*On the Generalized Radiation Problem of Weinstein**

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Introduction. In the introduction we do not state precisely the hypotheses made about the functions or distributions with which we are dealing. Precise hypotheses and results are stated later on.

The generalized radiation problem of WEINSTEIN may be formulated as follows: to find a distribution $u(x, y)$, in the half plane $y > 0$, which satisfies

$$(1) \quad \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{k}{y} \frac{\partial}{\partial y} \right) u = 0,$$

$$(2) \quad u(x, y) = 0 \quad \text{outside the domain} \quad x \geq y,$$

$$(3) \quad u(x, 0) = F(x) \quad \text{given} \quad (F(x) = 0 \quad \text{for} \quad x < 0),$$

for k a complex number (cf, A. WEINSTEIN [1]). Actually WEINSTEIN looks for u defined for $x > y$, with $u(x, x) = 0$; the two problems are equivalent.

WEINSTEIN has proved the existence of u for $-\infty < k < 1$ and F smooth enough. LIEBERSTEIN [1] has proved the uniqueness of u when $-\infty < k < 1$, and when u is subjected to some smoothness conditions.

In this paper, we shall prove (§2):

- (i) the existence of u for $-\infty < \operatorname{Re} k < 1$, without any condition on F ;
- (ii) the uniqueness of the solution for $-\infty < \operatorname{Re} k < 1$, with the most general hypotheses on u ;
- (iii) the impossibility of solving the problem for $\operatorname{Re} k \geq 1$.

The method given in this paper is considerably simpler than the previous ones. We give here an outline of the method. We introduce

$$(4) \quad v(x, y) = \frac{2}{\Gamma(\frac{1}{2}k)} \int_y^\infty z(z^2 - y^2)^{\frac{1}{2}k-1} u(x, z) dz \quad (\operatorname{Re} k > 0).$$

The reason for doing this is that

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$$\frac{2}{\Gamma(\frac{1}{2}k)} \int_y^\infty z(z^2 - y^2)^{\frac{1}{2}k-1} \left(\frac{\partial^2}{\partial z^2} + \frac{k}{z} \frac{\partial}{\partial z} \right) u(x, z) dz = \frac{\partial^2}{\partial y^2} v(x, y),$$

so that if u satisfies (1) and (2), then v satisfies

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = 0.$$

Note that $v(x, y) = 0$ outside the domain $x \geq y$ and consequently $v(x, y) = f(x - y)$, where $f(t) = 0$ for $t < 0$. Now (see §1) formula (4) can be inverted:

$$(5) \quad u(x, y) = \frac{2}{\Gamma(-\frac{1}{2}k)} \int_y^\infty z(z^2 - y^2)^{-\frac{1}{2}k-1} v(x, z) dz.$$

This is valid in the usual sense for $\text{Re } k < 0$; however, by analytic extension, formulas (4) and (5) are true for every $k \in C$. Since $v(x, y) = f(x - y)$, (5) can be written

$$(6) \quad u(x, y) = \frac{2}{\Gamma(-\frac{1}{2}k)} \int_y^\infty z(z^2 - y^2)^{-\frac{1}{2}k-1} f(x - z) dz.$$

This gives all the functions (or distributions) satisfying (1) and (2). By an integration by parts, we can write (6) in the form

$$(7) \quad u(x, y) = \frac{1}{\Gamma(-\frac{1}{2}k + 1)} \int_y^\infty (z^2 - y^2)^{-\frac{1}{2}k} f'(x - z) dz.$$

If now $y \rightarrow 0$, we prove (§1 and §2, No. 6) that $u(x, y)$ does not converge (unless $f \equiv 0$, hence $u \equiv 0$) when $\text{Re } k \geq 1$. For $\text{Re } k < 1$, it is obvious formally (and it is proved) that

$$u(x, y) \rightarrow \frac{1}{\Gamma(-\frac{1}{2}k + 1)} \int_0^\infty z^{-k} f'(x - z) dz, \quad y \rightarrow 0,$$

i.e.,

$$u(x, 0) = \frac{\Gamma(-k + 1)}{\Gamma(-\frac{1}{2}k + 1)} I^{-k+1} f',$$

where I^s denotes the Riemann Liouville integral of order s . Then

$$(8) \quad u(x, 0) = \frac{\Gamma(-k + 1)}{\Gamma(-\frac{1}{2}k + 1)} I^{-k} f.$$

We want this to equal F , hence the unique solution

$$f = \frac{\Gamma(-\frac{1}{2}k + 1)}{\Gamma(-k + 1)} I^k F.$$

With (7) we obtain

$$(9) \quad u(x, y) = \frac{1}{\Gamma(-k + 1)} \int_y^\infty (z^2 - y^2)^{-\frac{1}{2}k} I^{k-1} F(x - z) dz.$$

This is the main WEINSTEIN formula [1] (*cf.* §2, No. 7) and, furthermore, this proves the uniqueness.

The introduction of the operator (4) is closely related to a new family of transmutation operators, given in §1. The applications, with precise statements and hypotheses, are made in §2.

By our method, all the results extend to the equations

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - k\lambda (\cotg \lambda y) \frac{\partial}{\partial y} + \frac{\lambda^2 k^2}{4} \right) u &= 0, \\ \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - k\lambda (\coth \lambda y) \frac{\partial}{\partial y} - \frac{\lambda^2 k^2}{4} \right) u &= 0 \end{aligned}$$

(*cf.* M. THYSSEN [1]). We shall return later to more general problems.

§1. SOME TRANSMUTATION OPERATORS

1. Notations. We shall denote by Ω the open half line $]0, +\infty[$ and by $D'(\Omega)$ (resp. $D'_+(\Omega)$) the space of distributions on Ω (resp. with support limited to the right), *i.e.*, if $T \in D'_+(\Omega)$, then $T = 0$ for $x > a_T$, $a_T > 0$, depending on T . We provide these spaces with the topology of SCHWARTZ (*cf.* L. SCHWARTZ [1], [2]). We denote by $D(\Omega)$ the space of functions infinitely differentiable in Ω , with compact support; $D'(\Omega)$ is the dual of $D(\Omega)$.¹

We shall use a functional notation for the distributions (as physicists do): If $T \in D'(\Omega)$, we shall speak of the distribution $x \rightarrow T(x)$. If $\varphi \in D(\Omega)$ and $\langle T, \varphi \rangle$ is the scalar product of T and φ , we write $\langle T, \varphi \rangle = \langle T(x), \varphi(x) \rangle$.

When we consider $T(x)$, $T(y)$, $T(\xi)$, $\dots$, we shall write $T \in D'(\Omega_x)$, $D'(\Omega_y)$, $D'(\Omega_\xi)$, $\dots$. If $T \in D'(\Omega_x)$, we set

$$(1.1) \quad MT(\xi) = T(\sqrt{\xi}).$$

The precise meaning of (1.1) is: If $\varphi \in D(\Omega_\xi)$, then

$$\langle MT, \varphi \rangle = \langle T(x), 2x\varphi(x^2) \rangle;$$

it is easily verified that we define in this way a continuous linear mapping $T \rightarrow MT$ of $D'(\Omega_x)$ to $D'(\Omega_\xi)$. Actually M is an isomorphism, whose inverse M^{-1} is given by

$$(1.2) \quad M^{-1}S(x) = S(x^2), \quad S \in D'(\Omega_\xi).$$

The precise meaning of (1.2) is

$$\langle M^{-1}S, \psi \rangle = \langle S(\xi), \tfrac{1}{2}\xi^{-\frac{1}{2}}\psi(\sqrt{\xi}) \rangle \quad \text{if } \psi \in D(\Omega_x).$$

If $D_x = d/dx$, $D_\xi = d/d\xi$, one verifies easily that

$$(1.3) \quad MD_x^2T = (4\xi D_\xi^2 + 2D_\xi)MT,$$

¹ $D'_+(\Omega)$ is the dual of the space $D_+(\Omega)$ of infinitely differentiable functions, with non-compact support but zero near the origin.

for every $T \in D'(\Omega_x)$. Another remark, useful for the following, is that M is a continuous linear mapping of $D'_-(\Omega_x)$ to $D'_-(\Omega_t)$, with inverse M^{-1} .

Let us now recall some classical facts about the Riemann-Liouville integral. We use the notations of SCHWARTZ [1], pp. 42, 43, and [2], pp. 30, 31.

We denote by D' (resp. D'_+ , resp. D'_-) the space of distributions on the line (resp. with support limited to the left, resp. to the right). Then we set, with L. SCHWARTZ,

$$(1.4) \quad Y_p = \frac{1}{\Gamma(p)} \text{Pf} (x^{p-1})|_{x>0}, \quad p \in C.$$

$Y_p \in D'_+$ (Y_p is zero for $x < 0$). When $p = -l$, l positive integer, one has

$$(1.5) \quad Y_{-l} = \delta^{(l)} = D^l \delta,$$

where δ = Dirac measure. The fundamental properties of Y_p are

$$(1.6) \quad p \rightarrow Y_p \text{ is analytic with values in } D'_+,$$

$$(1.7) \quad Y_p * Y_q = Y_{p+q} \text{ for every } p \text{ and } q \in C.$$

We introduce now the distributions Z_p by

$$(1.8) \quad Z_p(x) = Y_p(-x).$$

Then $Z_p \in D'_-$ (Z_p is zero for $x > 0$) and the above properties imply

$$(1.9) \quad Z_{-l} = (-1)^l \delta^{(l)},$$

$$(1.10) \quad p \rightarrow Z_p \text{ is analytic with values in } D'_-,$$

$$(1.11) \quad Z_p * