Positive Operators

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Communicated by C. LOEWNER

The classical Frobenius theorems concerning matrices with non-negative elements have in recent years proved of fundamental importance and aid in the study of vibrations of mechanical systems [8], stochastic processes [7], and mathematical economics [5]. The first Frobenius theorem states that if $A = (a_{ij})$, $i, j = 1, \dots, n$, is a square non-singular matrix with non-negative elements, then among the eigenvalues of this matrix there exists a positive number ρ to which corresponds a non-zero eigenvector $v = (v_1, v_2, \dots, v_n)$ such that $Av = \rho v$ and $v_i \geq 0$. Furthermore, the magnitudes of all other eigenvalues of A do not exceed the value ρ .

One of the principal components of the analysis of this theorem revolves about the concept that a non-negative matrix may be regarded as a linear mapping which transforms the positive orthant into itself. The positivity preserving feature of the mapping can be exploited in a variety of ways. The newest trick is to use suitable contraction properties of projective mappings related to the matrix A [2], [14]. In this paper, we appeal to complex variable theorems about power series with positive coefficients. These appear to be easily susceptible to generalization (see Section 4 and Section 9).

In recent years numerous types of generalizations of the Frobenius theorem have been advanced. First, the positive orthant can be replaced by a general convex cone and we speak of linear transformations leaving the cone invariant. We shall still refer to such transformations as positive operators. Second, the extensions of the Frobenius theorem to infinite dimensional situations can be accomplished in varying degrees of generality. Some of the notable contributors in this respect include JENTZSCH, KAKUTANI, BIRKHOFF [2], BONNALL [4], KREIN & GANTMACHER [8], and KREIN & RUTMAN [12] (henceforth abbreviated K-R). (A thorough bibliography covering the years prior to 1948 is included in [12].)

The last reference embodies a comprehensive synthesis covering wide ramifications of the theory of positive mappings. This work serves as the chief inspiration for the studies in this paper and its influence on us will be abundantly manifest. We would also like to express our great indebtedness to H. F. BOHNENBLUST for our many valuable discussions which took place in the course

of developing the results of this paper. Some of his resourceful innovations are noted explicitly later.

In this manuscript we investigate several aspects of the theory of positive transformations. A brief outline follows. In Section 1 we summarize the relevant background of the theory of convex cones and partial orderings in Banach spaces. Section 2 presents some special results concerning monotone convergence of sequences of elements. In Section 3 we examine the relationships of linear mappings which leave a cone invariant and the property of continuity that these transformations inherently possess.

Section 4 is devoted to the first generalizations of the Frobenius theorems. Our results extend those of KREIN & RUTMAN. The novelty of our methods consists in utilizing appropriate classical theorems on power series having non-negative coefficients. For example, it is shown that the Pringsheim theorem (the Pringsheim theorem asserts that a power series of an analytic function $f(t)$ with non-negative coefficients and radius of convergence 1 has 1 as a singular point of $f(t)$) leads to and generalizes the first Frobenius theorem. Other Tauberian theorems of HARDY & LITTLEWOOD on power series with non-negative coefficients are similarly exploited.

In Section 5 we discuss some further refinements for the case of strictly positive operators (linear mappings which transform a cone into its interior).

A characterization of the spectral radius is developed in Section 6. Various applications of this characterization to the problem of the existence of eigenvalues are indicated. In the finite dimensional case the characterization of this section was originally suggested by BOHNENBLUST. The results of Section 6 are applied in Section 7 in the study of the variation of the spectral radius when positive operators are subjected to perturbations.

A special study of the nature of the spectrum with reference to the eigenvalues, lying on the boundary of the circle containing the spectrum, is undertaken in Section 8. Finally, the spectral properties of semi-groups of transformations leaving invariant a cone are briefly explored in Section 9.

1. Preliminaries. Let E be a real Banach space and let there be distinguished a closed set K possessing the following properties:

- (1) If x, y belong to K and λ, μ are real non-negative scalars then $\lambda x + \mu y \in K$.
- (2) If $x \in K$ and $-x \in K$ then $x = 0$.

We call K a cone. Some notation which will be used freely throughout the remainder of this article is now indicated. $x_n \rightharpoonup x$ shall mean that x_n converges weakly to x . $(f, x) = f(x)$ will denote the value of the functional f in E^* applied to the element x in E . Both notations will be interchangeably employed. If x belongs to K then we write $x \geq 0$, while if x is interior (with respect to the full Banach space; in the case of relative interior we adopt the language "inner") to K we write $x \gg 0$. If x is merely in K and non-zero we write $x > 0$. A natural

partial ordering can be defined in terms of K with $x \geq y$ being equivalent to $x - y \in K$.

The dual cone K^* in the conjugate space E^* is defined to be the set $K^* = \{f \mid f \in E^*, (f, x) \geq 0 \text{ for all } x \text{ in } K\}$. The existence of non-negative functionals is discussed in KREIN & RUTMAN [12]. For example, it is established that whenever K has an interior, K^* contains non-trivial elements.

Some classical important examples of cones K are the following:

(a) Let $E = C(\Omega)$ (the space of all continuous functions defined on a compact Hausdorff space Ω). Then K may be identified with the set of all non-negative continuous functions.

(b) Let $E = V(\Omega)$ (the space of all regular measures on Ω). Then K may be formed of all non-negative measures. This is the conjugate cone of example (a).

(c) Let $E = H$ (countable real Hilbert space where each element x in H is represented as a countable sequence of real numbers $x = (x_1, x_2, \dots)$ such that $\sum_{i=1}^{\infty} |x_i|^2 < \infty$). Here we may take $K = \{x \mid x \in H \text{ and } x_i \geq 0 \text{ for all } i\}$. Another useful choice of K is

$$K = \left\{ x \mid x \in H, x_1 \geq 0 \text{ and } x_1 \geq \sqrt{\sum_{i=2}^{\infty} x_i^2} \right\}.$$

The element $u = (1, 0, 0, \dots)$ is an interior point of this cone. This last cone is commonly referred to as the Lorentzian cone (see K-R, p. 48). Both cones are self reflexive, *i.e.*, $K = K^*$.

(d) Let E = the space of all bounded linear self-adjoint transformations defined on a Hilbert space $\mathcal{A}$. An element x in E is designated as positive if and only if x represents a positive definite transformation. Examples of elements f in K^* are obtained by specifying a finite collection of elements $\xi_1, \xi_2, \dots, \xi_r$ in $\mathcal{A}$, and (f, x) is interpreted as

$$(f, x) = \sum_{i=1}^r [x\xi_i, \xi_i]$$

where $[x\xi_i, \xi_i]$ represents the inner product of the Hilbert space $\mathcal{A}$.

Throughout the sequel we shall assume, unless explicitly stated otherwise, that every element of E can be represented as the difference of two elements of K . The representation is automatically guaranteed if K has an interior (K-R, p. 11). Under suitable conditions a similar representation of E^* in terms of K^* can be deduced (K-R, pp. 22-25). For example, if K has an interior element u with the property that the set of all x in E satisfying $-u \leq x \leq u$ is uniformly bounded in norm, then K^* spans E^* . We frequently place analogous requirements on K^* and occasionally on K^{**} relative to E^* and E^{**} , respectively, without investigating the relevant foundational questions concerned with the structure of these cones.

In dealing with questions of positive operators (linear mappings taking K into itself) and their spectra we shall speak, unless explicitly stated otherwise,

only about real bounded operators T defined on E and extended complex linearly to the complex Banach space $\tilde{E} = E + iE$. More precisely, $\tilde{E} = \{z\} = \{x + iy\}$ with $x, y \in E$. The norm in $\tilde{E}$ is defined as

$$\|z\| = \max_{0 \leq \theta \leq 2\pi} \|x \cos \theta + y \sin \theta\|.$$

If Tx is defined for all x in E with $\|T\|_E = \sup_{\|x\| \leq 1} \|Tx\|$ then $\tilde{T}z = Tx + iTy$, and it follows that $\|\tilde{T}\| = \|T\|_E$. In much that follows we shall not distinguish between E and $\tilde{E}$, T and $\tilde{T}$, and no ambiguities will arise.

In many instances, in order to obtain sharper statements about positive operators, it seems necessary to postulate the existence of an interior element u in K . Some immediate consequences are the following. If u is interior to K there exists a $\rho > 0$ such that, for any x with $\|x\| \leq 1$, $u \pm \rho x \in K$. Consequently, for f in K^* we have $(1/\rho)f(u) \geq |f(x)|$ for all x , $\|x\| \leq 1$. This states that the set H_u consisting of all f in K^* and satisfying $f(u) = 1$ is uniformly bounded ($\|f\| \leq 1/\rho$), and hence is a ω^* closed convex set contained in the sphere $\|f\| \leq 1/\rho$. But since the unit sphere in E^* is ω^* compact, we deduce that H_u is ω^* compact. Furthermore, it is clear that $\{\lambda H_u\} = K^*$ for $\lambda \geq 0$, and that no segment of H_u intersects the origin. Conversely, if a section of a plane H_u intersecting K^* has the properties

- (a) H_u is ω^* compact;
- (b) $\{\lambda H_u\} = K^*$ for $\lambda \geq 0$;
- (c) no segment of H_u intersects the origin;

then there exists an interior point u of K such that $H_u = [f] f \in K^*, (f, u) = 1]$. (For the proof see [3]).

In example (a) the function $x(t) = 1$ is an interior element. In example (c) (the Lorentzian cone) the element $x = (1, 0, 0, \dots)$ is an interior point. The identity operator serves as an interior element for the cone of example (d).

The existence of an interior point u for K does not imply the same for K^* . This can be seen by examining the Banach space $(c) =$ the space of all convergent sequences endowed with its natural ordering and the Banach space (l) which except for one dimension is $(c)^*$, the space of all absolutely convergent series. The cone of (l) has no interior points, but the cone of (c) has interior points, e.g., $(1, 1, 1, \dots)$. It is, however, important to note that there exist elements in (l) which are positive on all non-trivial non-negative elements of (c) . Thus $(f, x) > 0$ for any $x > 0$ does not necessarily imply that f is an interior point of K^* . Conversely, if x is an interior point, then $(f, x) > 0$ for any $f > 0$ in K^* . Furthermore, if x is on the boundary of K and K has an interior, then there exists a non-zero f_0 in K^* with $(f_0, x) = 0$ (K-R, p. 16). Finally, if K possesses an interior point u and $(f, x_0) > 0$ for every non-zero f in K^* , then x_0 is interior to K (K-R, p. 16).

2. Properties of K . In this section we present some results about convergence

of sequences of elements of K . First, we observe that K^{**} can be characterized in terms of K in the standard way. Explicitly,

Lemma 1. *The weak* closure of K in E^{**} is K^{**} .*

Proof. If we suppose the contrary then some ϕ in K^{**} is not in the ω^* -closure of K . Thus by the definition of the ω^* topology there exists a weak* neighborhood of ϕ not intersecting K . That is, we can find a finite number of linear functionals $f_1, -f_1, \dots, f_n, -f_n$, such that for any x in K there exists an index i such that $f_i(x) - \phi(f_i) \geq \epsilon > 0$. If we map x in K into E^n (Euclidean n -space) by means of

$$x \rightarrow \{f_1(x), f_2(x), \dots, f_n(x)\},$$

the closure of the image set is a closed convex cone C which does not include the point $c_i = \phi(f_i)$, $i = 1, 2, \dots, n$. Hence there exists a hyperplane with components of the normal $\{\xi_i\}$ which separates $c = \{c_1, c_2, \dots, c_n\}$ from C , or $f_0 = \sum \xi_i f_i$ has the properties that $f_0(x) \geq a$ for x in K and $\phi(f_0) < a$. Since K is a cone, we infer that $a = 0$ and hence f_0 is in K^* . But then $\phi(f_0) \geq 0$ for ϕ in K^{**} , which furnishes the desired contradiction.

