On the Preservation of Classes of Functions*

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Let P and Q be any two classes of periodic (of period 2π) functions, or, alternatively, the classes of Fourier series of such functions. We shall say that a two-way infinite sequence $\{\lambda_n\}$ of numbers is of type (P, Q) if whenever $\sum c_n e^{inx}$ is in P , $\sum c_n \lambda_n e^{inx}$ is in Q . The problem of necessary and sufficient conditions for $\{\lambda_n\}$ to be of type (P, Q) has been solved in a number of cases. For example, it is well known that if B, C, L denote respectively the classes of bounded, continuous and integrable functions, and if S is the class of Fourier-Stieltjes series, then $\{\lambda_n\}$ is of any one of the types $(B, B), (C, C), (L, L), (S, S)$ if and only if $\sum \lambda_n e^{inx}$ is in S or, what is the same thing, if

$$\sum' n^{-1} \lambda_n e^{inx}$$

is the Fourier series of a function of bounded variation (see *e.g.* **2**₁, p. 176).

Let Λ_α be the class of all periodic functions f satisfying a Lipschitz condition of order α , $0 < \alpha \leq 1$, that is, $f(x+h) - f(x) = O(h^\alpha)$ for $h > 0$ (uniformly in x). Since Λ_1 coincides with the class of the indefinite integrals of bounded periodic functions with mean value 0, it is clear that $(\Lambda_1, \Lambda_1) = (B, B)$, so that $\{\lambda_n\}$ is of type (Λ_1, Λ_1) if and only if $\sum \lambda_n e^{inx}$ is in S . The condition is still sufficient for $\{\lambda_n\}$ to be of type $(\Lambda_\alpha, \Lambda_\alpha)$ if $0 < \alpha < 1$ but is then no longer necessary. The main purpose of this note is to find a necessary and sufficient condition in this case. We shall also consider the class λ_α , $0 \leq \alpha < 1$, of periodic functions f such that $f(x+h) - f(x) = o(h^\alpha)$. (Clearly, $\lambda_0 = C$).

We shall be led to consider classes of functions which have proved useful in other connections, and we restrict ourselves once and for all to periodic functions. A continuous $F(x)$ will be said to be of the class Λ_* or λ_* if

$$\Delta^2 F(x, h) = F(x+h) + F(x-h) - 2F(x)$$

is $O(h)$ or $o(h)$, as the case may be, for $h \rightarrow 0$ and uniformly in x . Clearly Λ_* contains Λ_1 , and λ_* contains the class C' of continuously differentiable functions. It is not difficult to show that the WEIERSTRASS-HARDY continuous and nowhere differentiable function

$$\Phi(x) = \sum_1^\infty 2^{-n} \cos 2^n x$$

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is in Λ_* ; it is obviously not in Λ_1 . Similarly the function

$$\Psi(x) = \sum_1^\infty n^{-1} 2^{-n} \cos 2^n x$$

is in λ_* but clearly not in C' (Ψ is differentiable only in a set of measure 0; for all this see, for example, **2_I**, p. 47).

Classes Λ_* and λ_* appear in a natural way in the theory of best approximation of functions by trigonometric polynomials. If $E_n[F]$ denotes the best approximation of F by such polynomials of order n , then F is in Λ_* if and only if $E_n[F] = O(1/n)$; similarly F is in λ_* if and only if $E_n[F] = o(1/n)$. (See **1** or **2_I**, p. 114 sqq.)

Classes Λ_* and λ_* can be generalized in a way that is important for us. Let $1 \leq p \leq \infty$. We shall say that an $F \in L^p$ is in the class Λ_*^p if

$$||\Delta^2 F(x, h)||_p = \left\{ \int_0^{2\pi} |F(x+h) + F(x-h) - 2F(x)|^p dx \right\}^{1/p}$$

is $O(h)$ for $h \rightarrow 0$. Replacing here $O(h)$ by $o(h)$ we obtain class λ_*^p . If $E_n^p[F]$ denotes the best approximation of F by trigonometric polynomials of order n in the metric L^p , then the classes Λ_*^p and λ_*^p are characterized by the condition that $E_n^p[F]$ is $O(1/n)$ or $o(1/n)$, as the case may be (see **1**). Clearly $\Lambda_*^p \subset \Lambda_*^q$ if $p > q$, and similarly $\lambda_*^p \subset \lambda_*^q$. Classes Λ_* and λ_* may be considered as Λ_*^∞ and λ_*^∞ if we confine our attention to continuous functions. The functions Φ and Ψ defined above belong respectively to Λ_*^p and λ_*^p for any p .

