Unsteady Rectilinear Gas Flow

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In the unsteady isentropic flow of a gas with one space dimension the characteristic variables can be obtained explicitly in terms of the gas speed u and the speed of sound a , and the general problem is reduced to the integration of a linear equation of the Euler-Poisson type. When the flow becomes non-isentropic the characteristic equations cannot be integrated in general and the problem becomes more difficult. MARTIN [1], [2] has developed a theory of such flows by taking as the new independent variables the pressure p and a stream function ψ applicable to such flows, reducing the general problem to the solution of a Monge-Ampère equation for which a special integral was constructed. In a previous paper [3] the equations of steady rotational plane flow of a gas were investigated by employing the stream function of CROCCO, and, in the case of generalised simple waves, considerable simplifications resulted. The object of the present paper is to study the effect on the equations of unsteady non-isentropic rectilinear gas flow of the use of a modified stream function analogous to CROCCO's stream function. This function is defined by equations (9) in terms of the gas speed u and the acoustic speed a , and remains constant for each fluid element during the motion. The method consists of applying certain transformations to a Lagrangian equation for $x(\psi, t)$, the position at time t of the fluid element ψ . The general problem is reduced to that of a quasi-linear partial differential equation homogeneous in the second order derivatives. For suitably prescribed distributions of entropy this equation possesses special integrals, one of which is equivalent to MARTIN's exact solution [5]. Other integrals are obtained by concentrating on degenerate solutions analogous to simple waves.

1. General Flow Equations. The equations of one-dimensional inviscid flow of a perfect gas are

$$(1) \quad \rho_t + (\rho u)_x = 0,$$

$$(2) \quad \rho(u_t + uu_x) + p_x = 0,$$

$$(3) \quad S_t + uS_x = 0,$$

$$(4) \quad S = C_v \log (p\rho^{-\gamma}).$$

Defining the acoustic speed by

$$(5) \quad a^2 = \frac{\gamma p}{\rho},$$

the equation (2) may be converted into

$$(6) \quad u_t + uu_x + naa_x = \frac{a^2}{\gamma R} S_x$$

where

$$(7) \quad n = \frac{2}{\gamma - 1}.$$

Similarly using equations (4), (5) to express the density ρ in terms of a^2 and S , the equation of continuity (1) may be transformed into the relation

$$(8) \quad (a^n)_t + (ua^n)_x = 0$$

after employing the adiabatic relation (3). The equation (8) is identically satisfied if one introduces a function $\psi(x, t)$ such that

$$(9) \quad \psi_x = a^n, \quad \psi_t = -ua^n.$$

Then

$$(10) \quad \psi_t + u\psi_x = 0,$$

so that, by (3), the entropy may be regarded as a function of ψ only. Therefore the function $\psi(x, t)$ remains constant for each element of fluid. From (10) the fluid speed is given by

$$(11) \quad u = -\frac{\psi_t}{\psi_x}.$$

This relation may be inserted in (6) to yield a quasi-linear equation for ψ in the form

$$(12) \quad (u^2 - a^2)\psi_{xx} + 2u\psi_{xt} + \psi_{tt} = -\frac{a^{2n+2}}{\gamma R} S'(\psi).$$

In the Lagrangian theory the equation of motion for each fluid element ψ is constructed. Therefore a Lagrangian equation for x with the quantity ψ used to identify the fluid elements may be obtained simply by interchanging the roles of the dependent and independent variables ψ and x in the equation (12). From (9),

$$(13) \quad d\psi = a^n (dx - u dt);$$

therefore

$$a^n dx = d\psi + ua^n dt.$$

Then

$$(14) \quad a^n x_\psi = 1, \quad x_t = u,$$

$$(15) \quad x_\psi = \frac{1}{\psi_x}, \quad x_t = -\frac{\psi_t}{\psi_x}.$$

The equation (12) now transforms into

$$(16) \quad x_\psi^{\gamma+1} x_{t\psi} - x_{\psi\psi} = \frac{x_\psi S'(\psi)}{\gamma R}$$

which is effectively a Lagrangian equation for non-isentropic flow.

2. Construction of a Homogeneous Equation. The difference between the equations (12), (16) and the corresponding equations for isentropic flow lies in the presence of the term $S'(\psi)$ in both equations. This term renders the equations non-homogeneous in the second order derivatives, and the associated characteristic equations cannot be integrated. This difficulty may be partially overcome by constructing an equation for the fluid speed $u(x, t)$ in which the non-homogeneous term has been removed. The required transformations may be carried out by the following method.