We further assume in this section that for any real f in E^* ($\|f\| \leq 1$) there exists a representation $f = f_1 - f_2$ with $f_1, f_2 \in K^*$ and $\|f_i\| \leq C$ where C is a constant independent of f (usually $C = 1$). This condition is readily seen to be satisfied in most examples of classical Banach spaces.

Lemma 2. *If $x_1 \leq x_2 \leq x_3 \leq \dots$ and x_n converges weakly to x_0 , then x_n converges strongly to x_0 .*

Proof. We consider the functions $\phi_n(f) = f(x_n)$ ($n \geq 0$), defined for f in K^* , satisfying $\|f\| \leq C$. This set is weak* compact. Since $\phi_n(f)$ constitute an increasing sequence of continuous functions defined on a compact space converging to a continuous function $\phi_0(f)$, it follows by a familiar result of DINI that the convergence is uniform. Taking account of the assumption imposed prior to this lemma, we infer that the convergence is uniform for all f in E^* in which $\|f\| \leq 1$. Invoking the Hahn Banach theorem yields

$$\|x_n - x_0\| = \sup_{\|f\| \leq 1} |f(x_n) - f(x_0)| \rightarrow 0.$$

Corollary 1. *If E is weakly complete, $x_1 \leq x_2 \leq x_3 \leq \dots$, and $\sup_n f(x_n) < \infty$ for every f in K^* , then x_n converges strongly.*

In fact, for $f \in K^*$, we have $f(x_n) \leq f(x_{n+1}) \leq \dots < \infty$, and hence the sequence x_n converges weakly. By virtue of the hypotheses there exists a y_0 in E such that $x_n \xrightarrow{\omega} y_0$. Lemma 2 yields that $x_n \rightarrow y_0$ strongly.

In the case of E^* , a stronger result is obtained.

Theorem 1. *If E^* is separable and $f_1 \leq f_2 \leq f_3 \leq \dots \leq f_0$, then f_n converges strongly. (The assumption which was imposed on E^* prior to Lemma 2 is made on E^{**} .)*

Proof. It follows immediately that $f_n(x)$ converges monotonically for every x in K , and hence converges for any x in E . Let $\bar{f}$ denote its weak* limit. We now prove that if ϕ belongs to K^{**} , then $\phi(f_n) \rightarrow \phi(\bar{f})$. Since E^* is separable, we deduce with the aid of Lemma 1 the existence of a sequence x_r in K such that

$$\phi(f) = \lim_{r \rightarrow \infty} f(x_r)$$

for every f in E^* . We have

$$\lim_{n \rightarrow \infty} \phi(f_n) = \lim_{n \rightarrow \infty} \lim_{r \rightarrow \infty} f_n(x_r) = \lim_{r \rightarrow \infty} \lim_{n \rightarrow \infty} f_n(x_r) = \lim_{r \rightarrow \infty} \bar{f}(x_r) = \phi(\bar{f}),$$

provided the interchange of limits can be justified. To this end, consider the mapping U of E into (l) (the space of absolutely convergent series) defined by

$$x \xrightarrow{U} \{f_n(x) - f_{n-1}(x)\}, \quad n = 1, 2, \dots,$$

with f_0 defined $\equiv 0$. The series for x in K becomes

$$\sum |f_n(x) - f_{n-1}(x)| = \sum [f_n(x) - f_{n-1}(x)] = \lim f_n(x).$$

But any x in E is the difference of two x 's in K , and therefore the mapping is well defined. It is immediately verified that U is additive and closed and hence continuous. As x_r is weakly convergent and a bounded transformation preserves weak convergence, Ux_r is weakly convergent in (l) . But weak and strong convergence are equivalent in (l) (see [1], p. 137). This implies that for any prescribed positive ϵ there exists $r_0(\epsilon)$ such that for all integers M and any $r_1, r_2 \geq r_0(\epsilon)$ we have

$$\sum_1^M |\{f_n(x_{r_1}) - f_{n-1}(x_{r_1})\} - \{f_n(x_{r_2}) - f_{n-1}(x_{r_2})\}| \leq \epsilon.$$

Letting $r_2 \rightarrow \infty$ we obtain

$$\sum_1^M |\{f_n(x_{r_1}) - f_{n-1}(x_{r_1})\} - \{\phi(f_n) - \phi(f_{n-1})\}| \leq \epsilon.$$

But, valid for any M , this holds uniformly for $r_1 \geq r_0(\epsilon)$, and we have

$$|\lim_{r_1 \rightarrow \infty} \sum_1^\infty \{f_n(x_{r_1}) - f_{n-1}(x_{r_1})\} - \sum_1^\infty \{\phi(f_n) - \phi(f_{n-1})\}| \leq \epsilon.$$

But

$$\lim_{r \rightarrow \infty} \lim_{n \rightarrow \infty} f_n(x_r) = \lim_{r \rightarrow \infty} \sum_1^\infty \{f_n(x_{r_1}) - f_{n-1}(x_{r_1})\},$$

and

$$\lim_{n \rightarrow \infty} \lim_{r \rightarrow \infty} f_n(x_r) = \sum_1^\infty \{\phi(f_n) - \phi(f_{n-1})\}$$

means that

$$|\lim_{r \rightarrow \infty} \lim_{n \rightarrow \infty} f_n(x_r) - \lim_{n \rightarrow \infty} \lim_{r \rightarrow \infty} f_n(x_r)| \leq \epsilon.$$

Since ϵ is arbitrary we may let ϵ tend to zero, and the interchange of limit is thereby justified, yielding $\phi(f_n) \rightarrow \phi(\bar{f})$ for every ϕ in K^{**} and hence for any element of E^{**} . An application of Lemma 2 implies that f_n converges strongly to $\bar{f}$. *Q.E.D.*

Two simple applications, both of which are known, will now be deduced in order to illustrate the flexibility of these results. Let μ_n denote a sequence of regular measures defined over a Borel field of sets $\mathfrak{F}$ contained in a metric space. The measures μ_n are assumed to represent elements of the normed linear space consisting of all bounded sign measures. This Banach space is a special case of an abstract L space which is known to be weakly complete [10]. Here the cone K is usually identified with the set of all non-negative measures.

Suppose $\mu_n(A) \leq \mu_{n+1}(A)$ and $\sup_n \mu_n(A) < \infty$ for every set A in $\mathfrak{F}$. Applying Corollary 1, we conclude that $\mu_n(A)$ converges uniformly for A in $\mathfrak{F}$ to a regular measure $\mu_0(A)$.

A particular instance of this result is the fundamental monotone convergence theorem of Lebesgue integration.

Our second example deals with an increasing sequence of bounded self-adjoint operators defined on a Hilbert space H . Specifically, let A_i denote a countable family of self-adjoint operators on Hilbert space such that, for each i , $A_{i+1} - A_i$ is positive definite. Suppose also that $\sup_i (A_i \xi, \xi) < \infty$ for each ξ in H . The techniques of Lemma 2 can easily be adapted in order to prove that A_i converges in norm to a self-adjoint operator A .

3. Positive Operators. A bounded linear operator T is said to be positive if T transforms K into itself. In symbols we write $T > 0$ to indicate a non-trivial positive transformation T . It is of interest to relate the properties of positivity and boundedness. The simplest case is the assertion that positive linear functionals defined on a Banach space possessing a cone K with an interior are automatically bounded (continuous). We now point out some analogous results for positive operators.

It will be assumed, as before, that for any real f in E^* there exists a representation $f = f_1 - f_2$ with f_i in K^* and $\|f_i\| \leq C$ whenever $\|f\| \leq 1$. The corresponding assumption for E is required in Theorem 3.

Theorem 2. *If T is a positive linear mapping defined everywhere and K possesses an interior element, then T is continuous.*

Proof. Let u denote an interior element of K . There exists a real number $\rho > 0$ such that $u \pm \rho x/\|x\|$ belongs to K for all $x \neq 0$. Consequently $\|x\|/\rho Tu \geq \pm Tx$. For any f in K^* with $\|f\| \leq 1$, we conclude that $\|x\|/\rho f(Tu) \geq$

$|f(Tx)|$. Therefore for any f with $\|f\| = 1$ we find $(2C \|x\|/\rho) \|Tu\| \geq |f(Tx)|$. This yields $M\|x\| \geq \|Tx\|$, as was to be proved.

Another result in this direction is the following:

Theorem 3. *Let E possess the property that, for any $x, y \geq 0$, $\|x + y\| = \|x\| + \|y\|$. If T is a positive linear mapping defined everywhere, then T is bounded.*

Proof. If we suppose the contrary, then for each n there exists an element x_n in K such that

$$\|x_n\| = 1 \quad \text{with} \quad \|Tx_n\| \geq n.$$

Let

$$y = \sum_{r=1}^{\infty} \frac{x_r}{r^2}.$$

Note that $y \geq \sum_{r=1}^n x_r/r^2$. The hypothesis evidently implies that if $0 \leq x \leq y$ then $\|y\| \geq \|x\|$, and thus

$$\|Ty\| \geq \left\| \sum_{r=1}^n \frac{Tx_r}{r^2} \right\| = \sum_{r=1}^n \frac{\|Tx_r\|}{r^2} \geq \sum_{r=1}^n \frac{1}{r}.$$

Hence $\|Ty\| = \infty$, which is impossible.

Theorem 2 applies to linear transformations defined on Banach spaces of the form $C(Q) = \{\text{collection of all continuous functions on a compact Hausdorff space } Q\}$. On the other hand, Theorem 3 is useful when the Banach space E is an abstract Banach space of type (L) (essentially a space of signed measures) with the usual norm imposed on measures.

4. Spectral Properties of Positive Operators. In this section we investigate the structure of the spectrum and the nature of the eigenvalues possessed by positive operators. As stated in the introduction, our object is to extend the classical Frobenius theorems about matrices with positive elements to the case of general linear bounded positive transformations. It is interesting to note that most of the eigenvalue properties of these types of operators will be deduced by applying appropriate theorems concerning power series with non-negative coefficients. Specifically, the relevant power series theorems to be exploited are the following:

Theorem A (HARDY & LITTLEWOOD). *Let $f(\lambda) = \sum_1^{\infty} a_n \lambda^n$ have a radius of convergence 1 and $a_n \geq 0$. If, furthermore, $f(\lambda) \sim a/(1 - \lambda)^k$ as $\lambda \rightarrow 1-$, then*

$$\frac{\Gamma(k+1)}{n^k} \sum_{i=1}^n a_i \rightarrow a.$$

Theorem B. *Let $f(\lambda) = \sum_1^{\infty} a_n \lambda^n$ have a radius of convergence 1. If $\sum_1^n a_i \sim cn^k$, then $(1 - \lambda)^k f(\lambda) \rightarrow c \Gamma(k+1)$ as $\lambda \rightarrow 1-$.*

Theorem B is an Abelian theorem while Theorem A is a Tauberian theorem [16]. The importance for us is that Theorems A and B persist even when a_i , a ,

and c are replaced by positive elements of a Banach space, provided we substitute the asymptotic relations by norm-convergence in the sense of the Banach space. The proofs for the abstract versions run parallel to the complex variable case. For example, in Theorem A the method of KARAMATA [16, p. 227] may be suitably adapted in order to establish the proof of the vector-valued analogue.

The resolvent $R(\lambda, T)$ of an operator T is defined to be the bounded linear operator $(\lambda I - T)^{-1}$ wherever it exists. The set of λ where $R(\lambda, T)$ exists is called the resolvent set $\rho(T)$. The set of λ where $(\lambda I - T)x = 0$ for some $x \neq 0$ is referred to as the point spectrum $P\sigma(T)$. The set where $(\lambda I - T)^{-1}$ exists with domain non-dense is known as the residual spectrum $R\sigma(T)$. And, finally, the set where $(\lambda I - T)^{-1}$ exists as an unbounded operator with dense domain is called the continuous spectrum $C\sigma(T)$. The complement of $\rho(T)$ is denoted by $\sigma(T)$. $R(\lambda, T)$ represents an abstract analytic function in $\rho(T)$ and satisfies the functional equation

$$(\lambda - \mu)R(\lambda, T)R(\mu, T) = R(\mu, T) - R(\lambda, T).$$

Finally, $\sigma(T)$ is closed and bounded. (For further discussion of these ideas see HILLE & PHILLIPS [9, Chap. 51].)