We shall say that the series $\sum \lambda_n e^{inx}$, not necessarily a Fourier series, is in the class $(\Lambda_*)'$ if the integrated series,

$$\sum' (in)^{-1} \lambda_n e^{inx}$$

is the Fourier series of a function from Λ_* . Classes $(\lambda_*)'$, $(\Lambda_*^p)'$, $(\lambda_*^p)'$ are defined correspondingly. Since the Fourier coefficients of functions from Λ_*^p , $1 \leq p \leq \infty$, are $O(1/n)$, and those of functions from λ_*^p are $o(1/n)$ (see **2_I**, pp. 46–47), it follows that if $\sum \lambda_n e^{inx}$ is in $(\Lambda_*^p)'$ or $(\lambda_*^p)'$, then λ_n is $O(1)$ or $o(1)$ respectively.

Theorem I. Let $0 < \alpha < 1$. A necessary and sufficient condition for $\{\lambda_n\}$ to be of any of the types $(\Lambda_\alpha, \Lambda_\alpha)$, $(\lambda_\alpha, \lambda_\alpha)$, (Λ_*, Λ_*) , (λ_*, λ_*) is that $\sum \lambda_n e^{inx} \in (\Lambda_*^1)'$.

It is interesting to compare this condition with the necessary and sufficient condition stated at the beginning of the note. It is well known that a function $L(x)$ is equivalent to a function of bounded variation (that is, coincides almost everywhere with a function of bounded variation) if and only if

$$\int_0^{2\pi} |L(x+h) - L(x)| dx = O(h)$$

(see, e.g. **2_I**, p. 180, Example 8), so that the class of functions equivalent to functions of bounded variation is contained in Λ_*^1 , and the class S of trigono-

metric series is contained in $(\Lambda_*^1)'$. Hence the condition $\sum \lambda_n e^{inx} \in S$ is *sufficient* for $\{\lambda_n\}$ to be of any of the types considered in the theorem, a result which is easy to prove directly (see **2_I**, p. 178).

We confine our attention to the type $(\Lambda_\alpha, \Lambda_\alpha)$, $0 < \alpha < 1$; in the remaining cases the argument runs parallel and it will be enough to add brief comments.

We begin by showing that the condition is sufficient, that is, that if $f(x) \sim \sum c_n e^{inx}$ is in Λ_α and $\sum' (in)^{-1} \lambda_n e^{inx}$ is in Λ_*^1 , then $\sum c_n \lambda_n e^{inx}$ is in Λ_α .^{*} Write

$$G(x) = \sum' (in)^{-1} \lambda_n e^{inx}, \quad H(x) = \sum' (in)^{-1} c_n \lambda_n e^{inx}.$$

Then

$$H(x) = \frac{1}{2\pi} \int_0^{2\pi} f(t) G(x-t) dt.$$

We fix a $u > 0$ and write

$$\begin{aligned} \Delta f(x, u) &= f(x + \tfrac{1}{2}u) - f(x - \tfrac{1}{2}u) \sim \sum' c_n (2i \sin \tfrac{1}{2}u) e^{inx}, \\ \Delta^2 G(x, u) &= G(x+u) + G(x-u) - 2G(x) = \sum' (in)^{-1} \lambda_n (2i \sin \tfrac{1}{2}u)^2 e^{inx}, \\ \Delta^3 H(x, u) &= H(x + \tfrac{3}{2}u) - 3H(x + \tfrac{1}{2}u) + 3H(x - \tfrac{1}{2}u) - H(x - \tfrac{3}{2}u) \\ &= \sum' (in)^{-1} c_n \lambda_n (2i \sin \tfrac{1}{2}u)^3 e^{inx}. \end{aligned}$$

