The Isoperimetric Inequality and Associated Boundary Problems

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1. Introduction. For a simple closed plane curve Γ of length L and enclosed area A there have been many proofs of the isoperimetric inequality $4\pi A \leq L^2$, with the sign of equality holding if and only if Γ is a circle. In particular (see [5], pp. 185–187), one of the most elementary proofs uses the so-called “Wirtinger inequality”, which expresses the extremal property of the second proper value of the boundary problem

$$(1.1) \quad \eta'' + \lambda\eta = 0, \quad \eta(0) - \eta(2\pi) = 0, \quad \eta'(0) - \eta'(2\pi) = 0.$$

In Section 2 of the present paper there is presented an alternative elementary proof of this isoperimetric inequality, based on the extremal property of the smallest proper value of the boundary problem

$$(1.2) \quad \eta'' + \lambda\eta = 0, \quad \eta(0) = 0, \quad \eta(L) = 0.$$

In Section 3 it is shown that the extremal property of the smallest proper value of a boundary problem related to (1.2) affords a similar elementary proof of the modified isoperimetric inequality for the simplest instance of a plane variational problem of “Dido type”. Finally, with the aid of the extremal property of the smallest proper value of (1.2) there is established in Section 4 an integral inequality for simply connected pieces of regular surfaces, which provides an elementary proof of the result of CARLEMAN [4] (see also RADÓ [7; p. 45] and BECKENBACH & RADÓ [1]) that the isoperimetric inequality $4\pi A \leq L^2$ holds for a simply connected rectifiable piece of a regular minimal surface.

2. The classical isoperimetric inequality. Suppose that

$$(2.1) \quad \Gamma: x = x(s), \quad y = y(s), \quad 0 \leq s \leq L,$$

is a simple closed rectifiable curve in the xy -plane, where the parameter s denotes

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are length measured in the positive sense on Γ from a fixed point (x_0, y_0) on this curve. In particular, $x(s)$ and $y(s)$ are Lipschitzian on $[0, L]$ and $x'^2 + y'^2 = 1$ a.e. (almost everywhere) on this interval. For brevity, set $x_1(s) = x(s) - x_0$, $y_1(s) = y(s) - y_0$, and let $z(s)$ be the complex-valued function $z(s) = x_1(s) + iy_1(s)$. Then $|z'(s)|^2 = 1$ a.e. on $[0, L]$, and the area A enclosed by C is given by

$$(2.2) \quad A = \frac{1}{2} \int_0^L (x_1 y'_1 - y_1 x'_1) ds = -\frac{1}{4} i \int_0^L (z' \bar{z} - \bar{z}' z) ds.$$

Consequently, we have the equality

$$(2.3) \quad L^2 - 4\pi A = L \int_0^L \left[|z'|^2 + \frac{i\pi}{L} (z' \bar{z} - \bar{z}' z) \right] ds.$$

Now if the complex-valued function $w(s)$ is defined by $z(s) = w(s) \exp \{i\pi s/L\}$, then $w(s)$ is Lipschitzian on $[0, L]$ and $w(0) = 0 = w(L)$; moreover, by direct computation it follows that the integrand of the integral in (2.3) is equal to $|w'|^2 - (\pi^2/L^2)|w|^2$, so that

$$(2.3') \quad L^2 - 4\pi A = L \int_0^L \left[|w'|^2 - \frac{\pi^2}{L^2} |w|^2 \right] ds.$$

We shall now employ the result of the following lemma, which is the extremal property of the smallest proper value of the boundary value problem (1.2).

Lemma 2.1. *If $L > 0$ and $\eta(s)$ is a real-valued absolutely continuous function on $[0, L]$ satisfying $\eta(0) = 0 = \eta(L)$, and with $\eta'(s)$ of integrable square on this interval, then*

$$\int_0^L \left[\eta'^2 - \frac{\pi^2 \eta^2}{L^2} \right] ds \geq 0,$$

and the equality sign holds if and only if $\eta(s)$ is of the form $c \sin (\pi s/L)$.