In equation (16) write

$$(17) \quad S(\psi) = R \log g'(\psi).$$

Then the equation may be transformed into

$$\frac{\partial}{\partial \psi} [\gamma g(\psi) x_{t\psi} + g'(\psi) x_\psi^{-\gamma}] - \frac{\partial}{\partial t} [\gamma g(\psi) x_{\psi\psi}] = 0,$$

and will be automatically satisfied by introducing a function $v(\psi, t)$ defined by

$$(18) \quad v_t = \gamma g(\psi) x_{t\psi} + g'(\psi) x_\psi^{-\gamma}, \quad v_\psi = \gamma g(\psi) x_{\psi\psi}.$$

Then

$$dv = \gamma g(\psi) (x_{t\psi} dt + x_{\psi\psi} d\psi) + g'(\psi) x_\psi^{-\gamma} dt$$

or

$$(18a) \quad dv = \gamma g(\psi) du + g'(\psi) x_\psi^{-\gamma} dt.$$

Regarding u as a function of v and t it follows that

$$(19) \quad \gamma g(\psi) u_v = 1, \quad \gamma g(\psi) u_t = -g'(\psi) x_\psi^{-\gamma}.$$

From these relations by differentiation there result three equations connecting $x_{t\psi}$ and $x_{\psi\psi}$:

$$\begin{aligned} g'(\psi) u_v + \gamma [g(\psi)]^2 u_{vv} x_{\psi\psi} &= 0, \\ u_{v\psi} + u_{vv} [\gamma g(\psi) x_{t\psi} + g'(\psi) x_\psi^{-\gamma}] &= 0, \\ u_{t\psi} + u_{vt} [\gamma g(\psi) x_{t\psi} + g'(\psi) x_\psi^{-\gamma}] - \frac{g'(\psi)}{g(\psi)} x_\psi^{-\gamma-1} x_{\psi\psi} &= 0. \end{aligned}$$

Eliminating x_{tt} , $x_{\psi t}$ leads to the Monge-Ampère equation for $u(v, t)$:

$$(20) \quad u_{vv}u_{ttt} - u_{vt}^2 + f(u_v)u_t^{(\gamma+1)/\gamma} = 0$$

where

$$(21) \quad f(u_v) = [h(\psi)]^2,$$

$$(22) \quad h(\psi) = \frac{1}{g(\psi)} \left[-\frac{\gamma g(\psi)}{g'(\psi)} \right]^{(1-\gamma)/2\gamma}.$$

In equation (21) the relation connecting u_v and ψ is provided by the equation (19).

The final step necessary to achieve the required form homogeneous in the second derivatives consists of applying the contact transformation

$$(23) \quad w = u - tu_t$$

to equation (20). Writing $\lambda = u_t$, w is to be regarded as a function of v and λ with

$$(24) \quad w_v = u_v, \quad w_\lambda = -t, \quad u_{vv}u_{ttt} - u_{vt}^2 = -\frac{w_{vv}}{w_{\lambda\lambda}},$$

so that (20) reduces to the homogeneous differential equation

$$(25) \quad w_{vv} - \lambda^{(\gamma+1)/\gamma} f(w_v) w_{\lambda\lambda} = 0.$$

3. Characteristic Equations. Consider now the characteristic equations of the equation (25). Writing

$$(26) \quad t = -w_\lambda, \quad z = w_v,$$

so that, from (24), (19), z is a function of ψ only:

$$(27) \quad z = [\gamma g(\psi)]^{-1},$$

one obtains the characteristic systems at once:

$$(28) \quad \begin{cases} \lambda_\alpha - \lambda^{(\gamma+1)/2\gamma} h(\psi) v_\alpha = 0, \\ \lambda_\beta + \lambda^{(\gamma+1)/2\gamma} h(\psi) v_\beta = 0, \end{cases}$$

$$(29) \quad \begin{cases} z_\alpha + \lambda^{(\gamma+1)/2\gamma} h(\psi) t_\alpha = 0, \\ z_\beta - \lambda^{(\gamma+1)/2\gamma} h(\psi) t_\beta = 0. \end{cases}$$

To convert equations (29) to standard forms note that from (14), (19) the quantity $\lambda = u_t$ is given by

$$(30) \quad \lambda = -\frac{g'(\psi)}{\gamma g(\psi)} a^{n+2}.$$

Then, with the aid of (27), equations (29) reduce to

$$(31) \quad \begin{cases} \psi_\alpha - a^{n+1} t_\alpha = 0, \\ \psi_\beta + a^{n+1} t_\beta = 0. \end{cases}$$

Similarly employing (18a), (30) to obtain v_α , v_β , λ_α , λ_β in terms of u_α , u_β , t_α , t_β , ψ_α , ψ_β , the equations (28) reduce to

$$(32) \quad \begin{cases} u_\alpha + na_\alpha - (a/\gamma R)S'(\psi)\psi_\alpha = 0, \\ u_\beta - na_\beta + (a/\gamma R)S'(\psi)\psi_\beta = 0. \end{cases}$$

Also

$$\psi_\alpha = a^n(x_\alpha - ut_\alpha), \quad \psi_\beta = a^n(x_\beta - ut_\beta),$$

so that (31) is replaced by

$$(33) \quad x_\alpha = (u + a)t_\alpha, \quad x_\beta = (u - a)t_\beta.$$

Using the relations (4), (5) the characteristic forms (32) become

$$(34) \quad \rho au_\alpha + p_\alpha = 0, \quad \rho au_\beta - p_\beta = 0,$$

as required.