The last series is a convolution of the preceding two so that

$$\Delta^3 H(x, u) = \frac{1}{2\pi} \int_0^{2\pi} \Delta f(t, u) \Delta^2 G(x-t, u) dt.$$

By hypothesis,

$$\Delta f(t, u) = O(u^\alpha), \quad \int_0^{2\pi} |\Delta^2 G(x, u)| dx = O(u).$$

Hence

$$\begin{aligned} |\Delta^3 H(x, u)| &\leq O(u^\alpha) \int_0^{2\pi} |\Delta^2 G(x-t, u)| dt \\ &= O(u^\alpha) \int_0^{2\pi} |\Delta^2 G(t, u)| dt = O(u^{1+\alpha}). \end{aligned}$$

But it is well known (see **1**, p. 64) that if a continuous function $H(x)$ satisfies the condition $\Delta^3 H(x, u) = O(u^{1+\alpha})$, $0 < \alpha < 1$, then $H'(x)$ exists everywhere and is in Λ_α . In our case this means that $H'(x) = \sum c_n \lambda_n e^{inx}$ is in Λ_α and the sufficiency of the condition is established.

^{*}This result is stated in **2_{II}**, p. 159, Example 13, in the following equivalent form: The convolution of two functions which are in Λ_α and Λ_*^1 respectively (more generally, in Λ_α^p and Λ_*^q , where $1/p + 1/q = 1$) has a derivative in Λ_α . We give a complete proof here since the place indicated contains only a hint.

In the proof of the necessity we shall use facts from fractional integration and we recall relevant definitions and properties (for all this see 1 or 2_{II}, pp. 133–142). Given any integrable and periodic $\varphi(x)$ with mean value 0, so that $\varphi \sim \sum' \gamma_n e^{inx}$, and any $\alpha > 0$, we define the α^{th} integral φ^α of φ by the formula

$$\varphi^\alpha(x) \sim \sum' \gamma_n |n|^{-\alpha} \exp \{-\tfrac{1}{2}\pi i \operatorname{sign} n\} e^{inx}.$$

It is immediate that $\varphi^\alpha * \psi = \varphi * \psi^\alpha$. Moreover, if $0 < \alpha < 1$, we have:

- (i) If φ is bounded, then φ^α is in Λ_α .
- (ii) If φ is in Λ_α , then $\varphi^{1-\alpha}$ is in Λ_* .

Furthermore, if $\psi(r, x)$ designates the Poisson integral of $\psi(x)$:

- (iii) A necessary and sufficient condition for ψ to be in Λ_* is that

$$\psi''(r, x) = O\left(\frac{1}{1-r}\right)$$

uniformly in x (differentiation is always with respect to the angle).

- (iv) A necessary and sufficient condition for ψ to be in Λ_*^1 is that

$$\int_0^{2\pi} |\psi''(r, x)| dx = O\left(\frac{1}{1-r}\right).$$

Let now, as before, $G(x) \sim \sum' (in)^{-1} \lambda_n e^{inx}$, and let $\Gamma(x)$ be the indefinite integral of $G(x)$. We assume that whenever $f(x) \sim \sum c_n e^{inx}$ is in Λ_α , so is $\sum c_n \lambda_n e^{inx}$, or, what is the same thing, the convolution $f * G$ has a derivative in Λ_α . This implies, in view of (i), that if φ is bounded and has mean value 0, the convolution

$$\varphi^\alpha * G = \varphi * G^\alpha$$

has a derivative in Λ_α , so that, by (ii),

$$\varphi * (G^\alpha)^{1-\alpha} = \varphi * \Gamma$$

has a derivative in Λ_* , or, alternatively, that

$$\psi = \varphi * \Gamma' = \varphi * G$$

is in Λ_* . Since

$$\psi(r, x) = \varphi(x) * G(r, x),$$

(iii) implies that

$$\psi''(r, x) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) G''(r, x-t) dt = O\left(\frac{1}{1-r}\right)$$

uniformly in x , and, in particular, taking $\alpha = 0$, that

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi(-t) G''(r, t) dt = O\left(\frac{1}{1-r}\right),$$

for each bounded φ . This in turn leads to*

$$\int_0^{2\pi} (1-r) |G''(r, t)| dt = O(1),$$

so that, by (iv), $G(x)$ is in Λ_*^1 , and the necessity of the condition is established.

For type $(\lambda_\alpha, \lambda_\alpha)$ the argument is similar. The proof of sufficiency is exactly the same as for $(\Lambda_\alpha, \Lambda_\alpha)$. As regards the proof of necessity, it is enough to replace (i) and (ii) by:

(i') If φ is continuous, then φ^α is in λ_α ;

(ii') If φ is in λ_α , then $\varphi^{1-\alpha}$ is in λ_* ,

and in the rest of the argument use the fact that if $\{\lambda_n\}$ is of type $(\lambda_\alpha, \lambda_\alpha)$, then it is *a fortiori* of type $(\lambda_\alpha, \Lambda_\alpha)$.

The proof for type (Λ_*, Λ_*) is parallel to that for $(\Lambda_\alpha, \Lambda_\alpha)$, but the result may be also directly deduced from what has been proved if we use the fact that for periodic functions with mean value 0 the class Λ_* is *identical* with the class of integrals of order $1 - \alpha$ of functions in Λ_α , $0 < \alpha < 1$, so that the types $(\Lambda_\alpha, \Lambda_\alpha)$ and (Λ_*, Λ_*) are necessarily the same. Similar remarks apply to (λ_*, λ_*) .

Theorem II. *A necessary and sufficient condition for $\{\lambda_n\}$ to be of any one of the types $(\lambda_\alpha, \Lambda_\alpha)$, (λ_*, Λ_*) , $0 < \alpha < 1$, is that $\sum \lambda_n e^{inx} \in (\lambda_*^1)'$.*

The proof is similar to that of Theorem I and it is enough to observe that instead of (iii) and (iv) we use the following modifications:

(iii') A necessary and sufficient condition for ψ to be in λ_* is that $\psi''(r, x) = o\{1/(1-r)\}$ uniformly in x .

(iv') A necessary and sufficient condition for ψ to be in λ_*^1 is that

$$\int_0^{2\pi} |\psi''(r, x)| dx = o\left(\frac{1}{1-r}\right).$$

Let $1 \leq p < \infty$ and let Λ_α^p and λ_α^p denote respectively the classes of functions f such that $\|\Delta f(x, h)\|_p$ is $O(h^\alpha)$ or $o(h^\alpha)$, $h > 0$. The formula used in the proof of Theorem I and representing $\Delta^3 H(x, u)$ as a convolution of $\Delta f(x, u)$ and $\Delta^2 G(x, u)$ easily shows that $\{\lambda_n\}$ is of type $(\Lambda_\alpha^p, \Lambda_\alpha^p)$ if $\sum \lambda_n e^{inx} \in \Lambda_*^1$, and the same conclusion holds for the types $(\lambda_\alpha^p, \lambda_\alpha^p)$, $(\Lambda_*^p, \Lambda_*^p)$, $(\lambda_*^p, \lambda_*^p)$.

*We use here the following familiar fact: If a sequence of functions $\chi_n(t)$ has the property that $\int_0^{2\pi} \varphi(t) \chi_n(t) dt = O(1)$ for each bounded—or even only continuous (this fact is needed in the proof of Theorem II below)— φ , then $\int_0^{2\pi} |\chi_n(t)| dt = O(1)$. In our case the parameter r tends to 1 in a continuous way; but since $G''(r, x)$ is continuous for $r < 1$ it is enough to consider a sequence $\{r_n\}$ tending to 1 so slowly that the boundedness of $\int_0^{2\pi} (1-r)|G''(r, t)| dt$ for r tending to 1 through the values r_n implies boundedness for all $r < 1$.