Let T be an arbitrary bounded operator. Then $\lim_{n \rightarrow \infty} \|T^n\|^{1/n} = \max_{\lambda \in \sigma(T)} |\lambda|$ = the spectral radius of T , which will be denoted by $r(T)$.

Whenever $r(T) > 0$ we shall normalize the operator T so that $r(T) = 1$. The special analysis required to treat the case where $r(T) = 0$ is left to the interested reader. Henceforth, unless stated to the contrary, we assume $r(T) = 1$. This normalization does not imply that $\|T\| = 1$, but only that

$$1 = r(T) = \lim \|T^n\|^{1/n} \leq \|T\|.$$

We assume in the remainder of this section that K and K^* span E and E^* , respectively. This restriction is considerably mild and satisfied by the vast majority of Banach spaces possessing a natural distinguished cone (see also Section 1).

The next theorem represents the full generalization of the first Frobenius result. It asserts that $r(T)$, the spectral radius, belongs to the spectrum (not necessarily point spectrum) of a positive operator. The proof depends on the classical theorem of Pringsheim concerning power series with positive coefficients, which states that the real positive point on the circle of convergence is a singular point of the analytic function which the power series represents.

Theorem 4. *If $T > 0$, then 1 is in $\sigma(T)$.*

Proof. If we invert in the unit circle, we must show that $(I - \lambda T)^{-1} = \sum_{n=0}^{\infty} \lambda^n T^n$ has a singularity at the point 1 (the radius of convergence being equal to 1). If we assume the contrary, then $\lambda^{-1}R(\lambda^{-1}, T)$ is analytic in a neighborhood of 1 ($|\lambda - 1| < \delta$), whence $\lambda^{-1}(f, R(\lambda^{-1}, T)x)$ for $x \in K$, and $f \in K^*$ is an ordinary analytic function when $|\lambda - 1| < \delta$ with expansion about the origin given by

$$\sum \lambda^n (T^n x, f) = \sum \lambda^n a_n, \quad a_n \geq 0.$$

By virtue of Pringsheim's Theorem concerning complex power series with non-negative coefficients [16, p. 214], $\lambda^{-1}(f, R(\lambda^{-1}, T)x)$ is analytic for $|\lambda| \leq 1 + \delta$.

Since K and K^* span E and E^* respectively, and since weak analyticity is equivalent to uniform analyticity (see HILLE & PHILLIPS, pp. 93), we infer that $\lambda^{-1}R(\lambda^{-1}, T)$ can be extended analytically throughout $|\lambda| \leq 1 + \delta$. Since $R(\lambda, T)$ satisfies the resolvent equation, it follows that $(\lambda I - T)^{-1}$ exists wherever $R(\lambda, T)$ is analytic. This contradicts the assumption that $r(T) = 1$.

Our next theorem establishes the existence of eigenvectors belonging to K and proves an ergodic theorem for a large class of positive operators which embraces, particularly, quasi completely continuous mappings. An operator T is said to be quasi completely continuous if there exists an integer k_0 and a completely continuous operator V such that

$$\|T^{k_0} - V\| < 1$$

(see [19] for an enumeration of examples of quasi completely continuous operators).

Theorem 5. *If 1 is a pole of order k of the resolvent of T , then*

$$\frac{\Gamma(k+1)}{n^k} \sum_{i=1}^n T^i \rightarrow (T - I)^{k-1}P$$

where P is a projection operator. (Here convergence is understood to be in the uniform topology.) The point 1 belongs to the $P\sigma(T)$, and there exist, associated with 1, non-zero elements of K which are eigenvectors. Furthermore, there exists an element $f \neq 0$ in K^* for which $T^*f = f$.

Proof. By the hypothesis, $R(\lambda, T)$ has an expansion of the form $R(\lambda, T) = \sum_{n=-k}^{\infty} a_n(\lambda - 1)^n$ with

$$a_n = \frac{1}{2\pi i} \int_c \frac{R(\xi, T) d\xi}{(\xi - 1)^{n+1}},$$

integrated about a small circle enclosing the point $\lambda = 1$. The operator calculus implies that

$$a_{-l} = (T - I)^{l-1}P$$

where $P = (1/2\pi i) \int_c R(\xi, T) d\xi$ is a projection operator [9, p. 206]. The above expansion clearly implies that $(\lambda - 1)^k R(\lambda, T) \rightarrow a_{-k}$. Since $\mu^{-1}R(1/\mu, T) = \sum \mu^n T^n$ is valid for $|\mu| < 1$, with $T^n \geq 0$ and

$$(1 - \mu)^k \frac{1}{\mu} R\left(\frac{1}{\mu}, T\right) = (1 - \mu)^k \sum_{n=0}^{\infty} \mu^n T^n \rightarrow a_{-k},$$

Theorem A may be applied to yield

$$\frac{\Gamma(k+1)}{n^k} \sum_{i=1}^n T^i \rightarrow a_{-k} = (T - I)^{k-1}P.$$

In view of $(T - I)^{k-1} P \neq 0$, there exists an element x in K (as K spans E) with $y = (T - I)^{k-1} Px \neq 0$ and $(T - I)y = 0$, the last relation being true because $(T - I)^k P = a_{-(k+1)} = 0$. Since

$$\frac{\Gamma(k+1)}{n^k} \sum_{i=1}^n T^i x$$

belong to K and converge to y , we deduce that y is in K . A similar argument shows that

$$\frac{\Gamma(k+1)}{n^k} \sum_{i=1}^n T^{*i} \rightarrow (T^* - I)^{k-1} P^*$$

where

$$P^* = \frac{1}{2\pi i} \int_C R(\xi, T^*) d\xi,$$

and thus an $f \neq 0$ in K^* exists with $T^*f = f$. This completes the proof of the theorem.

Corollary 1. *If $T > 0$ is quasi completely continuous with $r(T) > 0$ then amongst the largest eigenvalues in modulus is a positive real number, and its characteristic manifold contains non-zero elements of K .*

This follows by Theorem 4 and Theorem 5 and the fact that the spectrum of T lying on the periphery of the circle of radius equal to the spectral radius consists of point spectrum with each point of $\sigma(T)$ a pole of $R(\lambda, T)$ [19].

On the basis of the hypothesis of Theorem 5 an estimate of the rate of growth of T^n can be secured. In fact, since

$$\frac{\Gamma(k+1)}{n^k} \sum_{i=1}^n T^i \rightarrow (T - I)^{k-1} P,$$

we get that

$$(I - T) \frac{\Gamma(k+1)}{n^k} \sum_{i=1}^n T^i = \left(\frac{T}{n^k} - \frac{T^{n+1}}{n^k} \right) \Gamma(k+1) \rightarrow (T - I)^k P = 0,$$

whence $T^n/n^k \rightarrow 0$, which is the desired rate of growth.

Following a method of DUNFORD, a converse of Theorem 5 can also be achieved by appealing to Theorem B.

Theorem 6. *If $(1/n) \sum_{i=1}^n T^i \rightarrow P$ where P is non-zero, then P is a projection and 1 is a simple pole of $R(\lambda, T)$.*

Proof. The hypothesis implies with the aid of an application of Theorem B that $(1 - \lambda) R(\lambda^{-1}, T) \rightarrow P$ as $\lambda \rightarrow 1^-$. Let $(1 - \lambda) R(\lambda^{-1}, T) = S(\lambda)$. Using the resolvent equation, we get

$$(1 - \mu)S(\lambda) - (1 - \lambda)S(\mu) = (\lambda - \mu)S(\mu)S(\lambda) \frac{1}{\lambda\mu}.$$

Letting $\lambda \rightarrow 1$ leads to

$$(1 - \mu)P = (1 - \mu) \frac{S(\mu)P}{\mu} \quad \text{or} \quad \frac{S(\mu)P}{\mu} = P.$$

Hence if $\mu \rightarrow 1$, it follows that $P^2 = P$. P is bounded as a consequence of the principle of uniform boundedness. Thus P is a projection. By virtue of a result of DUNFORD [6, Theorem 3.6], we deduce that 1 is either in $\rho(T)$ or else a pole of order one of $R(\lambda, T)$. Theorem 4 eliminates the first possibility. *Q.E.D.*

The following existence theorem for eigenvalues of T^* should be compared with that of Theorem 5.

Theorem 7. *Let K be a cone with an interior and suppose K^* spans E^* . If $T > 0$, then $r(T) = 1$ belongs to the point spectrum of T^* and there exist eigenelements in K^* corresponding to the eigenvalue $r(T)$.*

Remark. Since K has an interior, K spans E (see Section 1). As usual, let u denote an interior point of K and let H_u be the section of K^* defined by the condition: f is in H_u if and only if $(f, u) = 1$. We repeat: The hypothesis that K^* spans E^* is considerably weak and is fulfilled, for example, whenever the set of x satisfying $-u \leq x \leq u$ is uniformly bounded in norm.

Proof. We shall first prove that there exists a sequence f_n in H_u such that

$$(1) \quad \|T^*f_n - f_n\| \rightarrow 0.$$

To this end, set $\lambda_n = 1 + 1/n$ and observe that $(T^* - \lambda_n I)^{-1}$ exist but are not uniformly bounded in norm. Otherwise $(T^* - I)^{-1}$ would exist, which is impossible by virtue of Theorem 4. By the principle of uniform boundedness and the fact that K^* spans E^* we secure an element g of K^* for which $\|(T^* - \lambda_n I)^{-1}g\| \rightarrow \infty$.

Let

$$\tilde{f}_n = \frac{(T^* - \lambda_n I)^{-1}g}{\|(T^* - \lambda_n I)^{-1}g\|}.$$

Since $\lambda_n > 1$ it follows that $\tilde{f}_n$ are in K^* . (The formula $(T - \lambda_n I)^{-1} = \sum_{r=0}^{\infty} T^r / \lambda_n^{r+1}$ makes this obvious.) Furthermore,

$$T^*\tilde{f}_n - \tilde{f}_n = \frac{g}{\|(T^* - \lambda_n I)^{-1}g\|} + (\lambda_n - 1)\tilde{f}_n \rightarrow 0.$$

If we normalize $\tilde{f}_n$ to f_n in H_u the uniform boundedness is preserved and the assertion (1) is proved.

Let f^* denote a weak* limit point of f_n in H_u . It follows from (1) that $T^*f^* = f^*$ as was to be shown.

Corollary [12]. *Let E be reflexive and both K and K^* possess interior elements. If $T > 0$ with spectral radius 1, then $Tx = x$ for some $x \neq 0$ in K and $T^*f = f$ for some $f \neq 0$ in K^* .*

The following three simple examples are typical and illustrate the flexibility and scope of the preceding results.

Example 1. Let (D) represent the Banach space of all bounded functions defined on the unit interval possessing discontinuities only of the first kind. The norm of a function is the supremum of its absolute values. The cone K of (D) is here identified with the collection of all non-negative functions in (D) .

Set, for x in D ,

$$Tx(t) = \bar{x}(t - a) + \int_0^1 K(t, s)x(s) ds$$

where $K(t, s)$ is jointly continuous and strictly positive in both variables for $0 \leq t, s \leq 1$ and

$$\bar{x}(t - a) = \begin{cases} 0 & \text{for } 0 < t \leq a \\ x(t - a) & \text{for } a < t < 1 \end{cases} \quad (a > 0).$$

This equation arises in connection with certain models of branching processes and was suggested by BELLMAN & HARRIS.

The part of the operator involving the integral kernel is obviously completely continuous, and because $a > 0$ there exists an iterate T^n such that T^n is completely continuous. It is easy to verify (since $K(t, s)$ is strictly positive, see Theorem 17 below) that the spectral radius of T is positive. Hence, appealing to Corollary 1 of Theorem 5, we deduce the existence of a non-trivial eigenfunction for which $Tx_0 = \lambda_0 x_0$ with $x_0 \gg 0$ and $\lambda_0 > 0$. Moreover, it can be proved with the aid of Theorem 12 that T^n/λ_0^n converges in norm to the projection operator which maps all of (D) into the one-dimensional linear space spanned by x_0 .