Comparing the modified relations (28), (29) with the standard forms, the advantage of using the equation (25) becomes apparent, since it is clear that, from equations (28), an integration can be performed only when the function $h(\psi)$, which is a function of z only by (27), reduces to a constant. This possibility has been investigated exhaustively by MARTIN [5]; here we shall only construct the equations to reconcile the results. MARTIN has employed a technique successfully for non-isentropic flows using the pressure and stream function defined in terms of the density. In reconciling the results, however, it is to be recalled that the present paper employs the modified stream function analogous to CROCCO's function. If $h(\psi)$ reduces to a constant it is easily deduced from the equations (22) that the associated form for $g(\psi)$ is

$$(35) \quad g(\psi) = \left(\frac{\psi_0}{\psi} \right)^{1/(n+2)},$$

ψ_0 being a constant.

Integrating the equations (28) directly and using (30), one obtains the characteristic variables explicitly:

$$2k\alpha = ka\psi^{-(n+2)^{-1}} + v,$$

$$2k\beta = ka\psi^{-(n+2)^{-1}} - v,$$

k being an integration constant. The equations (31) now become

$$\psi_\alpha - (\alpha + \beta)^{n+1} \psi^{(n+1)/(n+2)} t_\alpha = 0,$$

$$\psi_\beta + (\alpha + \beta)^{n+1} \psi^{(n+1)/(n+2)} t_\beta = 0,$$

and elimination of ψ leads to

$$2(\alpha + \beta)t_{\alpha\beta} + (n + 1)(t_\alpha + t_\beta) = 0$$

which is equivalent to MARTIN's result. It is necessary to note that this equation for $t(\alpha, \beta)$ is applicable only for the prescribed distribution of entropy associated with the function $g(\psi)$ defined by (35). A different solution is considered in §5.

4. Degenerate Solutions. Returning now to the integration of the basic Lagrangian equation (16) written in the form

$$(36) \quad x_{\psi}^{\gamma+1} x_{t,t} - x_{\psi\psi} + x_{\psi} F(\psi) = 0,$$

when $F(\psi)$ does not take the form required for the direct integration of the characteristic equations (28) one attempts to obtain intermediate integrals by POISSON's method, based on the assumption that a relation of the type $f(x_t, x_{\psi}) = 0$ exists. The method is essentially equivalent to applying a contact transformation to the equation (36), as demonstrated in [4]. The advantage of the use of the modified stream function ψ defined by the equations (9) is that this assumption, in virtue of the equations (14), is equivalent to seeking simple wave solutions for which the fluid speed u and the acoustic speed a are functionally related, since this leads to a relation connecting u and a which does not explicitly introduce the modified function ψ itself. Therefore POISSON's method may be applied directly by associating with (36) the additional relation

$$(37) \quad x_{t,t} x_{\psi\psi} - x_{\psi,t}^2 = 0,$$

for which a general integral is expressible in the form

$$(38) \quad x = tf(\lambda) + \psi g(\lambda) + X(\lambda),$$

the functions f, g, X being arbitrary functions of a variable $\lambda(t, \psi)$ which may be regarded as being defined by the relation

$$(39) \quad 0 = tf_{\lambda} + \psi g_{\lambda} + X_{\lambda}.$$

From (39) by differentiation it follows that

$$(40) \quad \begin{cases} (tf_{\lambda\lambda} + \psi g_{\lambda\lambda} + X_{\lambda\lambda})\lambda_t + f_{\lambda} = 0, \\ (tf_{\lambda\lambda} + \psi g_{\lambda\lambda} + X_{\lambda\lambda})\lambda_{\psi} + g_{\lambda} = 0. \end{cases}$$

Let $c = x_{\psi}$. Then from (38) by differentiation there results

$$(41) \quad u = x_t = f(\lambda), \quad c = x_{\psi} = g(\lambda),$$

in virtue of (39). By further differentiation

$$x_{t,t} = f_{\lambda}\lambda_t, \quad x_{\psi\psi} = g_{\lambda}\lambda_{\psi}.$$

Using the relations (40) these become

$$\begin{aligned} (tf_{\lambda\lambda} + \psi g_{\lambda\lambda} + X_{\lambda\lambda})x_{t,t} + f_{\lambda}^2 &= 0, \\ (tf_{\lambda\lambda} + \psi g_{\lambda\lambda} + X_{\lambda\lambda})x_{\psi\psi} + g_{\lambda}^2 &= 0. \end{aligned}$$