Theorem III. Let $0 < \alpha < 1$. A necessary and sufficient condition for $\{\lambda_n\}$ to be of any one of the types $(\Lambda_\alpha^1, \Lambda_\alpha^1)$, $(\lambda_\alpha^1, \lambda_\alpha^1)$, $(\Lambda_*^1, \Lambda_*^1)$, $(\lambda_*^1, \lambda_*^1)$ is that $\sum \lambda_n e^{i n x} \in (\Lambda_*^1)'$.

In view of what has just been said it is enough to prove the necessity of the conditions, and we may restrict ourselves to the type $(\Lambda_\alpha^1, \Lambda_\alpha^1)$, which is representative enough. We may also add that the necessity in the case of $(\Lambda_*^1, \Lambda_*^1)$ is immediate: Since $\sum e^{i n x}$, as a Fourier-Stieltjes series, is in $(\Lambda_1^1)'$, it is also in $(\Lambda_*^1)'$, and so, if $\{\lambda_n\}$ is to be of type $(\Lambda_*^1, \Lambda_*^1)$ or, equivalently in $((\Lambda_*^1)', (\Lambda_*^1)')$, the series $\sum \lambda_n e^{i n x}$ must necessarily be in $(\Lambda_*^1)'$.

The result which we want to prove is an immediate consequence of Theorem I and of the special case $p = 1$, $\alpha = \beta$ of the following theorem where $p' = p/(p - 1)$.

Theorem IV. If $1 \leq p \leq \infty$, $0 < \alpha < 1$, $0 < \beta < 1$, the types $(\Lambda_\alpha^p, \Lambda_\beta^p)$ and $(\Lambda_\alpha^{p'}, \Lambda_\beta^{p'})$ are identical.

The proof of Theorem IV is based on the following lemma.

Lemma. Let $1 \leq p \leq \infty$, $q = p'$, $0 < \alpha < 1$, $0 < \beta < 1$. A necessary and sufficient condition for

$$f \sim \sum c_\nu e^{i \nu x}$$

to be in Λ_α^p is that for every $g \in \Lambda_\beta^q$, $g \sim \sum c'_\nu e^{i \nu x}$, we have

$$\sum c_\nu c'_\nu \rho^{|\nu|} = O(\delta^{\alpha+\beta-2}) \quad (\delta = 1 - \rho).$$

Necessity. If $f \in \Lambda_\alpha^p$, $g \in \Lambda_\beta^q$, and $f(\rho, x)$, $g(\rho, x)$ are the Poisson integrals of f and g , then, as is well known (see e.g. 21, p. 296, Example 11),

$$||f'(\rho, x)||_p = ||\sum \nu c_\nu \rho^{|\nu|} e^{i \nu x}||_p = O(\delta^{\alpha-1}),$$

$$||g'(\rho, x)||_q = ||\sum \nu c'_\nu \rho^{|\nu|} e^{i \nu x}||_q = O(\delta^{\beta-1}).$$

Hence, by Hölder's inequality,

$$|\sum c_\nu c'_\nu \rho^{2|\nu|}| = \left| \frac{1}{2\pi} \int_0^{2\pi} f'(\rho, x) g'(\rho, -x) dx \right| \leq O(\delta^{\alpha-1}) O(\delta^{\beta-1}) = O(\delta^{\alpha+\beta-2}),$$

which is the required result with ρ^2 for ρ .

Sufficiency. Let $\gamma(x)$ be any function from L^q with mean value 0, $\gamma \sim \sum d_\nu e^{i \nu x}$, and let $g(x)$ be the β^{th} integral of $\gamma(x)$,

$$g = \gamma^\beta \sim \sum' d_\nu \epsilon_\nu |\nu|^{-\beta} e^{i \nu x},$$

where ϵ_ν satisfies $|\epsilon_\nu| = 1$, $\epsilon_{-\nu} = \bar{\epsilon}_\nu$, and is constant for $\nu > 0$ and for $\nu < 0$. The function $g = \gamma^\beta$ is in Λ_β^q (this is a generalization of (i), and reduces to the latter for $q = \infty$; the proof remains essentially the same). Hence, by hypothesis,

$$\sum' c_\nu d_\nu \epsilon_\nu |\nu|^{2-\beta} \rho^{|\nu|} = O(\delta^{\alpha+\beta-2}).$$