The numerous extensions of the same sorts of operators to which the preceding results apply are clear.

Example 2. Let $K(t)$ denote a positive even continuous function defined for $-\infty < t < \infty$, and suppose $K'(t) < 0$ for $t > 0$. Consider the Banach space J composed of all continuous functions $x(t)$ defined for $0 \leq t \leq a$ ($a > 0$), each of which is even (symmetric) with respect to the point $\frac{1}{2}a$. The usual norm is imposed on J . We examine the positive completely continuous transformation

$$Tx(t) = \int_0^a K(t - s)x(s) ds$$

which maps J into itself, as the following representation clearly shows:

$$Tx(t) = \int_0^{\frac{1}{2}a} [K(t - s) + K(a - t - s)]x(s) ds.$$

We construct the cone K composed of all functions x in J for which $x(t) \geq 0$ on $0 \leq t \leq a$ and where $x(t)$ is non-decreasing when $0 \leq t \leq \frac{1}{2}a$. A simple calcu-

lation shows that T transforms K into itself. If we contract J to the space spanned by K and consider T acting on this smaller Banach space, then we may turn to Theorem 5 and secure the existence of a non-trivial eigenfunction of K . Specifically, we obtain the existence of a positive continuous function $x_0(t)$ defined on $[0, a]$, non-decreasing for $0 \leq t \leq \frac{1}{2}a$ and even with respect to the point $\frac{1}{2}a$ satisfying

$$Tx_0 = \lambda_0 x_0 \quad (\lambda_0 > 0).$$

Example 3. An important source of positivity-preserving transformations occurs in the study of Markoff processes. As typical we discuss a special matrix transformation $P = ||p_{ij}||$, $i, j = 0, 1, \dots$, defined on the Banach space (l) (absolutely convergent series) into itself, to which the above theory applies. Let

$$p_{ij} = \begin{cases} p_i & i = 0; \quad j = 0, 1, 2, \dots \\ q_i & j = i + 1 \\ 0 & \text{elsewhere} \end{cases}$$

where $p_i + q_i = 1$ and $0 < \alpha \leq q_i \leq 1 - \alpha < 1$. In order to determine the spectrum of P , we try to invert the resolvent equation

$$(I - \lambda P)x = y$$

where $y = (y_0, y_1, y_2, \dots)$ is an element of (l) . The solution of this equation system is easy and we get

$$(*) \quad x_n = \sum_{i=0}^{n-1} \lambda^i \left(\prod_{j=1}^i q_{n-j} \right) y_{n-i} + \lambda^n \left(\prod_{j=1}^n q_{n-j} \right) x_0, \quad n = 1, 2, \dots,$$

and

$$(**) \quad x_0 - \lambda \sum_{i=0}^{\infty} p_i x_i = y_0.$$

The hypothesis on q_i readily implies that x_n is a convergent series for $|\lambda| < 1/(1 - \alpha)$, provided x_0 can be determined from (**). Inserting the values of x_n into (**) we see that x_0 is calculable provided $|\lambda| < 1/(1 - \alpha)$ and $\lambda \neq 1$. The coefficient of x_0 in (**) reduces to $1 - \lambda (1 - \prod_{i=0}^{\infty} q_i) = 1 - \lambda$, which is zero only when $\lambda = 1$.

To sum up, the spectrum of P lies inside the circle of radius $1 - \alpha$ except for an isolated simple pole located at $\lambda = 1$. That 1 is in the spectrum is clear since the column sums of p_{ij} are identically one. The results of Theorems 5 and 12 are therefore applicable to the matrix P , and in particular we have the existence of a positive non-trivial eigenfunction in (l) for which the uniform ergodic theorem (Theorem 12) holds.

5. Strictly Positive Operators. An operator T is said to be strictly positive ($T \gg 0$) if T maps K into the interior of K . In this section we develop some of

the more detailed properties of the spectrum of strictly positive operators. Some of our results represent extensions of analogous results obtained by KREIN & RUTMAN for strictly positive completely continuous operators. Our proofs are simpler even for those cases. Throughout this section we assume $T \gg 0$ unless stated explicitly to the contrary. The same normalization as imposed previously on T still holds. Particularly, $r(T) = 1$. We remark in passing that *all* the results developed in this section prevail for operators which fulfill the following weaker condition: For each x in K there exists an $n(x)$ such that $T^n x \gg 0$. Such an operator will be called quasi strictly positive ($T \ggg 0$). For clarity of exposition we carry through the proofs mostly for the case $n = 1$.

Lemma 3. *If $Tx > \lambda x$, $\lambda > 0$, then there exists a z in K and an $\epsilon > 0$ with $Tz \geq (\lambda + \epsilon)z$.*

Proof. Since $Tx > \lambda x$, we secure $Tz \gg \lambda z$ with $z = Tx$, and the conclusion follows.

Theorem 8. *If $Tx \geq x$ for $x > 0$, then $Tx = x$.*

Proof. If we suppose the contrary, then by Lemma 3 $Tz \geq (1 + \epsilon)z$ which implies $\lim ||T^n||^{1/n} \geq 1 + \epsilon$, contradicting the fact that $r(T) = 1$.

Definition. The Banach space E is said to possess an absolute value if for each x in E an element $|x|$ in K is defined which possesses the following properties:

- (a) If $x \neq 0$, then $|x| \neq 0$.
- (b) If x is in K , then $|x| = x$.
- (c) If α is a complex number, then $|\alpha x| = |\alpha||x|$.
- (d) If T is any positive operator, then $|Tx| \leq T|x|$.

Corollary. *If E possesses an absolute value and $Tx = \lambda x$ for some λ where $|\lambda| = 1$ with $x \neq 0$, then 1 is in the point spectrum.*

Proof of the corollary. $Tx = \lambda x$ implies that $T|x| \geq |x|$, and the conclusion follows from Theorem 8.

The identical result to that of Theorem 8 is obtained if the assumption of strict positivity of T is discarded and in its place we substitute the hypothesis that f_0 exists satisfying $(f_0, x) > 0$ for every $x \neq 0$ in K and $T^*f_0 = f_0$.

This corollary supports the anticipated conclusion that for $T > 0$ the worst singularity of $R(\lambda) = (I - \lambda T)^{-1} = \sum \lambda^n T^n$ on the circle of convergence lies at 1. The sense of this last statement is related to the idea that the rate of growth of $(R(\lambda)x, f)$, for x in K and f in K^* , is at least as large when $\lambda \rightarrow 1$ as the growth of this function when λ approaches any other point of the unit circle.

It is sometimes possible to establish the truth of Theorem 8 even when T is not assumed to be strictly positive. We noted above one such condition expressed in terms of the existence of an eigenvector f_0 ($T^*f_0 = f_0$) where f_0 is sufficiently positive. Another result of this kind deals with positive operators in Hilbert spaces.

Let $\mathcal{H}$ be a separable Hilbert space and suppose K is a cone satisfying $K^* = K$ (see Section 1 for examples of this situation). Here K need not possess an interior. Let A be a bounded self-adjoint operator defined over $\mathcal{H}$ satisfying $A(K) \subset K$. We now prove that if $\|A\| = 1$ and $Ax \geq x$ for x in K then $Ax = x$. To this end, observe that $Ax \geq x$ implies $(Ax, x) \geq (x, x)$. On the other hand, $\|A\| = 1$ gives $(Ax, x) \leq (x, x)$ which requires that $(Ax, x) = (x, x)$. By a standard variational argument, since

$$\sup_{y \neq 0} \frac{(Ay, y)}{(y, y)} = 1$$

we deduce $Ax = x$.

This result may be extended to the case where A is assumed to be an arbitrary normal operator carrying K into itself.

If A is a bounded normal operator with $\|A\| = 1$ and $A(K) \subset K$, then $Ax \geq x$ for x in K implies $Ax = x$.

Proof. Since $Ax \geq x$, we have $(Ax, x) \geq (x, x)$. Also $(Ax, Ax) \geq (x, Ax) \geq (x, x)$ or $(A^*Ax, x) \geq (x, x)$. But since $\|A^*A\| = \|A\|^2 = 1$ it follows that $(A^*Ax, x) = (x, x)$. By iteration we achieve $(A^i x, A^i x) = (x, x)$ for every i . Moreover, as $x \leq Ax \leq A^2 x \leq \dots \leq A^i x$ and $\|A^i x\| = \|x\|$, we deduce from Theorem 1 that $A^i x \rightarrow y$ and $Ay = y$.

$$\begin{aligned} (x - y, x - y) &= \lim_{i \rightarrow \infty} (x - A^i x, x - A^i x) = (x, x) - 2 \lim_{i \rightarrow \infty} (A^i x, x) \\ &\quad + \lim_{i \rightarrow \infty} (A^i x, A^i x) = (x, x) - 2(x, x) + (x, x) = 0. \end{aligned}$$

Hence $x = y$ and $Ax = x$.

Other extensions of Theorem 8 will be indicated in Section 6. We now return to our discussion of the structure of the eigenvalues and eigenvectors of strictly positive mappings. In Theorem 9 we follow an argument of KREIN & RUTMAN which is given in detail in order to preserve completeness.

Theorem 9. *If 1 is an eigenvalue, then it is simple (i.e., the eigenmanifold associated with 1 is one-dimensional).*

Proof. If $x \gg 0$ and y are real linearly independent eigenvectors, then $x + \delta y$ is also an eigenvector for any scalar δ . There exists a δ such that $x + \delta y$ is on the boundary of K since x is interior to K . But $x + \delta y = T(x + \delta y) \gg 0$, which is a contradiction. Thus there exists a one-dimensional manifold of real eigenvectors. The case for complex eigenvectors follows immediately, since the real and imaginary parts are separately eigenvectors.

The index of the elementary divisor associated with a point λ of $P\sigma(T)$ is defined as the smallest integer v for which

$$\mathfrak{N}_v = [x \mid (T - \lambda I)^v x = 0] = \mathfrak{N}_{v+1} = [x \mid (T - \lambda I)^{v+1} x = 0].$$

Theorem 10. *The index of the elementary divisor at 1 is 1.*

Proof. Suppose $(T - I)^2 y = 0$ with y real and $(T - I)y = z \neq 0$. We may assume without loss of generality that $z \gg 0$ since, by Theorem 9, $z = ax_0$. Again, without restricting any generality take $a = 1$. Thus $Ty - y = x_0$, and hence $T^2 y - Ty = x_0, \dots, T^n y - T^{n-1} y = x_0$. Adding and dividing by n , $T^n y/n = x_0 + y/n$, which for n sufficiently large is $\gg 0$. Hence $z = T^n y \gg 0$ and $T(T^n y) - T^n y = x_0$, which is $Tz - z = x_0$ with $z \gg 0$. Thus $T(x_0 + z) \gg x_0 + z$, contradicting Theorem 8.

Remark. The existence of an $f > 0$ with $T^*f = f$ establishes Theorem 10 in a trivial manner. Explicitly, if $(T - I)y = x$, $x \gg 0$, then $0 < (f, x) = (f, (T - I)y) = ((T^* - I)f, y) = 0$. We already mentioned that a similar accounting works in connection with Theorem 8, provided it has been demonstrated that there exists a solution f of $T^*f = f$ satisfying $(f, x) > 0$ for $x \neq 0$ in K . KREIN & RUTMAN achieved this stronger result in the case of completely continuous operators by appealing to the Brouwer fixed point theorem.

Theorem 11. *If 1 is a pole of $R(\lambda, T)$, then the pole is simple and $(1/n) \sum_{i=1}^n T^i$ converges to a projection P .*

This is an immediate application of Theorems 10, 9, and 5, and the fact that the degree of the pole is equal to the dimension of $\mathfrak{N}_v$, where v is the index of the elementary divisor associated with 1. In fact, by Theorem 10 $\mathfrak{N}_v = \mathfrak{N}_1$, and according to Theorem 9 $\dim \mathfrak{N}_1 = 1$. It remains to assert the conclusions of Theorem 5.