The equation (36) may now be transformed into

$$c^{\gamma+1}f_\lambda^2 - g_\lambda^2 - c(tf_{\lambda\lambda} + \psi g_{\lambda\lambda} + X_{\lambda\lambda})F(\psi) = 0$$

or, in terms of u and c , using (41),

$$(42) \quad c^{\gamma+1}u_\lambda^2 - c_\lambda^2 - c(tu_{\lambda\lambda} + \psi c_{\lambda\lambda} + X_{\lambda\lambda})F(\psi) = 0.$$

It should be noted that the variable $\lambda(t, \psi)$ is not a characteristic variable of the equation (36), since (40), (42) show that

$$c^{\gamma+1}\lambda_t^2 - \lambda_\psi^2 - c(tu_{\lambda\lambda} + \psi c_{\lambda\lambda} + X_{\lambda\lambda})^{-1}F(\psi) = 0,$$

and $\lambda(t, \psi)$ is characteristic only when

$$c^{\gamma+1}\lambda_t^2 - \lambda_\psi^2 = 0,$$

which will never be satisfied for $F(\psi)$ non-zero. Therefore solutions of the type considered here may not be obtained directly from the characteristic equations (28), (29) of §3, or by considering the equivalent equations (31), (32), by taking u, a to be independent of one of the characteristic variables α, β . Since λ is not characteristic it must be regarded as a function of both α and β . It is of interest to note that, from (38), (41),

$$(43) \quad X = x - tx_t - \psi x_\psi,$$

so that X is actually the Legendre transform of x with respect to t and ψ .

Using (39) to eliminate t , the equation (42) becomes

$$(44) \quad u_\lambda(c^{\gamma+1}u_\lambda^2 - c_\lambda^2) - c[u_\lambda X_{\lambda\lambda} - X_\lambda u_{\lambda\lambda} + \psi(u_\lambda c_{\lambda\lambda} - c_\lambda u_{\lambda\lambda})]F(\psi) = 0.$$

In this equation u, c, X are functions of λ only, and we put aside the possibility that ψ itself degenerates into a function of λ , which would result in a solution for u independent of t . It is also necessary to dispose of the possibility that

$$(45) \quad u_\lambda c_{\lambda\lambda} - c_\lambda u_{\lambda\lambda} = 0,$$

which will be shown to result in flows that are essentially steady. To prove this integrate (45) in the form

$$(46) \quad u = kc + U$$

with k, U constants of integration. Using (41) the equations (38), (39) become

$$(47) \quad x = ut + c\psi + X,$$

$$(48) \quad 0 = u_\lambda t + c_\lambda \psi + X_\lambda.$$

Equations (47), (48) are true generally. Using (46) they give the relations

$$x - Ut = c(\psi + kt) + X,$$

$$0 = c_\lambda(\psi + kt) + X_\lambda.$$

Eliminating the combination $(\psi + kt)$ produces

$$x - Ut = X - \frac{c}{c_\lambda} X_\lambda ,$$

so that $(x - Ut)$ is a function of λ only. Since $u = u(\lambda)$ it follows that $u = u(x - Ut)$; therefore, relative to axes moving with constant speed U , the flow is essentially a steady flow.

Returning now to the general case when the relation (45) is not satisfied, the equation (44) is of the type

$$(49) \quad A(\lambda) - [\psi + B(\lambda)]F(\psi) = 0$$

where

$$(50) \quad A(\lambda) = \frac{u_\lambda(c^{\gamma+1}u_\lambda^2 - c_\lambda^2)}{c(u_\lambda c_{\lambda\lambda} - c_\lambda u_{\lambda\lambda})} ,$$

$$(51) \quad B(\lambda) = \frac{u_\lambda X_{\lambda\lambda} - X_\lambda u_{\lambda\lambda}}{u_\lambda c_{\lambda\lambda} - c_\lambda u_{\lambda\lambda}} .$$

It is easily verified that the only possible non-trivial solution of (49) is given by

$$(52) \quad A(\lambda) = \frac{2}{m} , \quad B(\lambda) = \psi_0 ,$$

$$(53) \quad F(\psi) = \frac{2}{m(\psi + \psi_0)}$$

in terms of constants m, ψ_0 .

The relation

$$u_\lambda X_{\lambda\lambda} - X_\lambda u_{\lambda\lambda} = \psi_0(u_\lambda c_{\lambda\lambda} - c_\lambda u_{\lambda\lambda})$$

then integrates to give

$$X_\lambda = \psi_0 c_\lambda + t_0 u_\lambda ,$$

and again

$$(54) \quad X = x_0 + \psi_0 c + t_0 u ,$$

x_0, t_0 being further constants. Combining (54) with the general relation (47)

$$X = x - \psi c - tu ,$$

there results

$$x - x_0 = (\psi + \psi_0)c + (t + t_0)u .$$

Therefore x_0, ψ_0, t_0 are only additive constants and may all be taken as zero. Then by (54) the function $X(\lambda)$ also vanishes identically so that

$$(55) \quad x = tx_t + \psi x_\psi .$$

By the inversion relations (16) this implies

$$(56) \quad \psi = x\psi_x + t\psi_t,$$

so that the Legendre transform of ψ may be taken to be zero. In fact, from (9)

$$(57) \quad \psi = a^n(x - ut).$$

The equations (55), (56) assert the equivalent results that x is a function homogeneous in (t, ψ) of degree unity and that ψ is a function homogeneous in (x, t) of degree unity. In addition, (48) reduces to

$$(58) \quad tu_\lambda + \psi c_\lambda = 0$$

while (47) is, with $X = 0$,

$$(59) \quad x = ut + c\psi.$$

Eliminating ψ between (58), (59) yields

$$(60) \quad xc_\lambda + t(cu_\lambda - uc_\lambda) = 0.$$

Since u, c are functions of λ only, this equation determines λ as a function only of the quantity x/t . Therefore u, c are reducible to functions of x/t only.

From (53) with $\psi_c = 0$,

$$F(\psi) = \frac{2}{m\psi}.$$

Comparing (16), (36) it follows that

$$F(\psi) = -\frac{S'(\psi)}{\gamma R}.$$

Therefore the associated entropy distribution is given by

$$S(\psi) = -\frac{2\gamma R}{m} \log \psi.$$

These results establish the following theorems:

1. *Unsteady "simple wave" solutions, for which the fluid speed and the acoustic speed are functionally related, of the Lagrangian equation of unsteady rectilinear gas flow with non-constant entropy exist only when the distribution of entropy is of the form $S = C \log \psi$, C being some constant.*
2. *In the (x, t) plane the curves of constant speed and constant acoustic speed comprise a system of straight lines concurrent through the origin, that is, the flows obtained are simple centered waves.*

If the flow is isentropic, simple wave solutions are in certain respects more general than the solutions obtained here. In the (x, t) plane the curves of constant speed are still straight but are not required to be concurrent; in fact, the function $X(\lambda)$ may be prescribed arbitrarily for such flows. The relation between

simple waves in isentropic and non-isentropic unsteady unidirectional flows is remarkably similar to that between isentropic and non-isentropic simple waves in steady plane flow. The curves of constant speed in the case of steady plane non-isentropic simple waves comprise a system of straight lines concurrent or parallel [3], but in isentropic flow these lines need not necessarily be concurrent or parallel. The case of parallel lines in non-isentropic steady plane flow corresponds closely to the discarded solution (45), since in this solution u reduced to a function of $(x - Ut)$ and the curves of constant speed in the (x, t) plane comprised a system of parallel straight lines.

It follows from (58) that we may select

$$(61) \quad \lambda = \frac{\psi}{t}.$$

Then

$$(62) \quad \lambda = -\frac{u_\lambda}{c_\lambda},$$

so that, from (57) or (60), the equation in the (x, t) plane of the lines of constant speed is

$$(63) \quad x = (u + \lambda a^{-n})t.$$

Therefore, along a line of constant speed,

$$\frac{dx}{dt} = u + \lambda a^{-n},$$

whereas, for the characteristics,

$$\frac{dx}{dt} = u \pm a.$$

To obtain the flow field, regarding x as a function of (t, ψ) , one may write

$$(64) \quad x = tk(\lambda).$$

Then

$$(65) \quad u = k(\lambda) - \lambda k'(\lambda), \quad c = k'(\lambda).$$

The equations (50), (52) give

$$2c(u_\lambda c_{\lambda\lambda} - c_\lambda u_{\lambda\lambda}) = mu_\lambda(c^{\gamma+1}u_\lambda^2 - c_\lambda^2).$$

Since

$$(66) \quad u_\lambda = -\lambda c_\lambda$$

this becomes

$$2c = m\lambda c_\lambda(1 - \lambda^2 c^{\gamma+1}),$$

the integral of which is

$$(67) \quad \frac{1}{\lambda^2} = \frac{m}{m + \gamma + 1} c^{\gamma+1} + m_0 c^{-m}$$

in terms of a constant m_0 .