This lemma, which is clearly equivalent to Theorem 257 of [5; p. 185], is a direct consequence of the readily verified identity

$$(2.4) \quad \eta'^2 - \frac{\pi^2 \eta^2}{L^2} = \left[\frac{\pi \eta^2}{L} \operatorname{ctn} \left(\frac{\pi s}{L} \right) \right]' + \left[\eta' - \frac{\pi \eta}{L} \operatorname{ctn} \left(\frac{\pi s}{L} \right) \right]^2$$

and the fact (see [5; Theorem 222, p. 164]) that under the hypotheses of the lemma we have $\eta(s) = o[s^{\frac{1}{2}}]$ as $s \rightarrow 0^+$ and $\eta(s) = o[(L-s)^{\frac{1}{2}}]$ as $s \rightarrow L^-$. In passing, it is to be commented that identity (2.4) is, in essence, the well-known Jacobi form of the Legendre transformation for the second variation of a special non-parametric simple integral variational problem (see, for example, [3; p. 64]).

With the aid of Lemma 2.1 it follows from (2.3') that the isoperimetric inequality $L^2 \geq 4\pi A$ holds for Γ , and that the equality sign holds if and only if $w(s) = \gamma \sin (\pi s/L) \exp \{i\alpha\}$, where α, γ are real constants with $\gamma \geq 0$. Moreover, since $z(s) = w(s) \exp \{i\pi s/L\}$ and $|z'| = 1$ a.e. on $[0, L]$, it follows that $\gamma = L/\pi$. That is, for Γ the equality $L^2 = 4\pi A$ holds if and only if

$$z(s) = \frac{L}{\pi} \sin\left(\frac{\pi s}{L}\right) \exp\left\{i\left(\alpha + \frac{\pi s}{L}\right)\right\},$$

which is the equation of the circle with radius $L/(2\pi)$ and center $(x_*, y_*) = (x_0 - (L \sin \alpha)/(2\pi), y_0 + (L \cos \alpha)/(2\pi))$.

3. Solution of a simple problem of "Dido type". For P_0, P_1 distinct points in a plane Π , the problem of finding a simple arc Γ of prescribed length which forms with the straight line segment P_0P_1 a simple closed curve in Π with maximum enclosed area was considered by JACOB BERNOULLI (see [3; p. 3]). For this problem it will now be shown that the extremal property of the smallest proper value of a boundary problem related to (1.2) yields the corresponding modified isoperimetric inequality. This particular problem is the simplest instance of the type of problem that Lord KELVIN termed the "problem of Queen Dido" (see [3; Prob. 1, p. 528]), and many modifications of which have been considered as examples in the calculus of variations (see, for example, [3; Prob. 35, p. 149; Prob. 8, p. 446; Ex. XXII, p. 523; Prob. 29, p. 538]). Specifically, the following result will be established.

Theorem 3.1. *If $P_0 : (x_0, y_0)$ and $P_1 : (x_1, y_1)$ are distinct points of the xy -plane at distance $D = |P_0P_1| > 0$ and Γ_{01} is a simple arc in this plane of length L ($L > D$) joining P_0 to P_1 and forming with the straight line segment P_0P_1 a simple closed curve C enclosing an area A , then*

$$(3.1) \quad 4\sigma A + D^2 \sigma \operatorname{ctn} \sigma \leq L^2,$$

where σ is the root of

$$(3.2) \quad \frac{\sin \sigma}{\sigma} = \frac{D}{L}, \quad 0 < \sigma < \pi.$$

Moreover, the equality sign in (3.1) holds if and only if Γ_{01} is a circular arc of radius $L/(2\sigma) = D/(2 \sin \sigma)$.