If it is known that the spectrum of T on the unit circle is composed of isolated points and that each of these spectral points represents a pole for $R(\lambda, T)$ which has a finite dimensional characteristic manifold, then we can demonstrate the further result that, in fact, there exists a unique eigenvalue for T of modulus 1, which is necessarily real and positive. More precisely, suppose T has a representation $T = T_1 + Q$ where T_1 is the contraction of T to the characteristic manifold $\mathfrak{N}$ spanned by the eigenvectors corresponding to the eigenvalues of modulus 1. According to the hypothesis, $\mathfrak{N}$ is finite dimensional. Since $\mathfrak{N}$ is invariant under T we may view T_1 as a transformation of $\mathfrak{N}$ into itself. Let $K_1 = K \cap \mathfrak{N}$ represent the contraction of the cone K to $\mathfrak{N}$. K_1 is a cone with an interior since the characteristic manifold of 1 contains interior points of K . Moreover, T_1 is a strictly positive transformation defined on $\mathfrak{N}$ into itself.

We now prove that a strictly positive transformation T_1 on a finite dimensional space with $\sigma(T_1)$ located on the unit circle consists of only the point 1. Indeed, since $\mathfrak{N}$ is finite dimensional and T_1 is strictly positive, $T_1 - \delta I$ for $\delta > 0$ and sufficiently small remains a positive operator on $\mathfrak{N}$. But $T_1 - \delta I = T_2$ translates the spectrum of T_1 by δ , and hence the largest real spectral point of $T_2 > 0$ is $1 - \delta$. Then $\lambda \in \sigma(T_1)$ and $|\lambda| = 1$ implies that $\lambda - \delta$ belongs to the spectrum of T_2 , verifying $|\lambda - \delta| > 1 - \delta$ and thus contradicting Theorem 4. Consequently, there exists no spectral point λ of modulus 1 except $\lambda = 1$. Summing up, we have established

Theorem 12. Let T ($\gg 0$) have no spectral points of modulus one except for a finite number of eigenvalues, one of which is $1 = r(T)$. Let each of these eigenvalues possess a finite number of independent eigenvectors. Let $\mathfrak{M}$ be the linear space spanned by these eigenvectors. If $T = T_1 + Q$ where T_1 is the contradiction of T to $\mathfrak{M}$, then T_1 has only the spectral value 1 (i.e., $\sigma(T_1)$ consists of the single value $\lambda = 1$).

If T is quasi completely continuous, then a decomposition of the form described in the hypothesis of Theorem 12 exists [19]. Hence

Corollary. If $T \gg 0$ is quasi completely continuous, then 1 is the only eigenvalue of modulus 1.

Subject to the conditions of the corollary, we let P denote the projection of E on the one-dimensional characteristic manifold of the eigenvalue 1. Then $T = P + R$ where the spectral radius of R is less than 1 and $P \cdot R = R \cdot P = 0$ with $P^2 = P$. By iteration, we obtain $T^n = P + R^n$. But $\|R^n\|^{1/n} < r < 1$ for $n \geq n_0$, and hence $\|R^n\| < r^n$, from which follows $\|T^n - P\| \leq r^n$ ($r < 1$). This conclusion is a considerable sharpening of the assertion of Theorem 11, applicable for example, when T is assumed to be quasi completely continuous.

A converse to propositions 8–11 can be obtained.

Theorem 13. If $T > 0$ and satisfies the properties (a) 1 is a pole of T , (b) the characteristic manifold on 1 is one-dimensional spanned by the eigenvector x_0 ($x_0 \gg 0$), (c) there exists $f_0 \in K$ such that $T^*f_0 = f_0$ with $(f_0, x) > 0$ for every $x \neq 0$ in K , then for each non-zero $x \in K$ there exists an integer n depending upon x such that

$$\frac{T + T^2 + \cdots + T^n}{n} x \gg 0.$$

Proof. From the remark following Theorem 10 and (b) and (c) we conclude that 1 is a simple pole, whence, by Theorem 5, $(T + T^2 + \cdots + T^n)/n \rightarrow P$. Now, the Banach space E may be decomposed as a direct sum in a standard fashion such that $E = R + S$ where R is the linear manifold, the range of P , and S the linear manifold, the zeros of P . The set S can also be characterized as the closure of the collection of elements $z = y - Ty$ with y ranging arbitrarily in E . We assert that, for x non-zero in K , $Px \gg 0$. In fact, $x = ax_0 + z$ with z in S . It is enough to show that $a > 0$. If $a = 0$, then $x = z = \lim (Ty_n - y_n)$. But $0 < (f_0, x) = \lim (f_0, Ty_n - y_n) = \lim (T^*f_0 - f_0, y_n) = 0$. Since $(1/n) \sum_1^n T^i x \rightarrow ax_0$, we conclude that $a > 0$.

If in Theorem 13 one assumes that T is completely continuous and the space E is reflexive, then one can secure the choice of n independent of x . This will follow from Lemma 4.

Lemma 4. If $T_n \rightarrow S$ (convergence is with respect to the uniform topology), $S \gg 0$ and completely continuous, and E is reflexive, then for all $n \geq n_0$, $T_n \gg 0$.

It is assumed here that both E and E^* contain interior points u and f_0 respectively.

Proof. If we suppose the contrary, then there exists a sequence x_n, f_n with $f_n \in H_u, x_n \in H_{f_0}$, such that $(T_n x_n, f_n) \leq 0$. A subsequence can be selected so that $x_n \rightharpoonup x, f_n \rightharpoonup f$ with $x, f \in K, K^*$ respectively and each non-zero. As S is completely continuous, we deduce that $(Sx_n, f_n) \rightarrow (Sx, f) \geq \delta > 0$. On the other hand, $|(T_n x_n, f_n) - (Sx_n, f_n)| \rightarrow 0$ and we infer a contradiction.

Theorem 13 cannot be improved as one can see by examining simple examples in finite dimensional spaces, e.g., let

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

then $T^{2n+1} = T$ and $T^{2n} = I$ so that no power of T is strictly positive.

As a simple application of this theory we discuss the following problem. If two linear bounded transformations A and B commute, then it is trivial to show that $r(AB) \leq r(A) r(B)$, and, in fact, inequality frequently occurs. We shall describe a situation such that A and B are positive mappings and equality holds. Let A and B denote two bounded linear operators where A is strictly positive and B is merely positive, possessing a positive spectral radius. Suppose, furthermore, that $BA = AB$ and that 1 belongs to the point spectrum of A with a positive eigenvector x_0 ($Ax_0 = x_0$). Since $A \gg 0$ we must have $x_0 \gg 0$. Also $ABx_0 = BAx_0 = Bx_0$. Moreover, Bx_0 cannot be zero, for otherwise the spectral radius of B is zero (see the proof of Theorem 17). Therefore Bx_0 belongs to the characteristic manifold of A associated with the eigenvalue 1. By Theorem 9 we infer $Bx_0 = \mu x_0$ for some $\mu > 0$. Again the method of Theorem 17 shows that $\mu = r(B)$ and $\mu = r(AB)$, as was to be proved.

6. Characterization of $r(T)$. Let T be any positive operator and assume that K has an interior point u . We also assume throughout this section that K^* spans E^* in such a way that a representation for each $f = f_1 - f_2$ of norm 1 can be found with $\|f_i\| \leq C$. The last condition can equivalently be expressed in terms of the condition: If $0 \leq x \leq y$ then $\|x\| \leq \|y\|$.

Definition 1. Let $\bar{\Omega}_s(T)$ be the set of λ for which there exists a non-zero x in K with $Tx \geq \lambda x$. Put $\bar{\lambda}_s(T) = \sup \lambda$ with λ ranging in $\bar{\Omega}_s(T)$. It is easy to see that $\bar{\lambda}_s(T) < \infty$. (The idea of the value $\bar{\lambda}_s(T)$ was suggested to the author by BOHNENBLUST in discussing the first Frobenius theorem in the finite dimensional matrix case.)

Definition 2. $\underline{\lambda}_s(T)$ $\underline{\Omega}_s(T)$ are defined analogously with inequality reversed and sup replaced by inf. Note that $\underline{\lambda}_s(T) \geq 0$.

Definition 3. Let $\bar{\Omega}_\omega(T) = \{\lambda \mid \text{for any } f_1, \dots, f_n \in H_u \text{ there exists a non-zero } x \text{ in } K \text{ with } (f_i, Tx) \geq \lambda(f_i, x)\}$. Set $\bar{\lambda}_\omega(T) = \sup_{\lambda \in \bar{\Omega}_\omega} \lambda$.

Definition 4. Let $\underline{\Omega}_\omega(T)$ and $\underline{\lambda}_\omega(T)$ be defined analogously where the inequalities in Definition 3 are reversed and inf substituted for sup. All the analogous Ω sets and λ values are defined for the operator T^* . Whenever the subscript for $\underline{\lambda}(T)$ and $\bar{\lambda}(T)$ is suppressed, we shall mean the strong values unless stated explicitly to the contrary.

Theorem 14. $\bar{\lambda}_s(T^*) = \bar{\lambda}_\omega(T^*)$, $\underline{\lambda}_s(T^*) = \underline{\lambda}_\omega(T^*)$.

Proof (on $\bar{\lambda}$). Obviously $\bar{\lambda}_s \leq \bar{\lambda}_\omega$. Let λ be in $\bar{\Omega}_\omega(T^*)$. Suppose there exists no f in H_u for which

$$(1) \quad T^*f \geq \lambda f.$$

Consider the collection of sets where

$$G(x) = [f \in H_u, (T^*f, x) < \lambda(f, x)]$$

for all possible x in K . The sets $G(x)$ are ω^* open and, because of the non-validity of (1), they cover the compact set H_u . Thus there exists a finite covering $G(x_{\alpha_1}), \dots, G(x_{\alpha_n})$. But the set of f in H_u for which

$$(T^*f, x_i) \geq \lambda(f, x_i), \quad i = 1, \dots, n,$$

is non-void by the very meaning of λ in $\bar{\Omega}_\omega(T^*)$, which is a contradiction. Thus $\lambda \in \bar{\Omega}_s$ and hence $\bar{\lambda}_s \geq \bar{\lambda}_\omega$.

Theorem 15. Generally, $\bar{\lambda}(T) \geq \underline{\lambda}(T^*)$, $\underline{\lambda}(T) \leq \bar{\lambda}(T^*)$. If $I + T \gg 0$ then $\underline{\lambda}(T) = \bar{\lambda}(T^*)$ and $\bar{\lambda}(T) = \underline{\lambda}(T^*)$.

Proof. We shall prove only the first relation, the second following from similar considerations. Let $\lambda = \bar{\lambda}(T) + \epsilon$. Then for no $x \neq 0$ in K is it true that $Tx - \lambda x \geq 0$. Thus, since $P = [Tx - \lambda x \mid x \in K]$ is a convex cone disjoint from K (a closed cone with an interior), there exists a non-zero f_0 which separates, or $(f_0, Tx - \lambda x) \leq c$, $(f_0, x) \geq c$ for all x in K . Since K is a cone, it follows that c can be taken as zero. Hence $f_0 \in K^*$. But then $(T^*f_0 - \lambda f_0, x) \leq 0$ for all x in K , and hence $T^*f_0 \leq \lambda f_0$. Thus λ is in $\Omega_s(T^*)$ or $\underline{\lambda}(T^*) \leq \bar{\lambda}(T) + \epsilon$. We may permit $\epsilon = 0$ since ϵ is arbitrary.

