The Stefan Problem for a Nonlinear Equation

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1. Introduction. The Stefan problem is the problem of solving the diffusion or heat equation for a function c in a region bounded in part by an unknown moving surface which must also be determined. Such a problem is also called a free boundary problem. Problems of this type arise in many phase change processes, such as the freezing of a liquid. Then the interface between the two phases (*e.g.* liquid and solid) is the unknown or free boundary which must be found. STEFAN [1] solved explicitly a special problem of this kind in which the heat conductivity or diffusion coefficient and the other data of the problem were constants. In his case the solution depended only upon the time t and one Cartesian coordinate x , and the interface was a plane. It turned out that the displacement of the interface was proportional to $t^{\frac{1}{2}}$ and that c was a function of the product $xt^{-\frac{1}{2}}$. These two conclusions are consequences of dimensional analysis. C. ZENER [2] has solved explicitly the corresponding two and three-dimensional problems in which x denotes radial distance from an axis or radial distance from a point. In these cases the interface is a circular cylinder or sphere which initially has zero radius. The radius grows proportionally to $t^{\frac{1}{2}}$ and c depends only upon $xt^{-\frac{1}{2}}$, as dimensional analysis again requires.

In this paper we consider a one-dimensional problem like that solved by STEFAN. However, we permit the heat conductivity or diffusion coefficient to depend upon the solution c . Then the diffusion equation becomes nonlinear, in contrast to the linear equation treated by STEFAN. Dimensional analysis still requires the displacement of the interface to be proportional to $t^{\frac{1}{2}}$ and the solution c to be a function of $xt^{-\frac{1}{2}}$. We prove that there does exist a solution of the problem satisfying these requirements, provided that the diffusion coefficient lies between positive bounds and has a continuous derivative. In addition, a restriction must be imposed upon one constant in the problem. This result was originally conjectured by H. FRISCH and proved by W. L. MIRANKER [3] in a rather more complicated manner than that given here.

It has been shown by W. T. KYNER [4] that the present result is also true for the two and three-dimensional cases analogous to those treated by ZENER. KYNER's proof is practically the same as the present one. J. MORRISON [5] has proved a comparison theorem for solutions of the present problem from which

it follows that the solution is unique. On the other hand, F. S. HAM [6] has shown that in two and three dimensions there are solutions with elliptical or ellipsoidal interfaces which grow like $t^{\frac{1}{2}}$. Thus the solution which develops from a boundary which is initially a line or a point is not uniquely determined—the interface may be an elliptic cylinder or ellipsoid of any eccentricity, including a circular cylinder or sphere. Thus the solutions of ZENER and KYNER are not unique, although they may be unique in the restricted class of solutions with circular or spherical boundaries.

2. Formulation. We seek two functions $c(x, t)$ and $R(t)$ which satisfy the following equations:

$$(1) \quad c_t = \frac{D}{2} \left[g\left(\frac{c}{c_0}\right) c_x \right]_x, \quad x > R(t), \quad t > 0,$$

$$(2) \quad c[R(t), t] = 0,$$

$$(3) \quad c(x, 0) = c_0, \quad x > 0,$$

$$(4) \quad R_t = k \frac{D}{2} g\left(\frac{c}{c_0}\right) c_x \quad \text{at} \quad x = R(t), \quad t > 0,$$

$$(5) \quad R(0) = 0.$$

In these equations $c(x, t)$ may denote either the concentration of a diffusing substance or the temperature. Then (1) is either the diffusion equation, which expresses conservation of matter, or the heat equation, which expresses conservation of energy. The diffusion coefficient or heat conductivity $\frac{1}{2}Dg(c/c_0)$ is written in terms of a dimensional constant D and a dimensionless function $g(c/c_0)$ of c . We assume that g has a continuous derivative and lies between positive bounds:

$$(6) \quad 0 < G_1 \leq g \leq G_2.$$

The function $R(t)$ denotes the position of the interface at time t . Equation (2) expresses the condition that c must have a constant value at the interface, which we may take as zero. In the freezing of a liquid, c denotes temperature and this constant value is the freezing temperature. Equation (3) is the condition that initially c have the constant value c_0 . Equation (4), which asserts conservation of mass or of energy, relates the velocity of the boundary to the rate at which mass or energy diffuses toward the boundary. In (4) k is a given positive constant with dimensions which are reciprocal to those of c . Equation (5) gives the initial position of the interface.

From dimensional analysis it follows that the dimensionless combination $u = c/c_0$ must be a function of the dimensionless variable $s = x/\sqrt{Dt}$ alone. It also follows that the dimensionless ratio $\alpha = R/\sqrt{Dt}$ must be a constant. Therefore we introduce the function $u(s)$ and the constant α , which have just

been defined, into (1)–(5) in place of c and R . Then (5) is automatically satisfied and (1)–(4) become

$$(7) \quad [g(u)u_s]_s + su_s = 0, \quad s > \alpha,$$

$$(8) \quad u(\alpha) = 0,$$

$$(9) \quad u(\infty) = 1,$$

$$(10) \quad \alpha = \lambda g(0)u_s \quad \text{at} \quad s = \alpha.$$

In (10) the dimensionless positive constant $\lambda = kc_0$ has been introduced. We now seek a function $u(s)$ and a constant α satisfying (7)–(10).

3. Solution. The one point boundary value problem (7), (8) and (10) has a unique solution $u(s)$ for any non-negative value of α and for all $s \geq \alpha$. This is guaranteed by the usual theorems for a sufficiently small s interval around $s = \alpha$. Thus we are asserting that this solution can be indefinitely extended. To prove this assertion we suppose that the solution can be extended uniquely up to s_0 but not beyond. We will show that $u(s_0)$ and $u_s(s_0)$ are both finite and therefore the usual theorems show that the solution can be extended further. This contradiction shows that no such s_0 exists.

To show that $u(s_0)$ and $u_s(s_0)$ are bounded, we integrate (7), using (8) and (10), to obtain

$$(11) \quad u(s) = \frac{\alpha}{\lambda} \int_{\alpha}^s \frac{d\eta}{g[u(\eta)]} \exp \left\{ - \int_{\alpha}^{\eta} \frac{\tau d\tau}{g[u(\tau)]} \right\},$$

$$(12) \quad u_s(s) = \frac{\alpha}{\lambda g[u(s)]} \exp \left\{ - \int_{\alpha}^s \frac{\tau d\tau}{g[u(\tau)]} \right\}.$$

By using (6) in (11) and (12) we obtain for any non-negative value of α

$$(13) \quad \frac{\alpha}{\lambda G_2} \int_{\alpha}^s d\eta \exp \left\{ - \int_{\alpha}^{\eta} \frac{\tau d\tau}{G_1} \right\} \leq u(s) \leq \frac{\alpha}{\lambda G_1} \int_{\alpha}^s d\eta \exp \left\{ - \int_{\alpha}^{\eta} \frac{\tau d\tau}{G_2} \right\},$$

$$(14) \quad \frac{\alpha}{\lambda G_2} \exp \left\{ - \int_{\alpha}^{\eta} \frac{\tau d\tau}{G_1} \right\} \leq u_s(s) \leq \frac{\alpha}{\lambda G_1} \exp \left\{ - \int_{\alpha}^s \frac{\tau d\tau}{G_2} \right\}.$$

Since the upper and lower bounds in (13) and (14) are finite for all $s \geq \alpha$, so are $u(s)$ and $u_s(s)$. Thus the above existence and uniqueness statement is proved.

The bounds in (13) and (14) show that $u \equiv 0$ for $\alpha = 0$, while for $\alpha > 0$ both u and u_s are positive. Therefore u increases monotonically with s . Since the upper bound on u in (13) is finite for s infinite, u approaches a constant value as s becomes infinite. It is easy to see that this limit depends continuously on α . Now if we set $s = \infty$ in (13) we obtain

$$(15) \quad \frac{\alpha}{\lambda G_2} \int_{\alpha}^{\infty} \exp \left\{ \frac{\alpha^2 - \eta^2}{2G_1} \right\} d\eta \leq u(\infty) \leq \frac{\alpha}{\lambda G_1} \int_{\alpha}^{\infty} \exp \left\{ \frac{\alpha^2 - \eta^2}{2G_2} \right\} d\eta.$$

The left side of (15) increases monotonically with α from the value zero at $\alpha = 0$ to the limit $G_1/\lambda G_2$ at $\alpha = \infty$. Similarly the right side increases from zero to the limit $G_2/\lambda G_1$ at $\alpha = \infty$. Let us now suppose that λ satisfies the inequality

$$(16) \quad 0 < \lambda < \frac{G_1}{G_2}.$$

Then it follows that the left side of (15) is greater than unity for some value of α , say $\alpha = \alpha_1$. Therefore, from (15), for $\alpha = \alpha_1$ we have $u(\infty) > 1$.

On the other hand, for some different value of α , say $\alpha = \alpha_2$, the right side of (15) is less than unity. Therefore for $\alpha = \alpha_2$ we have $u(\infty) < 1$. But since $u(\infty)$ varies continuously with α , there is some intermediate value of α , say $\alpha = \alpha_0$, for which $u(\infty) = 1$. This value $\alpha = \alpha_0$, which is positive, and the corresponding solution $u(s)$ are the desired solution of (7)–(10) and therefore also of (1)–(5). We may summarize our result in the

Theorem. *If the function $g(c/c_0)$ has a continuous derivative and satisfies (6) and if the constant $\lambda = kc_0$ satisfies (16), then the problem (1)–(5) has a solution of the form $c = c_0 u(x/\sqrt{Dt})$ and $R = \alpha \sqrt{Dt}$. The function u and the positive constant α satisfy (7)–(10).*

If λ satisfies

$$(17) \quad \frac{G_2}{G_1} \leq \lambda,$$

then the problem (7)–(10) has no solution. This follows because the right side of (15) is then less than unity for all finite α . Therefore when λ satisfies (17), the problem (1)–(5) has no solution.

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