Without loss of generality it may be supposed that the orientation is such that the tracing out of C in the positive sense is afforded by the arc Γ_{01} followed by the straight line segment P_1P_0 , and that Γ_{01} has the equation

$$(3.3) \quad \Gamma_{01} : x = x(s), \quad y = y(s), \quad 0 \leq s \leq L,$$

where s denotes arc length measured along Γ_{01} from P_0 . Corresponding to the notation of Section 2, let $x_1(s) = x(s) - x_0$, $y_1(s) = y(s) - y_0$, and $z(s) = x_1(s) + iy_1(s)$. Then $|z'(s)| = 1$ a.e. on $[0, L]$, $|z(L)| = D$, and again the area A enclosed by C is given by the integral (2.2), so that for $0 < \sigma < \pi$ we have

$$(3.4) \quad L^2 - 4\sigma A - D^2 \sigma \operatorname{ctn} \sigma = L \left\{ -|z(L)|^2 \frac{\sigma}{L} \operatorname{ctn} \sigma + \int_0^L \left[|z'|^2 + i \frac{\sigma}{L} (z' \bar{z} - \bar{z}' z) \right] ds \right\}.$$

Moreover, for $z(s) = w(s) \exp \{i\sigma s/L\}$ we have $w(0) = 0$, $|w(L)| = D$, and

$$(3.4') \quad L^2 - 4\sigma A - D^2\sigma \operatorname{ctn} \sigma = L \left\{ -|w(L)|^2 \frac{\sigma}{L} \operatorname{ctn} \sigma + \int_0^L \left[|w'|^2 - \frac{\sigma^2}{L^2} |w|^2 \right] ds \right\}.$$

In order to proceed with the proof of Theorem 3.1 along the lines of the preceding section, use is made now of the following result.

Lemma 3.1. *If $L > 0$, $0 < \sigma < \pi$, and $\eta(s)$ is a real-valued absolutely continuous function on $[0, L]$ satisfying $\eta(0) = 0$, and with $\eta'(s)$ of integrable square on this interval, then*

$$(3.5) \quad -\eta^2(L) \frac{\sigma}{L} \operatorname{ctn} \sigma + \int_0^L \left[\eta'^2 - \frac{\sigma^2}{L^2} \eta^2 \right] ds \geq 0,$$

and the equality sign holds in (3.5) if and only if $\eta(s)$ is of the form $c \sin (\sigma s/L)$.

The conclusion of Lemma 3.1 is a ready consequence of the identity

$$\eta'^2 - \frac{\sigma^2}{L^2} \eta^2 \equiv \left[\frac{\sigma}{L} \eta^2 \operatorname{ctn} \frac{\sigma s}{L} \right]' + \left[\eta' - \frac{\sigma}{L} \eta \operatorname{ctn} \frac{\sigma s}{L} \right]^2$$

and the fact that the hypotheses of the lemma imply $\eta(s) = o[s^{\frac{1}{2}}]$ as $s \rightarrow 0^+$. It is to be remarked that from the standpoint of differential equations the result of this lemma is equivalent to the statement that the smallest proper value of the boundary problem

$$(3.6) \quad \eta'' + \lambda \eta = 0, \quad \eta(0) = 0, \quad \eta'(L) - \eta(L) \frac{\sigma}{L} \operatorname{ctn} \sigma = 0$$

is $\lambda = \sigma^2/L^2$, with corresponding proper functions of the form $c \sin (\sigma s/L)$ (see, for example, MORSE [6; Ch. III]).

As an immediate consequence of relation (3.4') and Lemma 3.1 we have inequality (3.1) for $0 < \sigma < \pi$; moreover, the equality sign in (3.1) holds if and only if

$$w(s) = \gamma \sin \left(\frac{\sigma s}{L} \right) \exp \{i\alpha\}, \quad z(s) = \gamma \sin \left(\frac{\sigma s}{L} \right) \exp \left\{ i \left(\alpha + \frac{\sigma s}{L} \right) \right\},$$

where α, γ are real constants with $\gamma \geq 0$. Since $|z'(s)| \equiv 1$ and $|z(L)| = D$, it then follows that $\gamma = L/\sigma$, and σ is the root of the equation (3.2) on $0 < \sigma < \pi$. That is, the equality sign holds in (3.1) if and only if σ is determined by (3.2) and

$$(3.5) \quad z(s) = \frac{L}{\sigma} \sin \left(\frac{\sigma s}{L} \right) \exp \left\{ i \left(\alpha + \frac{\sigma s}{L} \right) \right\}, \quad 0 \leq s \leq L,$$

which is the equation of the circular arc joining $P_0 : (x_0, y_0)$ to $P_1 : (x_1, y_1) = (x_0 + D \cos (\alpha + \sigma), y_0 + D \sin (\alpha + \sigma))$, with center $(x_0 - (L \sin \alpha)/(2\sigma), y_0 + (L \cos \alpha)/(2\sigma))$ and radius $L/(2\sigma) = D/(2 \sin \sigma)$.

