Integral Formulas for Hypersurfaces in Euclidean Space and Their Applications to Uniqueness Theorems

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Introduction. Let E be the Euclidean space of dimension $n + 1$. By a hypersurface in E we mean a differentiable manifold M of dimension n and a differentiable mapping $x: M \rightarrow E$, whose functional matrix is everywhere of rank n . Since the strength of the differentiability assumptions will not be the issue, we suppose our manifolds and mappings to be of class C^∞ .

If M is compact, certain immediate integral formulas are valid. More generally, we will derive integral formulas for the hypersurface x and a second hypersurface $x': M \rightarrow E$. Such formulas are generalizations of well-known formulas in the theory of convex bodies, which express the mixed volumes of two convex bodies as integrals [1].

Given two compact hypersurfaces, a rigidity or uniqueness theorem gives a sufficient condition such that commutativity holds in the diagram

$$\begin{array}{ccc} M & \xrightarrow{x} & E \\ & \searrow_{x'} & \swarrow^T \\ & E & \end{array}$$

where T is a motion in E . Of interest in differential geometry are conditions expressed in terms of the relative curvature of the hypersurface. We review its definition as follows:

¶ Suppose M to be oriented. Then to $p \in M$ there is a uniquely determined unit normal vector $\xi(p)$ at $x(p)$. We put

$$(1) \quad \text{I} = dx^2, \quad \text{II} = d\xi \, dx, \quad \text{III} = d\xi^2,$$

where $dx, d\xi$ are vector-valued linear differential forms in M and multiplication is in the sense of the scalar product in E . These are three quadratic differential forms in M (the "fundamental forms"), of which I is positive definite. The eigenvalues $k_1, \dots, k_n$ of II relative to I are called the principal curvatures. If the Gauss-Kronecker curvature $K = k_1 \cdots k_n \neq 0$, the reciprocals $1/k_1, \dots, 1/k_n$ are called the radii of principal curvature; they are the eigenvalues of II

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relative to III, which is also positive definite under the assumption $K \neq 0$. In this case we introduce the l^{th} elementary symmetric function

$$(2) \quad \binom{n}{l} P_l = \sum \frac{1}{k_1} \cdots \frac{1}{k_l}, \quad 1 \leq l \leq n.$$

It seems that our integral formulas are more effective for hypersurfaces which are strictly convex, *i.e.*, hypersurfaces for which the Gauss-Kronecker curvature is everywhere > 0 . By using an inequality of L. GÅRDING [2], we can immediately derive the uniqueness theorem of ALEXANDROFF-FENCHEL-JESSEN [3, 4], to the effect that two closed strictly convex hypersurfaces differ by a translation if the function P_l (for a fixed l) takes the same value at points with the same normal vector. Actually this proof does not differ essentially from the one given by ALEXANDROFF. Its merit lies perhaps in stating the result in a more general form and in separating the geometrical from the analytical part. One can, however, derive in this way further uniqueness theorems. For instance, it will be proved that a closed strictly convex hypersurface is a hypersphere if, for a fixed l , $2 \leq l \leq n$, $P_{l-1}^\alpha P_l^\beta = \text{const.}$, $\alpha \geq 0, \beta \geq 0, \alpha + \beta > 0$.

In the proofs of more general uniqueness theorems for closed strictly convex (two-dimensional) surfaces the most important tool is the "index method". This does not seem to generalize to higher dimensions, at least not in an obvious way. The need and search for new methods should make the higher dimensional problems more challenging and interesting.

One immediate problem is the following: Is a closed strictly convex hypersurface defined up to a translation if the l^{th} ($1 \leq l \leq n-1$) elementary symmetric function of the principal curvatures is given as a function of the normal vector?

1. Integral formulas. Let M be an oriented differentiable manifold of dimension n , and let $x: M \rightarrow E$ be a hypersurface. Let $\xi(p)$, $p \in M$, be the unit normal vector at $x(p)$. We consider the orthonormal frames $e_1, \dots, e_n$ in the tangent hyperplane at $x(p)$, such that the determinant $(e_1, \dots, e_n, \xi) = +1$. The space of all $e_1, \dots, e_n$ can be identified with the principal fiber bundle B of M relative to the Riemannian metric I . We have

$$(3) \quad \begin{aligned} dx &= \omega_1 e_1 + \cdots + \omega_n e_n, \\ d\xi &= \theta_1 e_1 + \cdots + \theta_n e_n, \end{aligned}$$

so that $\omega_i, \theta_i, 1 \leq i \leq n$, are linear differential forms in B . Since

$$(4) \quad \xi \, dx = 0,$$

we get, by exterior differentiation,

$$(5) \quad d\xi \wedge dx = 0.$$

The left-hand side in (5) is the exterior product of two vector-valued linear differential forms; vectors are multiplied in the sense of scalar products in E .