The second half of the theorem is proved as follows: By the definition of $\underline{\lambda}(T^*)$ there exists $f_0 \neq 0$ in K^* such that

$$(2) \quad T^*f_0 \leq \lambda f_0$$

when $\lambda = \underline{\lambda}(T^*) + \epsilon$. Also, for $\mu = \bar{\lambda}(T) - \epsilon$ we find $x_0 \neq 0$ in K satisfying

$$(3) \quad Tx_0 \geq \mu x_0.$$

But (2) and (3) together imply that

$$(4) \quad \lambda(f_0, x_0) \geq (T^*f_0, x_0) = (f_0, Tx_0) \geq \mu(f_0, x_0).$$

If $(f_0, x_0) = 0$, then also $(f_0, Tx_0) = 0$, and $I + T$ cannot be strictly positive as postulated. Therefore $(f_0, x_0) > 0$, and we may conclude from (4) that $\lambda \geq \mu$.

This readily yields

$$\underline{\lambda}(T^*) \geq \bar{\lambda}(T),$$

as desired.

The assertion of Theorem 15 cannot in general be improved upon. Examples of positive mappings in finite dimensional space can be constructed to show the occurrence of actual inequality for the relation

$$\underline{\lambda}(T) < \bar{\lambda}(T^*).$$

Theorem 16. *If $I + T \gg 0$, then $\underline{\lambda}_\omega(T) = \underline{\lambda}_s(T)$ and $\bar{\lambda}_\omega(T) = \bar{\lambda}_s(T)$. If E is reflexive, the hypothesis may be weakened to requiring merely that $T > 0$.*

Proof. The second half of the statement follows by Theorem 14. The proof of the first part is similar to that of Theorem 15 and will be omitted.

Our next result describes a basic relationship between the spectral radius of an operator T and the $\bar{\lambda}$, $\underline{\lambda}$ values. We shall see later how we can exploit this relationship in order to establish the existence of non-trivial positive eigenvectors.

Theorem 17. *If $T \gg 0$, then $r(T) = \underline{\lambda}(T) \geq \bar{\lambda}(T)$.*

Proof. Consider $\lambda = \bar{\lambda}(T^*) + \epsilon = \underline{\lambda}(T) + \epsilon$ in $\Omega(T)$. Hence $Tx \leq \lambda x$ for some non-zero x in K . As $T \gg 0$, it follows that $x \gg 0$. Let y be an arbitrary element $\|y\| = 1$. There exists a $\rho > 0$ with $x \pm \rho y \in K$. Thus

$$\lambda x \geq Tx \geq \pm \rho Ty,$$

which by iteration yields

$$\lambda^n x \geq \pm \rho T^n y.$$

Now, using the condition stated at the start of this section, we get

$$\|T^n y\| \leq |\lambda|^n \frac{\|x\|}{\rho} c'$$

for an appropriate constant c' . Since y is arbitrary of norm 1, we have $\lim \|T^n\|^{1/n} \leq |\lambda|$. Thus $r(T) \leq \bar{\lambda}(T^*)$. Hence

$$\underline{\lambda}(T) = \bar{\lambda}(T^*) \geq r(T^*) = r(T).$$

The opposite inequality is obvious since $\lambda x \leq Tx$ or $\lambda f \leq T^*f$ for either x or f non-zero, respectively, in K and K^* easily implies $\|T^n\|^{1/n} \geq \lambda$. Q.E.D.

It is not known whether, in this case, also $\bar{\lambda}(T) = r(T)$. In this connection the reader may consult Theorem 19.

We now attempt to weaken the hypothesis of Theorem 16 and seek to determine further relations between $\bar{\lambda}(T^*)$, $\underline{\lambda}(T^*)$, and $r(T)$. With the aid of Theorem 17 we may show that if K possesses an interior and K^* has an element f_0 in H_u satisfying $(f_0, x) > 0$ for every non-zero x in K , then $T > 0$ implies

$r(T) \geq \bar{\lambda}(T^*) \geq \underline{\lambda}(T^*)$. To this end, we define $T_n x = Tx + (1/n)f_0(x)u$, where u is interior to K . Then $T_n \gg 0$ and T_n converges in norm to T . For T_n we observe by Theorems 15 and 17 that $r(T_n) = \underline{\lambda}(T_n) = \bar{\lambda}(T_n^*) \geq \bar{\lambda}(T_n) = \underline{\lambda}(T_n^*)$. Using weak* compactness we may produce f_n and g_n in H_u (see Lemma 5 below) such that

$$T_n^* f_n \geq \bar{\lambda}(T_n^*) f_n, \quad T_n^* g_n \leq \underline{\lambda}(T_n^*) g_n.$$

Selecting a weak* limit point f, g of f_n, g_n respectively and designating $a = \overline{\bar{\lambda}(T_n^*)}$ and $b = \underline{\underline{\lambda}(T_n^*)}$, we obtain in the limit

$$T^* f \geq af, \quad T^* g \leq bg.$$

Consequently

$$\bar{\lambda}(T^*) \geq \overline{\lim} \bar{\lambda}(T_n^*) = \overline{\lim} r(T_n^*)$$

while

$$\underline{\lambda}(T^*) \leq \underline{\lim} \underline{\lambda}(T_n^*) \leq \underline{\lim} r(T_n^*).$$

This yields that $\underline{\lambda}(T^*) \leq \bar{\lambda}(T^*)$, but since always $r(T^*) \geq \bar{\lambda}(T^*)$ we have established

Theorem 18. *If $T > 0$, K has an interior, and K^* possesses an element f_0 for which $(f_0, x) > 0$ for every $x \neq 0$ in K , then*

$$r(T) \geq \bar{\lambda}(T^*) \geq \underline{\lambda}(T^*).$$

Lemma 5. *If $T > 0$ and K has an interior, then $\bar{\lambda}(T^*)$ and $\underline{\lambda}(T^*)$ belong to $\bar{\Omega}(T^*)$ and $\underline{\Omega}(T^*)$, respectively.*

Proof (on $\bar{\lambda}$). Define $A_n = [f | f \in H_u, T^* f \geq (\bar{\lambda} - 1/n)f]$. Evidently A_n is a w^* closed, non-void subset of H_u . In fact, let f_0 denote a limit point of A_n . Consider, for x in K , $y = [T - (\bar{\lambda} - 1/n)I]x$. Then for any positive ϵ there exists an $\bar{f}$ in A_n with

$$|(y, \bar{f}) - (y, f_0)| < \epsilon$$

or

$$|([T^* - (\bar{\lambda} - 1/n)I]\bar{f}, x) - ([\dots]f_0, x)| \leq \epsilon.$$

But, since $T^* \bar{f} \geq (\bar{\lambda} - 1/n)\bar{f}$ and because ϵ is arbitrary, the same holds for f_0 , thus placing f_0 in A_n .

Now $A_1 \supset A_2 \supset A_3 \supset \dots$ in conjunction with the property of compactness yields $\Gamma = \bigcap A_n \neq 0$. Any point g of Γ has the property that $T^* g \geq \bar{\lambda}g$, as desired.

By altering the hypothesis slightly we may strengthen Theorem 18 as follows:

Theorem 19. If $T > 0$, K has an interior and K^* spans E^* , then

$$r(T) = \bar{\lambda}(T^*) \geq \underline{\lambda}(T^*).$$

Proof. By Theorem 7, a non-zero element f_0 in K^* exists for which $T^*f_0 = r(T)f_0$. Hence $\bar{\lambda}(T^*) \geq r(T)$. Since the opposite inequality is always true the result follows.

Unfortunately, no general relationship between $\underline{\lambda}(T)$ and $\bar{\lambda}(T)$ exists, as the following examples indicate.

Example 1. Let $E = (m)$ [space of bounded sequences $x = (x_i)$]. Put $(Tx)_i = x_{i-1}$, $(Tx)_1 = 0$. For $(x_{i-1}) = Tx \geq \lambda(x_i)$, let n be the first index where $x_n > 0$. Then $0 = x_{n-1} \geq \lambda x_n$ implies $\lambda = 0$. Hence $\bar{\lambda}(T) = 0$. On the other hand, $Tx \leq \lambda x$ means $x_{n-1} \leq \lambda x_n$, and through iteration $x_1 \leq \lambda^n x_n$. But, as $\max |x_n| \leq c$ for some constant c , we get for $\lambda < 1$ that $x_1 = 0$ and similarly every $x_i = 0$. We may therefore conclude that $\underline{\lambda}(T) = 1$ since $\lambda = 1$ is an admissible $\underline{\lambda}$ value. This example displays the possible gap $0 = \bar{\lambda} < \underline{\lambda} = 1$.

Example 2. This example will show that $\underline{\lambda} < \bar{\lambda}$. Let T be the direct product mapping $\{\alpha I; \beta I\}$ (I = identity operator) acting on $E \times E$ with $\alpha > 0$ and $\beta > 0$ ($\alpha \neq \beta$). Clearly, $\bar{\lambda}(T) = \max(\alpha, \beta) > \min(\alpha, \beta) = \underline{\lambda}(T)$.

One important instance when the characterization $r(T) = \bar{\lambda}(T)$ applies is the following.

Theorem 20. If $T > 0$ has a pole at $r(T) = 1$ (e.g. if T is completely continuous), then $\underline{\lambda}(T) \leq \bar{\lambda}(T) = r(T)$ and $\underline{\lambda}(T^*) \leq \bar{\lambda}(T^*) = r(T)$.

(The reader should recall that $r(T)$ belongs to the spectrum of T (Theorem 4).)

Proof. This is an immediate consequence of Theorem 5 in conjunction with the observations that T^* has a pole at 1 if and only if T does, and that the characteristic manifold of 1 contains elements of K and K^* , respectively.

Combining this with Theorems 15 and 17, we have

Corollary. If $T \gg 0$ has a pole at $r(T) = 1$, then $\underline{\lambda}(T) = \underline{\lambda}(T^*) = \bar{\lambda}(T) = \bar{\lambda}(T^*) = r(T)$.

Theorem 21 [12]. If $T \gg 0$ and $Tx = x$ with x non-zero in K , then there exists an f in K^* such that $T^*f = f$.

Proof. By Theorem 15 and Lemma 5 we may find $f_0 \neq 0$ in K^* with $T^*f_0 \geq f_0$. Since $Tx^0 = x^0 \gg 0$ we obtain $(T^*f_0 - f_0, x^0) = 0$, which implies that $T^*f_0 = f_0$.

Subject to the conditions of this theorem we also have $\underline{\lambda}(T) = \bar{\lambda}(T) = 1$.

These same ideas can be used to explore the existence of eigenvalues for special kinds of positivity preserving non-linear operators. We briefly discuss one such class—studied previously by KREIN & RUTMAN [pp. 111–120]. Our proofs are more direct and relate to the characterizations of the $\bar{\lambda}$ values of this section.

Let T denote a completely continuous operator *not necessarily linear*. Here T

is assumed to be homogeneous, *i.e.*, for $\lambda > 0$, $T\lambda x = \lambda Tx$ and $T(K) \subset K$. To begin with we postulate that T is strictly positive and we, furthermore, assume the existence of a number $c > 0$ and x non-zero in K such that $Tx \geq cx$. (This restriction is also made by KREIN & RUTMAN.) Under these conditions we establish the existence of $\lambda_0 > 0$ and $x_0 \gg 0$ satisfying $Tx_0 = \lambda_0 x_0$.

To this end, following the pattern indicated at the start of this section, we call a positive value λ admissible if there exists a non-zero x in K such that $Tx \geq \lambda x$. Since T is homogeneous we may always select x , associated with an admissible λ value, normalized so that $\|x\| = 1$.

The hypothesis imposed on T assures us that the set Ω of all admissible values is non-void and bounded above. Let $\lambda_0 = \sup \lambda$ for λ in Ω . We prove that λ_0 is also in Ω . In fact, let $\lambda_n \rightarrow \lambda_0$ and denote by x_n ($\|x_n\| = 1$) the corresponding element of K satisfying

$$Tx_n \geq \lambda_n x_n.$$

Since T is completely continuous, we infer the existence of a subsequence Tx_{n_m} converging to $x_0 \neq 0$ in K with $Tx_0 \geq \lambda_0 x_0$. If equality does not hold, then since T is strictly positive and homogeneous we obtain

$$T(Tx_0) \gg \lambda_0(Tx_0) \geq (\lambda_0 + \epsilon)Tx_0,$$

contradicting the meaning of λ_0 . Hence $Tx_0 = \lambda_0 x_0$, as desired.