Let us divide both sides by ρ , multiply by $\{\log (r/\rho)\}^{-\beta'}$ and integrate over $0 \leq \rho \leq r$, $r < 1$. Using the relations

$$\int_0^r \rho^{k-1} \left(\log \frac{r}{\rho} \right)^{-\beta} d\rho = \Gamma(1 - \beta) k^{\beta-1} r^k \quad (k > 0),$$

$$\int_0^r \left(\log \frac{r}{\rho} \right)^{-\beta} (1 - \rho)^{\alpha+\beta-2} d\rho \leq A (1 - r)^{\alpha-1},$$

we obtain (replacing r by ρ again)

$$\sum' c_\nu d_\nu \epsilon_\nu |\nu| \rho^{|\nu|} = \frac{1}{2\pi} \int_0^{2\pi} \gamma(x) (\sum' c_\nu \epsilon_\nu |\nu| \rho^{|\nu|} e^{-i\nu x}) dx = O(\delta^{\alpha-1}).$$

Since this holds for any γ in L^1 (with mean value 0), it follows that

$$\|\sum' c_\nu \epsilon_\nu |\nu| \rho^{|\nu|} e^{i\nu x}\|_p = O(\delta^{\alpha-1}).$$

The same inequality is satisfied if on the left we retain only the terms with $\nu < 0$ or $\nu > 0$ (we use the fact that if $u(\rho, x)$ and $v(\rho, x)$ are conjugate harmonic functions and if $\|u(\rho, x)\|_p = O(\delta^{-k})$, $k > 0$, $1 \leq p \leq \infty$, then also $\|v(\rho, x)\|_p = O(\delta^{-k})$; see 21, p. 258), and this, in view of the properties of the ϵ_ν , implies that

$$\|\sum c_\nu \rho^{|\nu|} e^{i\nu x}\|_p = O(\delta^{\alpha-1}),$$

that is, $\sum c_\nu e^{i\nu x} \in \Lambda_\alpha^p$. This completes the proof of the lemma.

To obtain Theorem IV it is now enough to observe that if $\{\lambda_\nu\} \in (\Lambda_\alpha^p, \Lambda_\beta^p)$ then for any $\sum c_\nu e^{i\nu x} \in \Lambda_\alpha^p$ we have $\sum \lambda_\nu c_\nu e^{i\nu x} \in \Lambda_\beta^p$, so that for any series $\sum c'_\nu e^{i\nu x} \in \Lambda_\alpha^{p'}$,

$$|\sum (\lambda_\nu c_\nu) c'_\nu \rho^{|\nu|}| = |\sum c_\nu (\lambda_\nu c'_\nu) \rho^{|\nu|}| = O(\delta^{\alpha+\beta-2}),$$

which implies that $\sum \lambda_\nu c'_\nu e^{i\nu x} \in \Lambda_\beta^{p'}$, that is, $\{\lambda_\nu\} \in (\Lambda_\alpha^{p'}, \Lambda_\beta^{p'})$.

Remarks. (i) Theorem IV holds if we replace one of the indices α and β , or both, by a star *.

(ii) A necessary and sufficient condition for $\{\lambda_n\}$ to be of any one of the types $(\lambda_*^1, \Lambda_*^1)$, $(\lambda_\alpha^1, \Lambda_\alpha^1)$, $0 < \alpha < 1$, is that $\sum \lambda_n e^{in x} \in (\lambda_*^1)'$.

The proofs of these facts do not require new ideas.

BIBLIOGRAPHY

- 1 A. ZYGMUND, Smooth Functions, *Duke Math. Journal* 12 (1945), pp. 47-76.
- 2 ———, *Trigonometric Series*, Second Edition, vol. I, pp. 1-383, vol. II, pp. 1-354, Cambridge University Press, 1959.

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