4. An isoperimetric inequality for surfaces. Let R be a Jordan region in the (u, v) -plane bounded by a simple closed curve C , and consider a real surface S_1 in three dimensional $\mathbf{r} = (x, y, z)$ space defined for (u, v) in an open region $R_1 \supset R + C$ by the vector equation

$$(4.1) \quad S_1 : \mathbf{r} = \mathbf{r}(u, v), \quad (u, v) \in R_1.$$

The symbol S will denote the portion of S_1 defined by

$$(4.2) \quad S : \mathbf{r} = \mathbf{r}(u, v), \quad (u, v) \in R + C.$$

Throughout the following discussion we shall make the following assumptions:

(i) $\mathbf{r}(u, v)$ is continuous and has continuous partial derivatives of the first two orders for $(u, v) \in R_1$.

(ii) C is rectifiable, and if $u = u(t)$, $v = v(t)$, $t_1 \leq t \leq t_2$, is an equation of C traced out in a counterclockwise sense, then the boundary curve

$$(4.3) \quad \Gamma : \mathbf{r} = \mathbf{r}(u(t), v(t)) = \mathbf{r}(t), \quad t_1 \leq t \leq t_2,$$

of S is a simple closed curve.

The first fundamental quantities of the surface will be denoted by $E = \mathbf{r}_u^2$, $F = \mathbf{r}_u \cdot \mathbf{r}_v$, $G = \mathbf{r}_v^2$, and it will be supposed that S_1 is regular, that is,

$$(iii) \quad W = (EG - F^2)^{\frac{1}{2}} > 0 \text{ for } (u, v) \in R_1.$$

In particular, the above conditions imply that Γ is rectifiable with length $L = \int_{t_1}^{t_2} |d\mathbf{r}/dt| dt$, and that the area A of the surface S of (4.2) is given by $A = \iint_R W du dv$. The second fundamental quantities of the surface S_1 will be designated by $L = \xi \mathbf{r}_{uu}$, $M = \xi \mathbf{r}_{uv}$, $N = \xi \mathbf{r}_{vv}$, where ξ is the unit normal $\xi = W^{-1}(\mathbf{r}_u \times \mathbf{r}_v)$ of S_1 , and H will denote the mean curvature $(EN - 2FM + GL)/(2W^2)$.

If s is the arc length measured along Γ in the sense prescribed by the positive sense along C , and $t = t[s]$, $0 \leq s \leq L$, relates the parameter value t along C to the arc length s , then for general vector functions $\mathbf{f}(u, v)$ defined for $(u, v) \in C$ we set $\mathbf{f}(s) = \mathbf{f}(u(t[s]), v(t[s]))$. The result of the following theorem is an extension of the classical isoperimetric problem in the plane to surfaces satisfying the above conditions (i), (ii), (iii).

Theorem 4.1. *If hypotheses (i), (ii), (iii) hold and $\mathbf{r}_0$ is an arbitrary point on Γ , then*

$$(4.4) \quad L^2 - 4\pi A - 4\pi \iint_R HW(\mathbf{r} - \mathbf{r}_0)\xi du dv \geq 0;$$

moreover, if s , $0 \leq s \leq L$, denotes the arc length measured along Γ in the sense prescribed by the positive sense on C with $\mathbf{r}_0 = \mathbf{r}(0) = \mathbf{r}(L)$, the equality sign in (4.4) holds if and only if

$$(4.5) \quad (\mathbf{r}(s) - \mathbf{r}_0)\xi(s) \equiv 0, \quad \mathbf{r}(s) - \mathbf{r}_0 \equiv \mathbf{g}(s) \sin\left(\frac{\pi s}{L}\right), \quad 0 \leq s \leq L,$$

where $\mathbf{g}(s)$ is a solution of the vector differential equation

$$(4.6) \quad \mathbf{g}'(s) = \frac{\pi}{L} (\xi(s) \times \mathbf{g}(s)), \quad 0 \leq s \leq L.$$

In particular, if the equality sign holds in (4.4) for a curve Γ along which $\xi(s)$ is constant, then Γ is a circle of radius $L/(2\pi)$.