In view of (3), equation (5) can be written

$$(6) \quad \sum_i \omega_i \wedge \theta_i = 0.$$

This is the fundamental relation in the theory of hypersurfaces in E .

Since ω_i are linearly independent, we can put, in view of (6),

$$(7) \quad \theta_i = \sum_k \mu_{ik} \omega_k, \quad \mu_{ik} = \mu_{ki}, \quad 1 \leq i, k \leq n.$$

If $\det(\mu_{ik}) \neq 0$, we introduce the matrix (λ_{ik}) inverse to (μ_{ik}) , so that we have

$$(8) \quad \omega_i = \sum_k \lambda_{ik} \theta_k.$$

By (1) and (3) we can also write

$$(9) \quad \begin{aligned} \text{I} &= \sum_i \omega_i^2, & \text{III} &= \sum_i \theta_i^2, \\ \text{II} &= \sum_i \theta_i \omega_i = \sum_{i,k} \mu_{ik} \omega_i \omega_k = \sum_{i,k} \lambda_{ik} \theta_i \theta_k. \end{aligned}$$

Let

$$(10) \quad \det(\delta_{ik} + \lambda_{ik} y) = \sum_{0 \leq l \leq n} \binom{n}{l} P_l(\lambda) y^l,$$

where y is a parameter. Then $P_l(\lambda)$ is a homogeneous polynomial of degree l in λ_{ik} , and it is easy to see that it is equal to the invariant P_l defined in (2).

We now introduce the differential forms

$$(11) \quad \begin{aligned} A_r &= (x, \xi, \underbrace{d\xi, \dots, d\xi}_{n-1-r}, \underbrace{dx, \dots, dx}_r), & 0 \leq r \leq n-1, \\ C_s &= (x, d\xi, \dots, d\xi, dx, \dots, dx), \\ D_s &= (\xi, d\xi, \dots, d\xi, dx, \dots, dx), & 0 \leq s \leq n. \end{aligned}$$

Each of these expressions is a determinant of order $n+1$, whose columns are the components of the respective vectors or vector-valued differential forms, with the convention that in the expansion of the determinant the multiplication of differential forms is in the sense of exterior multiplication. A_r is a differential form of degree $n-1$ in M , and C_s , D_s are differential forms of degree n ; the subscripts r , s in each case denote the number of entries dx in these determinants. It is to be observed that A_r , C_s depend on the choice of the origin in the space E , while D_s depends only on the hypersurface $x(M)$.

Let $h = \xi x$ be the support function, the distance from the origin to the tangent hyperplane at $x(p)$. Since $d\xi$ and dx are linear combinations of $e_1, \dots, e_n$ only, we have immediately

$$(12) \quad C_s = h D_s.$$

Exterior differentiation gives

$$dA_r = C_r - D_{r+1} = hD_r - D_{r+1}, \quad 0 \leq r \leq n-1.$$

Hence, by Stokes' Theorem, we have, under the assumption that M is compact,

$$(13) \quad \int_M hD_r - D_{r+1} = 0, \quad 0 \leq r \leq n-1.$$

More generally, for a pair of hypersurfaces $x, x': M \rightarrow E$ we introduce the differential forms

$$(14) \quad \begin{aligned} A_{rs} &= (x, \xi, \underbrace{d\xi, \dots, d\xi}_{n-1-(r+s)}, \underbrace{dx, \dots, dx}_r, \underbrace{dx', \dots, dx'}_s), \\ A'_{rs} &= (x', \xi, d\xi, \dots, d\xi, dx, \dots, dx, dx', \dots, dx'), \\ B_{rs} &= (x, x', d\xi, \dots, d\xi, dx, \dots, dx, dx', \dots, dx'), \quad 0 \leq r+s \leq n-1 \\ C_{uv} &= (x, \underbrace{d\xi, \dots, d\xi}_{n-(u+v)}, \underbrace{dx, \dots, dx}_u, \underbrace{dx', \dots, dx'}_v), \\ C'_{uv} &= (x', d\xi, \dots, d\xi, dx, \dots, dx, dx', \dots, dx'), \\ D_{uv} &= (\xi, d\xi, \dots, d\xi, dx, \dots, dx, dx', \dots, dx'), \quad 0 \leq u+v \leq n. \end{aligned}$$

As above, we have

$$(15) \quad C_{uv} = hD_{uv}, \quad C'_{uv} = h'D_{uv}, \quad h' = \xi x'.$$

By introducing the unit normal vector $\xi'(p)$ of the second hypersurface, we could have more general differential forms, but we will not write them down here.

Exterior differentiation gives

$$(16) \quad \begin{aligned} dA_{rs} &= C_{rs} - D_{r+1,s} = hD_{rs} - D_{r+1,s}, \\ dA'_{rs} &= C'_{rs} - D_{r,s+1} = h'D_{rs} - D_{r,s+1}, \\ dB_{rs} &= C_{r,s+1} - C'_{r+1,s} = hD_{r,s+1} - h'D_{r+1,s}, \quad 0 \leq r+s \leq n-1. \end{aligned}$$

From these follow the integral formulas

$$(17) \quad \begin{aligned} \int hD_{rs} - D_{r+1,s} &= 0, \\ \int h'D_{rs} - D_{r,s+1} &= 0, \\ \int hD_{r,s+1} - h'D_{r+1,s} &= 0, \quad 0 \leq r+s \leq n-1, \end{aligned}$$

the integrations being over M , supposed to be compact.

These integral formulas take a particularly simple form when the hypersurfaces are strictly convex. Then the Gauss-Kronecker curvatures are strictly

> 0 , and the normal mapping $\xi: M \rightarrow S_0$, where S_0 is the unit hypersphere in E , is a differentiable homeomorphism with functional determinant everywhere different from zero. We can therefore identify M with S_0 and define the hypersurface by $x: S_0 \rightarrow E$. Geometrically $x(\xi)$, $\xi \in S_0$, is the coordinate vector of the point of the hypersurface at which ξ is the normal vector. Similarly, the second hypersurface will be defined by the mapping $x': S_0 \rightarrow E$. The functions and differential forms previously defined in M are now in S_0 , and the integrals in (17) are over S_0 . We can suppose the hypersurfaces so oriented that Π is everywhere positive definite.

To apply the formulas (17) we wish to find more explicit expressions for D_{rs} . With two parameters y, y' we have

$$\begin{aligned} & \sum_{0 \leq r+s \leq n} \frac{n!}{r!s!(n-r-s)!} y^r y'^s D_{rs} \\ &= (-1)^n \sum_{1 \leq i_1, \dots, i_n \leq n} \epsilon_{i_1 \dots i_n} (y\omega_{i_1} + y'\omega'_{i_1} + \theta_{i_1}) \wedge \dots \wedge (y\omega_{i_n} + y'\omega'_{i_n} + \theta_{i_n}) \\ &= (-1)^n n! (y\omega_1 + y'\omega'_1 + \theta_1) \wedge \dots \wedge (y\omega_n + y'\omega'_n + \theta_n) \\ &= (-1)^n n! \det (y\lambda_{ik} + y'\lambda'_{ik} + \delta_{ik}) dV, \end{aligned}$$

where

$$dV = \theta_1 \wedge \dots \wedge \theta_n$$

is the volume element of S_0 . Let

$$(18) \quad \det (y\lambda_{ik} + y'\lambda'_{ik} + \delta_{ik}) = \sum_{0 \leq r+s \leq n} \frac{n!}{r!s!(n-r-s)!} y^r y'^s P_{rs}$$

so that P_{rs} is a polynomial in $\lambda_{ik}, \lambda'_{ik}$, homogeneous of degrees r and s respectively. In particular, $P_{i0} = P_i(\lambda)$, the latter being defined in (10). The integral formulas (17) can then be written

$$\begin{aligned} & \int (hP_{rs} - P_{r+1,s}) dV = 0, \\ (19) \quad & \int (h'P_{rs} - P_{r,s+1}) dV = 0, \\ & \int (hP_{r,s+1} - h'P_{r+1,s}) dV = 0, \quad 0 \leq r+s \leq n-1. \end{aligned}$$

The third equation in (19) is, for $0 \leq r+s \leq n-2$, a consequence of the first two equations.

An important consequence of the third equation of (19) consists in the formulas

$$\int (hP_{0l} - h'P_{1,l-1}) dV = 0, \quad \int (hP_{l-1,1} - h'P_{l0}) dV = 0, \quad 1 \leq l \leq n.$$

From these we derive

$$(20) \quad 2 \int h(P_{0l} - P_{l-1,1}) dV = \int \{h'(P_{1,l-1} - P_{l0}) - h(P_{l-1,1} - P_{0l})\} dV.$$

Let $F(u_1, \dots, u_n)$ be a function in n positive variables. We will say that F is of type $l \geq 2$ if the following conditions are satisfied: (1) $F(P_{10}, \dots, P_{n0}) = F(P_{01}, \dots, P_{0n})$ implies $P_{l-1,1} - P_{0l} \geq 0$; (2) $F(P_{10}, \dots, P_{n0}) = F(P_{01}, \dots, P_{0n})$, $P_{l-1,1} - P_{0l} = 0$ if and only if $\lambda'_{ik} = \lambda_{ik}$.

Theorem 1. *Let $F(u_1, \dots, u_n)$ be a function of type $l \geq 2$. If two closed strictly convex hypersurfaces have the property that at points with the same unit normal vector the functions $F(P_{10}, \dots, P_{n0})$ and $F(P_{01}, \dots, P_{0n})$ have the same value, then the hypersurfaces differ from each other by a translation.*

Proof. It suffices to prove that $\lambda'_{ik} = \lambda_{ik}$ at points with the same unit normal vector. Choose the origin in E such that $h > 0$. Then the integral at the left-hand side of (20) is ≤ 0 . The same is therefore true of the right-hand side of (20). But the latter is anti-symmetric in the two hypersurfaces, and hence must be zero. It follows that $P_{l-1,1} - P_{0l} = 0$. By the second property of the function F , we get $\lambda'_{ik} = \lambda_{ik}$.

2. The uniqueness theorem of Alexandroff-Fenchel-Jessen.

Theorem 2 (ALEXANDROFF-FENCHEL-JESSEN). *Two closed strictly convex hypersurfaces differ by a translation if the function $P_l(\lambda)$, $1 \leq l \leq n$, takes the same value at points with the same unit normal vector.*

For $2 \leq l \leq n$ we will prove that the function $F = u_l$ is of type l . Then the theorem for $2 \leq l \leq n$ will follow from Theorem 1.

We will need the following inequality of L. GÄRDING [2]:

To the polynomial $P_l(\lambda)$ defined in (10) let $P_l(\lambda^{(1)}, \dots, \lambda^{(l)})$ be its completely polarized form, so that $P_l(\lambda, \dots, \lambda) = P_l(\lambda)$. Then, for positive definite symmetric matrices $(\lambda_{ik}^{(1)}), \dots, (\lambda_{ik}^{(l)})$, the following inequality is valid:

$$(21) \quad P_l(\lambda^{(1)}, \dots, \lambda^{(l)}) \geq P_l(\lambda^{(1)})^{1/l} \dots P_l(\lambda^{(l)})^{1/l}.$$

The equality sign holds if and only if the l matrices are pairwise proportional.

Suppose now $P_{l0} = P_{0l}$. This condition can be written $P_l(\lambda) = P_l(\lambda')$. By definition $P_{l-1,1} = P_l(\lambda, \dots, \lambda, \lambda')$. Since (λ_{ik}) and (λ'_{ik}) are positive definite, it follows from (21) that

$$P_l(\lambda, \dots, \lambda, \lambda') \geq P_l(\lambda)^{(l-1)/l} P_l(\lambda')^{1/l} = P_l(\lambda'),$$

which implies the first condition for a function of type l . The equality sign holds only if $\lambda'_{ik} = \rho \lambda_{ik}$. Since $P_l(\lambda) = P_l(\lambda')$, we have $\rho = 1$.

The proof of Theorem 2 for $l = 1$ is different, but is actually easier in the sense that an inequality such as (21) will not be needed. The proof has been given on a previous occasion [5]; we present it here in our notation for the sake of completeness.

By definition we have

$$\begin{aligned}
 n(n-1)P_{20} &= \left(\sum_i \lambda_{ii}\right)^2 - \sum_{i,k} \lambda_{ik}^2, \\
 n(n-1)P_{02} &= \left(\sum_i \lambda'_{ii}\right)^2 - \sum_{i,k} \lambda'_{ik}{}^2, \\
 2n(n-1)P_{11} &= \sum_{i,k} (\lambda_{ii}\lambda'_{kk} + \lambda'_{ii}\lambda_{kk} - 2\lambda_{ik}\lambda'_{ik}).
 \end{aligned}
 \tag{22}$$

From these it follows that

$$\Lambda \equiv \sum_{i,k} \begin{vmatrix} \lambda'_{ii} - \lambda_{ii} & \lambda'_{ik} - \lambda_{ik} \\ \lambda'_{ki} - \lambda_{ki} & \lambda'_{kk} - \lambda_{kk} \end{vmatrix} = n(n-1)(P_{20} + P_{02} - 2P_{11}).$$

Our integral formulas (19) give

$$\begin{aligned}
 \int h(P_{01} - P_{10}) dV &= \int (P_{11} - P_{20}) dV, \\
 \int h'(P_{01} - P_{10}) dV &= \int (P_{02} - P_{11}) dV.
 \end{aligned}
 \tag{23}$$

From the hypothesis $P_{10} = P_{01}$ we therefore get

$$\int \Lambda dV = 0.$$

On the other hand, under the same hypothesis we have

$$\begin{aligned}
 \Lambda &= 2 \sum_{i < k} \begin{vmatrix} \lambda'_{ii} - \lambda_{ii} & \lambda'_{ik} - \lambda_{ik} \\ \lambda'_{ki} - \lambda_{ki} & \lambda'_{kk} - \lambda_{kk} \end{vmatrix} \\
 &= 2 \sum_{i < k} (\lambda'_{ii} - \lambda_{ii})(\lambda'_{kk} - \lambda_{kk}) - 2 \sum_{i < k} (\lambda'_{ik} - \lambda_{ik})^2 \\
 &= \sum_{i,k} (\lambda'_{ii} - \lambda_{ii})(\lambda'_{kk} - \lambda_{kk}) - \sum_i (\lambda'_{ii} - \lambda_{ii})^2 - 2 \sum_{i < k} (\lambda'_{ik} - \lambda_{ik})^2 \\
 &= - \sum_i (\lambda'_{ii} - \lambda_{ii})^2 - 2 \sum_{i < k} (\lambda'_{ik} - \lambda_{ik})^2.
 \end{aligned}$$

Therefore

$$\int \left\{ \sum_i (\lambda'_{ii} - \lambda_{ii})^2 + 2 \sum_{i < k} (\lambda'_{ik} - \lambda_{ik})^2 \right\} dV = 0,$$

which is possible only when $\lambda'_{ik} = \lambda_{ik}$.

Remark 1. The case $l=n$ of Theorem 2 is called the uniqueness of MINKOWSKI'S problem. The theorem for $l=1, n=2$ goes back to CHRISTOFFEL and HURWITZ. We refer to BONNESEN & FENCHEL'S book for historical facts concerning the problem and for its relation with the fundamental Brunn-Minkowski inequalities in convex bodies.

Remark 2. The uniqueness of the Minkowski problem can be proved without using GÄRDING'S inequality as follows: When we replace the matrix $\Delta = (y\lambda_{ik} + y'\lambda'_{ik} + \delta_{ik})$ by ${}^tT \Delta T$, where T is an arbitrary non-singular matrix; P_{rs} will be multiplied by $(\det T)^2$. Since (λ_{ik}) and (λ'_{ik}) are both positive definite, we can choose T to make $(\lambda'_{ik}) = I =$ unit matrix and $(\lambda_{ik}) =$ diagonal matrix, while (δ_{ik}) will then be transformed to a general positive definite symmetric matrix. If $\lambda_1, \dots, \lambda_n$ denote the diagonal elements of (λ_{ik}) , we have

$$P_{n-1,1} = \frac{1}{n-1} \sum \lambda_1 \cdots \lambda_{n-1},$$

where the sum denotes the elementary symmetric function of order $n-1$ in $\lambda_1, \dots, \lambda_n$. On the other hand, our condition $P_{n0} = P_{0n}$ gives $\lambda_1, \dots, \lambda_n = 1$. By NEWTON'S inequality we get $P_{n-1,1}^{1/(n-1)} \geq 1$ or $P_{n-1,1} - P_{0n} \geq 0$. Moreover, $P_{n-1,1} - P_{0n} = 0$ implies $P_{n-1,1}^{1/(n-1)} = 1$, which is possible only when $\lambda_1 = \dots = \lambda_n = 1$. This proves that the function $F = u_n$ is of type n , and the uniqueness follows from Theorem 1.