We may weaken the hypothesis to the extent that T is no longer assumed to be a strictly positive mapping but merely positive, provided we postulate the existence of an interior element u in K and the existence of a positive functional f with the property that $(f, x) > 0$ for all $x \neq 0$ in K .

Define $T_n x = Tx + (1/n)(f, x)u$. Then $T_n \gg 0$ and obeys all the requirements of the operator analyzed in the preceding paragraph. Hence, by what was proved, we secure the existence of x_n ($\|x_n\| = 1$, $x_n \in K$) satisfying $T_n x_n = \mu_n x_n$. Clearly $\|T\| + c \geq \mu_n \geq c$. We select a limit point μ_0 for μ_n and a limit z_0 of a subsequence of Tx_n . The last is possible by virtue of the fact that T is completely continuous. It remains to invoke a standard argument in order to verify $Tz_0 = \mu_0 z_0$.

7. Perturbation of Positive Operator. Conditions for convergence of eigenvalues and eigenvectors under perturbation of positive operators are now investigated.

Since the spectral radius is a value of the spectrum (Theorem 4), the convergence of these eigenvalues can be expressed as convergence of the spectral radius. For this analysis we will rely heavily on the characterization of the spectral radius in terms of the λ values as described in Theorems 17 and 18. Throughout this section we postulate that K has an interior and that for any interior element u the set of all x satisfying $-u \leq x \leq u$ is uniformly bounded in norm.

Theorem 22. *Let $T \gg 0$ be such that $\bar{\lambda}(T) = \underline{\lambda}(T) = r(T)$. If $T_n \rightarrow T$ ($T_n > 0$) in the strong topology of operators, then $r(T_n) \rightarrow r(T)$.*

Proof. Indeed, for a given $\epsilon > 0$ there exists $x_0 \gg 0$ such that

$$Tx_0 \leq (\lambda_0 + \epsilon)x_0$$

where $\lambda_0 = r(T)$. Thus

$$T_n x_0 = Tx_0 + (T_n - T)x_0 \leq (\lambda_0 + \epsilon)x_0 + (T_n - T)x_0.$$

But for $n \geq n_0(\epsilon, x_0)$, $(T_n - T)x_0 \leq \epsilon x_0$, and hence

$$T_n x_0 \leq (\lambda_0 + 2\epsilon)x_0.$$

Since $x_0 \gg 0$, we deduce that $r(T_n) \leq \lambda_0 + 2\epsilon = r(T) + 2\epsilon$. (See the proof of Theorem 17.)

The opposite inequality can be derived by a simpler argument using $\bar{\lambda} = r(T)$ and the fact that $\bar{\lambda}(T_n) \leq r(T_n)$.

A slightly more elaborate argument will establish the same conclusion with the weaker hypothesis of weak convergence replacing strong convergence. Another theorem of this kind is the following:

Theorem 23. *Let $T_n \geq T_{n+1} \geq \dots \geq T$ where $T > 0$, T_n^* converges weak* to T^* , and $r(T_n) = \bar{\lambda}(T_n^*)$, $r(T) = \bar{\lambda}(T^*)$. Then $r(T_n) \rightarrow r(T)$.*

Proof. It follows immediately from the definitions that $\bar{\lambda}(T_n^*) \geq \bar{\lambda}(T_{n+1}^*) \rightarrow \mu \geq \bar{\lambda}(T^*)$. Let $\Gamma_n = \{f \mid f \text{ in } H_u, T_n^* f \geq \bar{\lambda}(T_n^*)f\}$. Then clearly $\Gamma_n \supset \Gamma_{n+1}$. Since Γ_n are w^* compact and non-void (see the proof of Lemma 5), $\bigcap_n \Gamma_n$ is non-empty. Let f_0 be an element of the common intersection. It follows that, for $\mu - \epsilon$, $T_n^* f_0 \geq (\mu - \epsilon)f_0$. From the weak* convergence of $T_n f_0$ to $T f_0$ we deduce easily that $T^* f_0 \geq (\mu - \epsilon)f_0$. Consequently $\bar{\lambda}(T^*) \geq \mu - \epsilon = \lim \bar{\lambda}(T_n^*) - \epsilon$. Since ϵ is arbitrary we conclude that $\lim_{n \rightarrow \infty} r(T_n) = \lim \bar{\lambda}(T_n^*) = \bar{\lambda}(T^*) = r(T)$.

In a similar manner we have

Theorem 24. *If $T_n \leq T_{n+1} \leq \dots \leq T$, $\underline{\lambda}(T_n^*) = r(T_n^*)$ for all $n = 1, 2, \dots$, $r(T) = \underline{\lambda}(T^*)$, and T_n^* converges weak* to T^* , then*

$$\lim_{n \rightarrow \infty} r(T_n) = r(T_0).$$

We now turn to a brief discussion of the convergence properties of eigenvectors. Let u specify an interior element of K and H_u designate the usual section in K^* .

Theorem 25. *Suppose $T_n > 0$, $T \gg 0$, $T_n \rightarrow T$ strongly and T possesses a pole at $r(T) = \lambda_0$. Let f_n be a vector such that $T_n^* f_n \geq \bar{\lambda}(T_n^*)f_n$ (see Lemma 5), f_n in H_u , and let f_0 in H_u be such that $T^* f_0 = \lambda_0 f_0$. Then $f_n \rightarrow f_0$ (weak*).*

Proof. By reasoning similar to that of Theorem 22 and appeal to the corollary of Theorem 20, we deduce that $\bar{\lambda}(T_n^*) \rightarrow \bar{\lambda}(T^*) = r(T^*) = \lambda_0$. Let f' be any w^* limit point of f_n in H_u . It is easy to verify that for x in K

$$(T^* f' - \lambda_0 f', x) \geq 0.$$

Thus $T^*f' \geq \lambda_0 f'$. But $x_0 \gg 0$ exists with $Tx_0 = \lambda_0 x_0$, and hence $T^*f' = \lambda_0 f'$. Because of our next lemma and Theorem 5 this compels $f' = f_0$, and hence $f_n \xrightarrow{\omega^*} f_0$.

Lemma 6. *If $T \gg 0$ has a pole at $r(T) = \lambda_0$, then the solutions of $T^*f = \lambda_0 f$ form a one-dimensional space.*

Proof. In view of Theorem 8, the linear manifold $\mathcal{R} = [x \mid Tx = \lambda_0 x]$ is one-dimensional. Because of that same theorem we obtain the direct sum representation $E = \mathcal{R} + \mathcal{S}$ where $\mathcal{S} = \text{closure } [y \mid y = \lambda_0 z - Tz \text{ for some } z]$. A direct sum composition of E induces a direct sum decomposition for $E^* = \mathcal{R}' \oplus \mathcal{S}'$ where $\mathcal{R}' = [f \mid T^*f = \lambda_0 f]$ and also $\mathcal{R}' = \mathcal{S}^\perp$, the set of all functionals annihilating $\mathcal{S}$. It can be easily checked that $\mathcal{R}' \simeq \mathcal{R}^*$ (i.e. $\mathcal{R}'$ is isomorphic to the conjugate space $\mathcal{R}^*$). Since $\mathcal{R}$ is one-dimensional so is $\mathcal{R}'$.

Theorem 26. *Let $T_n > 0$ and T_0 be completely continuous and strictly positive with $T_n \rightarrow T_0$ uniformly. Let x_n be a positive eigenvector for $r(T_n)$ and let x_0 be the unique positive eigenvector of T for $r(T)$ all of norm 1. Then $x_n \rightarrow x_0$ strongly.*

Proof. Consider

$$\begin{aligned} \bar{\lambda}(T_n)x_n - \bar{\lambda}(T_m)x_m &= (T_n x_n - T_0 x_n) + (T_0 x_n - T_0 x_m) + (T_0 x_m - T_m x_m) \\ &= \text{I} + \text{II} + \text{III}. \end{aligned}$$

Since $T_n \rightarrow T_0$, I and III $\rightarrow 0$, and by the complete continuity of T_0 choose x_{n_i} so that $T_0 x_{n_i}$ converges. Since $r(T_n) \rightarrow r(T)$, by Theorem 22 we deduce that x_{n_i} converges to y where $Ty = \lambda_0 y$, $y \in K$ and $\|y\| = 1$. Hence $y = x_0$ (Theorem 8). Since this is valid for every subsequence, it follows that x_n converges in norm to x_0 .

For applications of the results of this section we refer the reader to SHIFFMAN [15] and KARLIN [11].

8. Further Study of Spectral Values on the Unit Circle. We begin with a discussion of the finite dimensional matrix operator of the form $\|a_{ij}\| = A$ where $a_{ij} \geq 0$. Many of the tricks of this section can be imitated and thus extended to apply to situations in which integral operators are defined by sufficiently smooth kernels. Some of the arguments of this section were suggested by BOHNENBLUST.

We assume as usual that $r(A) = 1$. Theorem 4 shows that 1 is an eigenvalue of A , and Theorem 5 guarantees an element x^0 of K (the non-negative orthant) belonging to the characteristic manifold of 1. Suppose, for definiteness, that in terms of components $x^0 = (x_1^0, \dots, x_r^0, 0, \dots, 0)$ where $x_1^0, x_2^0, \dots, x_r^0 > 0$. Since $Ax^0 = x^0$ we find that

$$A = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix}$$

where A_1 is a non-negative $r \times r$ matrix and A_2 is a non-negative $n - r \times n - r$ matrix. To investigate the nature of the eigenvalues of A it is clearly sufficient to study the spectra of A_1 and A_2 separately. By performing reductions of the above type on the conjugate operator (here A^* = the transpose matrix) we can assume without loss of generality that A has the property that $A^*f_0 = f_0$ with $f_0 \gg 0$ (all the components of f are positive).

Suppose now that λ is an eigenvalue of A of modulus 1. Then $Ax = \lambda x$ for some vector $x \neq 0$, i.e., $\sum_{i=1}^n a_{ij}x_j = \lambda x_i$ ($i = 1, \dots, n$). Hence $\sum_{i=1}^n a_{ij} |x_j| \geq |x_i|$ or $A|x| \geq |x|$. (Here $|x|$ represents the vector whose components are the corresponding absolute values of x .) But since $f_0 \gg 0$ we must have $A|x| = |x|$. Consequently $\sum_{i=1}^n a_{ij} |x_j| = |\sum_{i=1}^n a_{ij} x_j|$; that is, for each i all the values $a_{ij} x_j$ lie along the same ray in the complex plane. Hence we have the existence of complex numbers u_i ($|u_i| = 1$) such that $u_i a_{ij} |x_j| = a_{ij} x_j$ for all i and j . Let xy denote the vector whose components are $(x_i y_i)$. Multiplying the preceding relation by u_i and summing shows that $uA(|x|u) = A(xu)$ and $u|x| = uA|x| = Ax = \lambda x$. Consider the vector xu . Using the two relations just established gives $Axu = uA(|x|u) = \lambda uAx = \lambda^2 ux$, and hence ux is a characteristic vector corresponding to λ^2 . In the same way, since $u_i a_{ij} |x_j| u_j^2 = a_{ij} x_j u_j^2$, for all i and j , we get $uA(|x|u^2) = A(xu^2)$. But $A(|x|u^2) = \lambda A(xu) = \lambda^3 ux$, and thus $Axu^2 = \lambda^3 xu^2$, which shows that xu^2 is an eigenvector for λ^3 . Proceeding in this way we find that if λ is a characteristic value of modulus 1, then λ is a root of unity and $\lambda, \lambda^2, \lambda^3, \dots, \lambda^n, \dots$, are all eigenvalues with corresponding eigenvectors $x, xu, xu^2, \dots, xu^n, \dots$.