Let $\mathbf{r}_1(s) = \mathbf{r}(s) - \mathbf{r}_0$, where, for the moment, $\mathbf{r}_0$ is an arbitrary constant vector, and denote differentiation with respect to s by a prime. Then by Green's lemma we have

$$\int_0^L (\mathbf{r}_1 \times \mathbf{r}') \xi \, ds = \iint_R \{[(\mathbf{r}_1 \times \mathbf{r}_u) \xi]_u - [(\mathbf{r}_1 \times \mathbf{r}_v) \xi]_v\} \, du \, dv.$$

Since

$$[(\mathbf{r}_1 \times \mathbf{r}_v) \xi]_u - [(\mathbf{r}_1 \times \mathbf{r}_u) \xi]_v = (\mathbf{r}_1 \times \mathbf{r}_v) \xi_u - (\mathbf{r}_1 \times \mathbf{r}_u) \xi_v + 2(\mathbf{r}_u \times \mathbf{r}_v) \xi,$$

in view of the Weingarten equations (see, for example, [2; p. 112])

$$W^2 \xi_u = (FM - GL) \mathbf{r}_u + (FL - EM) \mathbf{r}_v,$$

$$W^2 \xi_v = (FN - GM) \mathbf{r}_u + (FM - EN) \mathbf{r}_v,$$

it follows immediately that

$$[(\mathbf{r}_1 \times \mathbf{r}_v) \xi]_u - [(\mathbf{r}_1 \times \mathbf{r}_u) \xi]_v = 2HW \xi \mathbf{r}_1 + 2W,$$

and consequently we have the relation (see [8; p. 329])

$$(4.7) \quad A = \frac{1}{2} \int_0^L (\mathbf{r}_1 \times \mathbf{r}') \xi \, ds - \iint_R HW \xi \mathbf{r}_1 \, du \, dv.$$

Throughout the remainder of the proof of Theorem 4.1 it will be supposed that the constant vector $\mathbf{r}_0$ has been chosen equal to $\mathbf{r}(0)$. Then $\mathbf{r}_1(0) = 0 = \mathbf{r}_1(L)$, and since $\mathbf{r}'_1(s) \equiv \mathbf{r}'(s)$ and $|\mathbf{r}'(s)| = 1$ a.e. on $[0, L]$, it follows from (4.7) that the left-hand member of (4.4) is equal to

$$(4.8) \quad J = L \int_0^L \left[|\mathbf{r}'_1(s)|^2 - \frac{2\pi}{L} (\mathbf{r}_1(s) \times \mathbf{r}'_1(s)) \xi(s) \right] ds.$$

Now let $\mathbf{g}_\alpha(s)$ ($\alpha = 1, 2, 3$) be real solutions of the vector differential equation (4.6) such that the 3×3 matrix

$$(4.9) \quad ||\mathbf{g}_1(s) \quad \mathbf{g}_2(s) \quad \mathbf{g}_3(s)||$$

is orthogonal for $s = 0$. If $\mathbf{g}_\alpha(s)$, $\mathbf{g}_\beta(s)$ are solutions of (4.6), then clearly $(\mathbf{g}_\alpha \mathbf{g}_\beta)' = \mathbf{g}_\alpha \mathbf{g}'_\beta + \mathbf{g}'_\alpha \mathbf{g}_\beta \equiv 0$, and consequently the matrix (4.9) is orthogonal on $[0, L]$. In particular, the equation

$$(4.10) \quad \mathbf{r}_1(s) = \mathbf{g}_\beta(s) w_\beta(s) = \sum_{\beta=1}^3 \mathbf{g}_\beta(s) w_\beta(s)$$

determines uniquely the scalar functions $w_\beta(s)$; moreover, $w_\beta(0) = 0 = w_\beta(L)$ ($\beta = 1, 2, 3$) since $\mathbf{r}_1(0) = 0 = \mathbf{r}_1(L)$. If $\mathbf{w}(s)$ denotes the vector function $(w_1(s), w_2(s), w_3(s))$, then

$$\mathbf{r}'_1 = \mathbf{g}'_\beta w_\beta + \mathbf{g}_\beta w'_\beta = \frac{\pi}{L} (\xi \times \mathbf{g}_\beta) w_\beta + \mathbf{g}_\beta w'_\beta = \frac{\pi}{L} (\xi \times \mathbf{r}_1) + \mathbf{g}_\beta w'_\beta,$$

and since $\mathbf{g}_\alpha \mathbf{g}_\beta = \delta_{\alpha\beta}$ ($\alpha, \beta = 1, 2, 3$), we have

$$|\mathbf{r}'_1|^2 = \frac{\pi^2}{L^2} |\xi \times \mathbf{r}_1|^2 + 2 \frac{\pi}{L} (\xi \times \mathbf{r}_1) \mathbf{g}_\beta w'_\beta + |\mathbf{w}'|^2.$$

Moreover,

$$(\mathbf{r}_1 \times \mathbf{r}'_1) \xi = (\xi \times \mathbf{r}_1) \mathbf{r}'_1 = \frac{\pi}{L} |\xi \times \mathbf{r}_1|^2 + (\xi \times \mathbf{r}_1) \mathbf{g}_\beta w'_\beta,$$

so that

$$\begin{aligned} |\mathbf{r}'_1|^2 - 2 \frac{\pi}{L} (\mathbf{r}_1 \times \mathbf{r}'_1) \xi &= |\mathbf{w}'|^2 - \frac{\pi^2}{L^2} |\xi \times \mathbf{r}_1|^2, \\ &= |\mathbf{w}'|^2 - \frac{\pi^2}{L^2} [|\mathbf{r}_1|^2 - (\xi \mathbf{r}_1)^2], \\ &= |\mathbf{w}'|^2 - \frac{\pi^2}{L^2} |\mathbf{w}|^2 + \frac{\pi^2}{L^2} (\xi \mathbf{r}_1)^2, \end{aligned}$$

the last relation following from the preceding, in view of the condition $|\mathbf{w}|^2 = |\mathbf{r}_1|^2$ implied by the orthogonal character of the matrix (4.9). Consequently, the functional J of (4.8) satisfies the relation

$$(4.11) \quad J \geq L \int_0^L \left[|\mathbf{w}'|^2 - \frac{\pi^2}{L^2} |\mathbf{w}|^2 \right] ds,$$

and the equality sign in (4.11) holds only if $\xi \mathbf{r}_1 \equiv 0$ for $0 \leq s \leq L$. Finally, in view of Lemma 2.1, it follows that the right-hand member of (4.11) is non-negative, and is equal to zero if and only if $\mathbf{w}(s) = \mathbf{k} \sin(\pi s/L)$, where $\mathbf{k} = (k_1, k_2, k_3)$ is a constant vector. Consequently, we have $J \geq 0$, and the equality sign holds if and only if relations (4.5) hold with $\mathbf{g}(s) = \mathbf{g}_\alpha(s) k_\alpha$.

Now suppose that the equality sign holds in (4.4) for a curve Γ along which the unit normal vector $\xi(s)$ is a constant vector ξ . Then Γ is a plane curve, and if $\mathbf{p}$ is a constant unit vector such that $\xi \mathbf{p} = 0$, then $\mathbf{q} = \xi \times \mathbf{p}$ is a unit vector satisfying $\xi \times \mathbf{q} = -\mathbf{p}$ and $(\xi \times \mathbf{p}) \mathbf{q} = 1$. Moreover, it follows readily that the solutions $\mathbf{g}_\alpha(s)$ of the corresponding vector differential equation (4.6) such that the matrix (4.9) is orthogonal at $s = 0$ may be chosen to be