Remark 3. GÄRDING'S proof of the inequality (21) makes use of general results on hyperbolic polynomials. It may be of interest to observe that for our purpose it would be sufficient to have the inequality for the case that $P_l(\lambda)$ is the l^{th} elementary symmetric function in $n (\geq l)$ positive numbers. It would be interesting to have an elementary proof of this particular case of the inequality.

Remark 4. As is well-known, we can determine our hypersurface by the support function $h(\xi)$, $\xi \in S_0$. Then P_l can be expressed in terms of $h(\xi)$ and its first and second partial derivatives. If $h(\xi)$ is considered to be an unknown function, this relation is, for $l \geq 2$, a non-linear partial differential equation of the second order in $h(\xi)$. Our uniqueness theorem can be interpreted as asserting that two solutions of this equation on the unit hypersphere S_0 differ from each other by a linear homogeneous combination of the coordinates of the space. In this respect the fact that the cases $l = 1$ (linear case) and $l \geq 2$ have to be treated differently is of significance.

Remark 5. With integral formulas as the main tool in the proofs the theorems can be immediately extended to hypersurfaces with boundaries. The extension amounts merely to a writing of the formula (19) with a term pertaining to the boundary, and to an interpretation of this term. It is necessary that the boundaries differ by a translation and that corresponding points have the same normal vectors. This condition, together with the same conditions as in the case of hypersurfaces without boundary, is then sufficient.

3. Characterization of the hypersphere. When the second hypersurface is a hypersphere of radius r , Theorem 1 gives

Theorem 3. Let $F(u_1, \dots, u_n)$ be a function in n positive variables which has the following properties ($l \geq 2$): (1) $F(P_1, \dots, P_n) = F(r, \dots, r^n) = \text{const.}$

implies that $P_{l-1} - r^{l-1} \geq 0$; (2) $F(P_1, \dots, P_n) = F(r, \dots, r^n)$, together with $P_{l-1} = r^{l-1}$, implies $\lambda_{ik} = r\delta_{ik}$. Then a closed strictly convex hypersurface with $F(P_1, \dots, P_n) = \text{const.}$ is a hypersphere of radius r .

As a consequence of this theorem we have the following characterization of the hypersphere:

Theorem 4. A closed strictly convex hypersurface with $P_{l-1}^\alpha P_l^\beta = \text{const}$, $2 \leq l \leq n$, $\alpha + \beta > 0$, $\alpha \geq 0$, $\beta \geq 0$, is a hypersphere.

Proof. In the inequality (21) we put

$$\lambda_{ik}^{(1)} = \dots = \lambda_{ik}^{(l-1)} = \lambda_{ik}, \quad \lambda_{ik}^{(l)} = r\delta_{ik}.$$

Then we get

$$(24) \quad P_{l-1}^{1/(l-1)} \geq P_l^{1/l},$$

where the equality sign holds if and only if $\lambda_{ik} = s\delta_{ik}$.

In proving Theorem 4 we can suppose $\beta > 0$. Let $r > 0$ be defined by

$$(25) \quad P_{l-1}^\alpha P_l^\beta = r^{\alpha(l-1) + \beta l},$$

so that

$$P_l = r^{(\alpha/\beta)(l-1) + l} P_{l-1}^{-\alpha/\beta}.$$

From (24) we get

$$P_{l-1} \geq P_l^{(l-1)/l} = r^{(l-1)\{(\alpha/\beta)\{(l-1)/l\} + 1\}} P_{l-1}^{-\alpha/\beta \{(l-1)/l\}},$$

which gives

$$P_{l-1} \geq r^{l-1}.$$

Moreover, the equality sign holds here if and only if it holds in (24). Hence $P_{l-1} = r^{l-1}$ implies $\lambda_{ik} = s\delta_{ik}$. But then (25) gives $s = r$. Our theorem therefore follows from Theorem 3.

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