To integrate further suitable values must be selected for the constants m , m_0 . Using (66) the relation between u and c is given by

$$(68) \quad c_u^2 = \frac{m}{m + \gamma + 1} c^{\gamma+1} + m_0 c^{-m}.$$

To obtain a special integral of (68) take $m = -2\lambda$ and write $m_0 = -(n+2)a_0^2$ in terms of a further constant a_0 . Then (68) becomes

$$c_u^2 = (n+2)[c^{\gamma+1} - a_0^2 c^{2\gamma}].$$

Since $c = a^{-n}$ this converts into

$$n^2 a^2 a_u^2 = (n+2)(a^2 - a_0^2),$$

so that

$$(69) \quad u = u_0 \pm \frac{n}{(n+2)^{\frac{1}{2}}} (a^2 - a_0^2)^{\frac{1}{2}}.$$

The constant of integration u_0 represents a constant velocity imposed on the entire flow field and will be subsequently omitted. Since $\lambda = \psi/t$, $c = a^{-n}$, $m = -2\gamma$, the equation (67) is

$$(70) \quad t = \pm \psi(n+2)^{\frac{1}{2}} (a^2 - a_0^2)^{\frac{1}{2}} a^{-(n+2)}$$

in which the sign chosen must be the same as that chosen in (69). To complete the solution substitute for u , t in terms of a from (69), (70) in (57). Then

$$(71) \quad x = \psi[(n+1)a^2 - na_0^2]a^{-(n+2)}.$$

The equations (69), (70), (71) provide u , t , x in terms of ψ and a . Taken together these equations comprise a solution of the equations of flow in parametric form. This representation is sufficient to obtain the variation of x with t for each fluid element ψ . Using (70), (71) the curves of t and x are drawn against a for constant values of ψ . When they are combined Fig. 1 is obtained. This curve possesses cusps at the points $A(x_1, t_1)$, $B(x_1, -t_1)$ associated with the value $a = a_1$ given by the equations

$$\frac{\partial x}{\partial a} = 0, \quad \frac{\partial t}{\partial a} = 0.$$

From (70), (71) this value of a is found to be

$$(72) \quad a_1 = \left(\frac{n+2}{n+1} \right)^{\frac{1}{2}} a_0,$$

FIGURE 1.

and the associated values of x , t are given by

$$(73) \quad x_1 = 2\psi a_0^2 a_1^{-(n+2)}, \quad t_1 = \psi a_1^{-(n+1)}.$$

In addition, the two branches which connect at the origin also form a third cusp there. However, at the origin both u and a become infinite. Therefore these two branches will not be considered.

At zero time the gas is everywhere at rest with stagnation acoustic speed equal to a_0 . The element identified by the value ψ is located initially at the position $x_0 = \psi a_0^{-n}$. The elements $\psi > 0$ always occupy the region $x > 0$, the elements $\psi < 0$ occupying the region $x < 0$. The separating plane $\psi = 0$ remains permanently at rest. At $t = 0$ the gas $\psi > 0$ commences to move towards $x = +\infty$, the elements originally at $x_0(\psi)$ arriving at the point $x_1(\psi)$ at the later time $t_1(\psi)$. Similarly the gas for which $\psi < 0$ moves towards $x = -\infty$. Eliminating ψ between (70), (71) results in the equation

$$(74) \quad x = \pm t(n+2)^{-\frac{1}{2}}(a^2 - a_0^2)^{-\frac{1}{2}}[(n+1)a^2 - na_0^2].$$

The positive sign is taken for $t > 0$, the negative sign for $t < 0$. Substituting for a in terms of u from (69), with $u_1 = 0$, yields the equation of the lines of constant speed:

$$(75) \quad x - ut = \frac{t}{n(n+2)u} [n^2 a_0^2 + (n+2)u^2].$$

It is easily verified from this equation by differentiation with respect to u that the lines of constant speed concentrate at the value $u = u_1$ given by

$$(76) \quad u_1 = \frac{na_0}{(n+1)^{\frac{1}{2}}(n+2)^{\frac{1}{2}}},$$

which corresponds to the value a_1 given by (72). In fact, the equation (75) defines real values of u only when

$$\left| \frac{x}{t} \right| \geq \frac{2a_0^2}{a_1},$$

that is, using equations (73),

$$\left| \frac{x}{t} \right| \geq \frac{x_1}{t_1} ,$$

which, for $\psi > 0$, corresponds to the region above the lines $x/t = \pm x_1/t_1$ in the (x, t) plane.

By equations (72), (76),

$$(77) \quad \frac{2a_0^2}{a_1} = u_1 + a_1 ,$$

so that the critical lines are given by $x = \pm (u_1 + a_1)t$. This wave line starts at the plane $x = 0$ at $t = 0$ and for $t > 0$ moves towards $x = +\infty$ with constant speed $(u_1 + a_1)$, which exceeds that of the accelerating fluid in front of it. The element ψ is overtaken by this wave at time t_1 when it has reached the position x_1 . The motion for $t > t_1(\psi)$ cannot be obtained from the solution considered here.

Since, for the critical wave,

$$\frac{dx}{dt} = u_1 + a_1$$

and u_1, a_1 denote the gas speed and acoustic speed on this wave, then the critical wave is a characteristic of the equation (36). It is the only line of constant speed which is a characteristic of this equation. The solution could be extended across this characteristic by employing in the adjoining region a solution of (36), which takes the prescribed constant values of the first derivatives

$$x_t = u_1 , \quad x_\psi = a_1^{-n}$$

along the characteristic, and satisfying in addition the first of the relations (73). It may be verified using equation (69) that the flow is everywhere subsonic for the solution considered, since only the branch $a_0 \leq a \leq a_1$ has been considered.

5. Direct Use of Equation (25). In the two solutions considered the function $g(\psi)$ in both cases was of the form

$$g(\psi) = \psi^m ,$$

the index m being determined in terms of γ for MARTIN'S solution but being arbitrary for generalised simple waves. By equations (26), (27) the quantity w_* associated with this form of $g(\psi)$ is given by

$$\gamma w_* = \psi^{-m} .$$

In addition, from equations (21), (22) the function $f(w_*)$ also assumed the form

$$f(w_*) = \psi^{m'} ,$$

m' being some index defined in terms of m and γ . Combining these results it

follows that in both cases the function $f(w_*)$ occurring in equation (25) was of the form

$$f(w_*) = w_*^\gamma .$$

Therefore to illustrate the direct use of the equation (25) we confine attention to the particular form

$$w_{vv} - \lambda^{(\gamma+1)/\gamma} w_*^\gamma w_{\lambda\lambda} = 0 .$$

Special integrals of this equation may be obtained by a variety of methods. We shall concentrate on the solution of the type

$$w = V(v) + \Lambda(\lambda) .$$

Then V, Λ satisfy the equations

$$(78) \quad V_{vv} = A V_v^\gamma , \quad \lambda^{(\gamma+1)/\gamma} \Lambda_{\lambda\lambda} = A ,$$

A being some constant.

The integral of (78) may be taken to be

$$(79) \quad \Lambda_\lambda = -t_0 \lambda^{-1/\gamma} ,$$

t_0 being a constant to replace A . Since, by (24), $-t = w_\lambda = \Lambda_\lambda$, then (79) gives

$$\lambda = (t_0/t)^\gamma .$$

Since $\lambda = u_t$, the equation (19) shows that

$$x_\psi = tk'(\psi)$$

where

$$(80) \quad t_0 k'(\psi) = - \left[\frac{g'(\psi)}{\gamma g(\psi)} \right]^{1/\gamma} .$$

Then $x(t, \psi)$ must be of the form

$$(81) \quad x = tk(\psi) + H(t) .$$

To determine the functions $g(\psi)$, $H(t)$ insert the proposed form (81) directly in

$$x_\psi^{\gamma+1} x_{tt} - x_{\psi\psi} = \frac{g''(\psi)}{\gamma g'(\psi)} x_\psi$$

obtained by combining (16), (17). Then

$$t^\gamma H''(t) [k'(\psi)]^\gamma - \frac{k''(\psi)}{k'(\psi)} = \frac{g''(\psi)}{\gamma g'(\psi)} .$$

Using (80) this becomes

$$(t/t_0)^\gamma H''(t) = 1 - \frac{2g(\psi)g''(\psi)}{g'(\psi)^2} .$$

Both sides of this equation must reduce to a constant. We therefore write

$$mg(\psi)g''(\psi) + (1 - m)g'(\psi)^2 = 0,$$

which produces an integral of the form

$$(83) \quad g(\psi) = (\psi/\psi_0)^m,$$

so that from (80)

$$k(\psi) = k_0\psi^{(\gamma-1)/\gamma},$$

k_0 being some constant.

The integral of the equation for $H(t)$,

$$\frac{t}{t_0} H''(t) = \frac{2 - m}{m},$$

will be taken in the form

$$H(t) = \frac{(m - 2)t^2}{m(\gamma - 1)(2 - \gamma)} \left[\frac{t_0}{t} \right]^\gamma + Ut.$$

Redefining the constants of integration, the solution (81) may be exhibited as

$$x = a_1 t \psi^{(\gamma-1)/\gamma} + a_2 t^{2-\gamma} + Ut,$$

where the index m occurring in (83) is now given by

$$(84) \quad m = 2 - \gamma(\gamma - 1)(2 - \gamma) \left[\frac{\gamma - 1}{\gamma} a_1 \right]^\gamma a_2.$$

Attention is directed to the solution

$$(85) \quad x = a_1 t \psi^{(\gamma-1)/\gamma} + a_2 t^{2-\gamma}.$$

Another solution may be obtained in the usual way by regarding a_1 and a_2 as functions of a variable $s(t, \psi)$ to be eliminated between (85) and its s -derivative;

$$(86) \quad 0 = a_1' \psi^{(\gamma-1)/\gamma} + a_2' t^{1-\gamma}$$

provided the variable s so defined is characteristic. Since m is constant, (84) gives a_2 in terms of a_1 . From (86) by differentiation,

$$(87) \quad [a_1'' \psi^{(\gamma-1)/\gamma} + a_2'' t^{1-\gamma}] s_t + (1 - \gamma) a_2' t^{-\gamma} = 0,$$

$$(88) \quad \gamma [a_1'' \psi^{(\gamma-1)/\gamma} + a_2'' t^{1-\gamma}] s_\psi + (\gamma - 1) a_1' \psi^{-1/\gamma} = 0.$$

Using (85), (87), (88) the condition on $s(t, \psi)$,

$$x_\psi^{\gamma+1} s_t^2 - s_\psi^2 = 0,$$

requires that

$$(89) \quad \gamma^2 \left[\frac{\gamma - 1}{\gamma} a_1 \right]^{\gamma+1} a_2' + a_1' = 0$$

after using (86).