$$\mathbf{g}_1(s) = \xi, \quad \mathbf{g}_2(s) = \mathbf{p} \cos\left(\frac{\pi s}{L}\right) + \mathbf{q} \sin\left(\frac{\pi s}{L}\right), \quad \mathbf{g}_3(s) = \mathbf{q} \cos\left(\frac{\pi s}{L}\right) - \mathbf{p} \sin\left(\frac{\pi s}{L}\right),$$

and the most general real solution $\mathbf{g}(s)$ of (4.6) such that $\xi \mathbf{g}(s) \equiv 0$ is of the form

$$\mathbf{g}(s) = \gamma \left[\mathbf{p} \cos \left(\alpha + \frac{\pi s}{L} \right) + \mathbf{q} \sin \left(\alpha + \frac{\pi s}{L} \right) \right],$$

where α, γ are real constants with $\gamma \geq 0$. Moreover, since $\mathbf{r}(s) - \mathbf{r}_0 = \mathbf{g}(s) \sin(\pi s/L)$, it follows from the relation $|\mathbf{r}'(s)| = 1$ that $\gamma = L/\pi$, so that Γ has the equation

$$\mathbf{r}(s) = \mathbf{r}_0 + \frac{L}{\pi} \left[\mathbf{p} \cos \left(\alpha + \frac{\pi s}{L} \right) + \mathbf{q} \sin \left(\alpha + \frac{\pi s}{L} \right) \right] \sin \left(\frac{\pi s}{L} \right), \quad 0 \leq s \leq L;$$

that is, Γ is the circle with radius $L/(2\pi)$ and center

$$\mathbf{r}_* = \mathbf{r}_0 + (L/(2\pi)) (-\mathbf{p} \sin \alpha + \mathbf{q} \cos \alpha).$$

Theorem 4.2. *If hypotheses (i), (ii), (iii) hold and S is a minimal surface, then*

$$(4.12) \quad L^2 - 4\pi A \geq 0;$$

moreover, the equality sign in (4.12) holds only if Γ is a circle and S is the circular disc bounded by Γ .

If S is a minimal surface then $H \equiv 0$ for $(u, v) \in R$, and (4.4) reduces to the inequality (4.12). Moreover, if $L^2 - 4\pi A = 0$ then the equality sign holds in (4.4) for arbitrary $\mathbf{r}_0$ on Γ , and consequently conditions (4.5) hold for $\mathbf{r}_0$ an arbitrary point on Γ . In particular, if $\mathbf{r}, \mathbf{r}_0$ are arbitrary points of Γ and ξ is the normal to S at $\mathbf{r}$, then $(\mathbf{r} - \mathbf{r}_0)\xi = 0$, and consequently Γ is a curve along which $\xi(s)$ is a constant unit vector ξ_0 . From the latter part of Theorem 4.1 it then follows that Γ is a circle of radius $L/(2\pi)$. Now if A_0 is the area of the circular disc S_0 bounded by Γ we have $A_0 = L^2/(4\pi)$, and hence $A_0 = A$. Finally, the conclusion of Theorem 4.2 is obtained by noting that, in view of the plane character of Γ , we have $A = A_0$ if and only if $\xi(u, v)$ is constant throughout S , so that S is coincident with S_0 . Indeed, application of (4.7) to the plane disc S_0 yields the result

$$A_0 = \frac{1}{2} \left| \int_0^L (\mathbf{r}(s) \times \mathbf{r}'(s))_{\xi_0} ds \right|,$$

which, in view of Green's lemma, is equal to $|\iint_D (\mathbf{r}_u \times \mathbf{r}_v)_{\xi_0} du dv|$, so that by the Schwarz inequality the condition $A_0 = A = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| du dv$ can hold if and only if $\mathbf{r}_u \times \mathbf{r}_v \equiv \xi_0$ on D .

It is to be remarked that the elementary method of this section for the proof of the Carleman result of Theorem 4.2 avoids any representation theorem for minimal surfaces as occurs in CARLEMAN [4] (see also RADÓ [7; p. 45]), or in the proof of the isoperimetric inequality on surfaces of non-positive Gaussian curvature as given by BECKENBACH & RADÓ [1].

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