*$ is a pole of some element of $[a_{jk}(s)]$, then each element of $[a_{jk}(s)]$ has a real residue at $i\omega_*$. This fact is a direct consequence of the following lemma.

Lemma 2. *Let $[a_{jk}(s)]$ be an n^{th} -order symmetric matrix of real rational functions such that:*

- (1) $a_{jj}(s)$ is a p.r. function for $j = 1, 2, \dots, n$.
- (2) Same as (2) of Theorem 5.
- (3) Each element of the matrix has no multiple-order poles on the finite imaginary axis.

Let ω_* be real such that $i\omega_*$ is a pole of some element of $[a_{jk}(s)]$. Then each element of $[a_{jk}(s)]$ has a real residue at $i\omega_*$.

Proof. In view of (1) pick integers j and k with $1 \leq j \leq n$, $1 \leq k \leq n$, and $j < k$, and observe that it suffices to show that $a_{jk}(s)$ has a real residue at $i\omega_*$.

Suppose the contrary. Then

$$a_{jk}(s) = \frac{\lambda + i\mu}{s - i\omega_*} + F(s)$$

where λ and μ are real, $\mu \neq 0$, and $F(s)$ is analytic in a neighborhood of $i\omega_*$. Then for each real ω

$$\operatorname{Re} [a_{jk}(i\omega)] = \frac{\mu}{\omega - \omega_*} + \operatorname{Re} [F(i\omega)].$$

But condition (2) of this lemma states that for each real ω

$$\begin{vmatrix} r_{jj}(\omega) & \frac{\mu}{\omega - \omega_*} + \operatorname{Re} [F(i\omega)] \\ \frac{\mu}{\omega - \omega_*} + \operatorname{Re} [F(i\omega)] & r_{kk}(\omega) \end{vmatrix} \geq 0.$$

However, $\operatorname{Re} [F(i\omega)]$ is bounded for all $i\omega$ in some closed interval I on the imaginary axis with $i\omega_*$ in its interior. Also $r_{jj}(\omega)$ and $r_{kk}(\omega)$ are bounded for all $i\omega$ in I since the real part of a p.r. function has no poles on the imaginary axis [7]. Furthermore, expansion of the above determinant gives

$$r_{jj}(\omega)r_{kk}(\omega) - \left(\frac{\mu}{\omega - \omega_*}\right)^2 - 2 \operatorname{Re} [F(i\omega)] \frac{\mu}{\omega - \omega_*} - \{\operatorname{Re} [F(i\omega)]\}^2 \geq 0.$$

As ω approaches ω_* , the term

$$-\left(\frac{\mu}{\omega - \omega_*}\right)^2$$

will dominate the left side of the above inequality giving rise to a contradiction. Thus $\mu = 0$ and the proof is complete.

REFERENCES

- [1] BRUNE, O., Synthesis of a Finite Two-Terminal Network Whose Driving-Point Impedance is a Prescribed Function of Frequency, *J. Math. Phys.*, **10**, pp. 191–236, 1931.
- [2] BOTR, R. & R. J. DUFFIN, Impedance Synthesis Without Use of Transformers, *J. Appl. Phys.*, **20**, 1949.
- [3] McMILLAN, B., Introduction to Formal Realizability Theory, *Bell Sys. Tech. J.*, **31**, pp. 217–279, 541–600, 1952; Monograph No. 1994.
- [4] WEINBERG, L. & P. SLEPIAN, Realizability Conditions on n -Port Networks, *IRE Trans. PGCT*, CT-5, No. 3, pp. 217–221, 1958.
- [5] GUILLEMIN, E. A., A Summary of Modern Methods of Network Synthesis, *Advances in Electronics*, **3**, pp. 261–303, Academic Press Inc., New York, 1951.
- [6] WEINBERG, L., *Linear Network Synthesis*, notes for a networks course taught at the University of Southern California, 1955; to be published by McGraw-Hill Book Company.
- [7] GUILLEMIN, E. A., *The Mathematics of Circuit Analysis*, John Wiley & Sons, Inc., New York, 1949.
- [8] TALBOT, A., A New Method of Synthesis of Reactance Networks, Monograph No. 77, *Proc. I.E.E.*, Part 4, **101**, pp. 46–48, 1954.
- [9] CAUER, W., Ein Reaktanztheorem, *Sitzungsberichte Preuss. Akad. Wissenschaft*, **30/32**, pp. 673–681, 1951.

Hughes Research Laboratories
Culver City, California