The integral of the relation (89) is

$$\gamma^2(\gamma - 1)a_2 - \left[\frac{\gamma - 1}{\gamma} a_1 \right]^{-\gamma} = \text{constant}.$$

The constant of integration must be taken to be zero to be reconciled with the equation (84). Then

$$(90) \quad \gamma^2(\gamma - 1)a_2 = \left[\frac{\gamma - 1}{\gamma} a_1 \right]^{-\gamma},$$

and the associated value of m is given by

$$m = \frac{3\gamma - 2}{\gamma}.$$

Eliminating a_1' , a_2' between (86), (89) produces

$$(\gamma - 1)a_1 = \left[\frac{t\psi^{1/\gamma}}{\gamma} \right]^{-(\gamma-1)/(\gamma+1)}.$$

Equation (90) then gives

$$\gamma(\gamma - 1)a_2 = [\gamma\psi t^\gamma]^{(\gamma-1)/(\gamma+1)}.$$

The solution alternative to (85) is then of the form

$$x = C[\psi^{\gamma-1}t^3]^{1/(\gamma+1)},$$

C being an arbitrary constant, as may be verified by direct substitution into the equation for $x(t, \psi)$.

6. Chaplygin Gas. The general integration of the equation (25) or the equivalent Lagrangian equation (16) presents a formidable problem. There is, however, one case of interest for which considerable simplifications result. This occurs when the hypothetical gas of Chaplygin is considered, in which the adiabatic law embodied in equations (3), (4) is modified for variation of pressure and density in the region of a suitable point p_0 , ρ_0 by using a linear relation connecting p and ρ^{-1} . This is equivalent to selecting $\gamma = -1$.

The equation (25) reduces to

$$w_{,vv} - f(w_{,v})w_{,\lambda\lambda} = 0.$$

The characteristic equations (31), (32) reduce to

$$\begin{aligned}\psi_\alpha - t_\alpha &= 0, \\ \psi_\beta + t_\beta &= 0, \\ u_\alpha - a_\alpha + \frac{a}{R} S'(\psi) \psi_\alpha &= 0, \\ u_\beta + a_\beta - \frac{a}{R} S'(\psi) \psi_\beta &= 0.\end{aligned}$$

The characteristic variables may now be obtained explicitly:

$$\alpha = \psi + t, \quad \beta = \psi - t.$$

Writing

$$S'(\psi) = 2RF(\alpha + \beta),$$

the equations for u, a become

$$\begin{aligned}u_\alpha - a_\alpha + aF(\alpha + \beta) &= 0, \\ u_\beta + a_\beta - aF(\alpha + \beta) &= 0.\end{aligned}$$

Equations (33) give

$$2x_\alpha = u + a, \quad 2x_\beta = -(u - a),$$

so that

$$u = x_\alpha - x_\beta, \quad a = x_\alpha + x_\beta.$$

The equation determining $x(\alpha, \beta)$ is the linear equation

$$2x_{\alpha\beta} - (x_\alpha + x_\beta)F(\alpha + \beta) = 0,$$

solutions of which can be obtained when $F(\alpha + \beta)$ takes the form

$$F(\alpha + \beta) = k(\alpha + \beta)^m,$$

so that the associated distribution takes the form given by

$$S'(\psi) = 2kR(2\psi)^m,$$

k being some constant.

In this paper methods of attacking the problem of obtaining solutions of the equations of non-isentropic unsteady one-dimensional flow of a gas have been discussed. In contrast to the equations of isentropic flow, even the problem of constructing special integrals is one of considerable difficulty, due largely to the fact that the characteristic variables cannot be obtained explicitly in the general case of arbitrary distribution of entropy. However, considerable progress was possible in the discussion on generalised simple waves, due to the simplifications

that result through the use of the modified particle function ψ in terms of which the gas speed and acoustic speed may be directly expressed. The general theory of simple waves yielded the important result that all simple waves in non-isentropic flow are "centered" in the (x, t) plane, the gas speed and speed of sound essentially depending only on the quantity x/t . This theorem brings the theory into line with the corresponding theory of steady rotational simple waves in plane flow. It should be noted that the specific example considered in §4 is only one of a large number of possibilities obtained by assigning different values to the arbitrary constants m, m_0 of equation (68).

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