The preceding argument proves more. Specifically, we show that $\dim M_\lambda \leq \dim M_{\lambda^k}$ where $M_\lambda = \{x | Tx = \lambda x\}$ is the eigenmanifold corresponding to the eigenvalue λ . In fact, let $x^1, x^2, \dots, x^k$ be k linearly independent vectors in M_λ . By relabeling the components of x^i and forming suitable linear combinations of x^i we can achieve $y^1, y^2, \dots, y^k$ in M_λ in such a way that the first coordinate of $y^2, \dots, y^k$ is zero while the first y^1 is different from zero, the second coordinate of $y^3, \dots, y^k$ is zero, and the second coordinate of y^2 is different from zero, etc. Now, according to the above construction, $y^i u^i$ belong to M_{λ^k} for some suitable vector u^i which has every component of absolute magnitude 1. By examining the first k components successively, we find it impossible that

$$\sum_{i=1}^k a_i (u^i y^i) = 0$$

and a_i not all zero. Hence $\dim M_{\lambda^k} \geq \dim M_\lambda$, as was to be proved. Consequently $\dim M_1 \geq \dim M_\lambda$ for any eigenvalue λ ($|\lambda| = 1$).

Another interesting property associated with the eigenvalues λ lying on the unit circle other than 1 has to do with comparing the order of the pole $l(\lambda_0)$ of $R(\lambda, T)$ at λ_0 with the order of the pole $l(1)$ of $R(\lambda, T)$ at $\lambda = 1$. It is shown in the discussion following Theorem 7 that $l(\lambda_0) \leq l(1)$. These facts highlight the distinctive role of the spectral value 1 for positive operators.

We now direct attention to the task of extending these properties of the

spectrum to the case of the linear bounded positive operator in a Banach space.

Let T be a quasi completely continuous positive operator with $r(T) = 1$, and let T have the form $T = T_1 + R$ where T_1 is a contraction of T to the finite dimensional space $\mathfrak{M}$ spanned by the characteristic vectors corresponding to eigenvalues of modulus 1, so that $r(R) < 1$. We note that T_1 defines a positive operator on $\mathfrak{M}$ to itself. Suppose that K is a lattice with an interior (i.e., $\sup(x, y)$ is in K whenever x and y are in K). It follows that $K_{\mathfrak{M}} = K \cap \mathfrak{M}$ is a lattice. Suppose, furthermore, that $K_{\mathfrak{M}}$ has a interior. By theorem of YUDIN [18] any positive operator defined on a finite dimensional space with a cone $K_{\mathfrak{M}}$, which contains $\sup(x, y)$ whenever x, y belong to $K_{\mathfrak{M}}$, can be represented as a positive matrix and the cone can be made to correspond to the positive orthant. Invoking the consequences of the above discussion for positive matrices, we obtain

Theorem 27. *Suppose K is a lattice with an interior. Let T be quasi completely continuous and positive with a representation of the form*

$$T = T_1 + R$$

where T_1 is the contraction of T to the manifold $\mathfrak{M}$ of characteristic vectors corresponding to characteristic values of modulus 1, and hence $r(R) < 1$. Suppose $K_{\mathfrak{M}}$ contains an interior element. If λ ($|\lambda| = 1$) is an eigenvalue, then $\lambda, \lambda^2, \lambda^3, \dots$, are all eigenvalues. If $\mathfrak{M}_{\lambda}$ is the characteristic manifold of λ , then $\dim \mathfrak{M}_{\lambda} \leq \dim \mathfrak{M}_{\lambda^2} \leq \dim \mathfrak{M}_{\lambda^3} \leq \dots \leq \dim \mathfrak{M}_{\lambda^1}$.

The representation $T = T_1 + R$ exists whenever, for instance, $\|T^n\| \leq C$ or when there exists a pseudo norm $\mu(x)$ (i.e. μ satisfies $\mu(x_1 + x_2) \leq \mu(x_1) + \mu(x_2)$ and $\mu(\lambda x) = |\lambda| \mu(x)$), so that $\mu(T^n x) \leq C$ for $\|x\| \leq 1$.

The result of this theorem constitutes a slight extension of analogous results developed by KREIN & RUTMAN [Sec. 8]. The proofs here are simpler.

We now show that the result which asserts that λ in the spectrum implies $\lambda^2, \lambda^3, \dots$ in the spectrum extends to another class of positive operators. Let T denote any operator with $\|T^n\| = 1$ such that the spectrum of T has the property that whenever μ belongs to $\sigma(T)$ and $|\mu| = 1$, then $\mu^2, \mu^3, \dots$ also belong to the spectrum of T . Let us consider the algebra of all operators of the form $A = \sum_{n=0}^{\infty} a_n T^n$ where $\sum |a_n| < \infty$. The operator calculus implies that λ is in the spectrum of A if and only if $\lambda = \sum_{n=0}^{\infty} a_n \mu^n$ and μ belongs to the spectrum of T . Suppose that

$$R = \sum_{n=0}^{\infty} r_n T^n$$

where $r_n \geq 0$, $\sum_{n=0}^{\infty} r_n = 1$.

If λ_0 belongs to the spectrum of R with $|\lambda_0| = 1$, then $\lambda_0 = \sum_n r_n \mu_0^n$ and μ_0 is in $\sigma(T)$. Consequently $\lambda_0 r_n = r_n \mu_0^n$ whenever r_n is positive, and therefore $\sum_n a_n \mu_0^{2n} = \lambda_0^2$. By the hypothesis, μ_0^2 is in the spectrum $\sigma(T)$, and we obtain that λ_0^2 is in the spectrum of R . Similarly, $\sum a_n \mu_0^{kn} = \lambda_0^k$ shows that λ_0^k belongs to the spectrum of R .

An example of such an operator is obtained by letting the Banach space E be (m) ((m) = bounded sequences) and defining $(Tx)_i = x_{i-1}$ when $i \geq 1$ and $(Tx)_0 = 0$, the familiar shift transformation. It is easy to verify that the closed unit circle belongs to the spectrum of T and $r(T) = 1$. The mapping $R = \sum_0^\infty r_n T^n$ has the form

$$(Rx)_i = \sum_{j=0}^i r_{i-j} x_j .$$

Thus we find that if λ ($|\lambda| = 1$) belongs to the spectrum of R , then $\lambda^2, \lambda^3, \text{etc.}$ are all eigenvalues.

A second example is obtained by choosing the bounded sequence space $x = \{x_i\}$, $i = 0, \pm 1, \pm 2, \dots$, and $R = \sum_{n=-\infty}^\infty r_n T^n$ where $(Tx)_i = x_{i-1}$. Writing R explicitly we have

$$(Rx)_i = \sum_{j=-\infty}^\infty r_{i-j} x_j .$$

If $r_i \geq 0$ and $\sum_{-\infty}^\infty r_i = 1$, then the previous analysis applies and we deduce that if λ is in the spectrum of R with $|\lambda| = 1$, then $\lambda^2, \lambda^3, \dots$, are also values of the spectrum of R .

Other continuous convolution examples may be exhibited for which the same holds. It remains an open question whether this property of the spectrum is true for every positive operator.

9. Positive Semi-Groups. Much of the analysis valid for the positive operator T and its iterates T^n extends to the case of semi-groups of operators $T(t)$ in which $T(t) > 0$ for $t \geq 0$. Theorems about Laplace transforms substitute for the power series theorems employed previously. We assume that $T(t)$ is a strongly continuous semi-group of bounded linear operators which are positive; i.e., $T(t)$ for each t transforms K into itself. The usual assumptions regarding K and K^* are imposed. Specifically, K and K^* span E and E^* respectively.

Let A denote the infinitesimal transformation of $T(t)$ and suppose without loss of generality that A has a dense domain. We now develop some results for $T(t)$ and A analogous to those of Theorems 4 and 5.

Suppose the spectrum of A lies in the right half-plane given by $R(\lambda) \geq \lambda_0$ and that λ_0 is the smallest value (algebraically) for which this is true.

Theorem 28. *If the smallest half-plane containing the spectrum $\sigma(A)$ of A is $R(\lambda) \geq \lambda_0$, then λ_0 belongs to $\sigma(A)$.*

Proof. It is well known [9, page 337] that

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t) dt$$

for $\lambda > \lambda_0$. Thus, for x in K and f in K^* , we get

$$(R(\lambda, A)x, f) = \int_0^\infty e^{-\lambda t} (T(t)x, f) dt.$$

If this function is analytic for λ in a neighborhood of λ_0 , then since $(T(t)x, f) = \phi(t) \geq 0$ we infer, with the aid of a classical result concerning Laplace transforms of non-negative functions, that $(R(\lambda, A)x, f)$ is analytic for a small vertical strip S containing λ_0 in its interior. Of course, the strip does not depend on x and f . This implies that $R(\lambda, A)$ exists for λ in S (see Theorem 4), contradicting the hypothesis. Thus λ_0 is in $\sigma(A)$.

From now on we assume for definiteness that $\lambda_0 = 0$ and that $\|T(t)\| \leq 1$. This can always be achieved by considering the semi-group $Ce^{-\lambda t}T(t)$.

Theorem 29. *Under the conditions of Theorem 28, if 0 is a pole of $R(\lambda, A)$, then there exists an element $y \neq 0$ in K with $Ay = 0$.*

Proof. The expansion (if $R(\lambda, A)$ has a pole at 0 then it necessarily is a simple pole because of the restriction $\|T(t)\| \leq 1$) $R(\lambda, A) = Q/\lambda + \dots$ is a consequence of the hypothesis. Furthermore, we note that Q is not identically zero. Consequently $\lim_{\lambda \rightarrow 0+} \lambda R(\lambda, A) = Q$. A standard Tauberian theorem implies, since $T(t) \geq 0$, that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t T(u) du = Q.$$

Suppose $T(t)x$ belong to the domain of A . Then

$$A \frac{1}{t} \int_0^t T(u)x du = \frac{1}{t} \int_0^t \frac{dT(u)}{du} x du = \frac{T(t)x - x}{t} \rightarrow 0$$

as $t \rightarrow \infty$ because $\|T(t)\| \leq 1$. Since $(1/t) \int_0^t T(u)du x$ tends to Qx we deduce, recalling that A is a closed mapping, that Qx belongs to the domain of A and $AQx = 0$. The linear span of all elements of the form $T(t)x$ with x in K is dense in the domain of A , and therefore, since $Q \neq 0$, there exists a non-zero x in K such that $Qx \neq 0$ belongs to the domain of A and $AQx = 0$. But $Qx = y$ is in K by virtue of the fact that

$$Qx = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t T(u)x du \geq 0.$$

Q.E.D.

If $R(\lambda, A)$ is completely continuous for a specific value of λ , then the spectrum of A is discrete. Moreover, the spectral points represent singular points of the resolvent operator $R(\lambda, A)$ which are poles [9, p. 210]. Therefore

Corollary. *If $R(\lambda, A)$ is a completely continuous operator for some value of λ , then there exists a $y \neq 0$ in K with $Ay = 0$.*

Many of the classical Sturm-Liouville differential equations which serve as the infinitesimal generators of positive semi-groups possess the property that $R(\lambda, A)$ is completely continuous for λ in the resolvent set (e.g., the regular case of Sturm-Liouville operators).

Under the conditions of Theorem 29 we may establish a strong ergodic theorem. We already have found the convergence $(1/t) \int_0^t T(u)du$. Consider

$$||T(t+s)x - T(t)x|| \leq ||T(t)|| ||T(s)x - x|| \leq ||T(s)x - x||$$

which tends to zero for each x . Hence the function $\varphi(t) = (f, T(t)x)$ is bounded and is slowly oscillating, for each x in E and f in E^* , and $(1/t) \int_0^t \varphi(u)du$ converges. It follows from a classical Tauberian theorem that $\varphi(t)$ itself must converge. But the weak convergence of the semi-group $T(t)$ is equivalent to its strong convergence.

We remark in closing, without elaborating on details, that much of the other structure of the spectrum of positive operators can be carried over to the semi-group version.

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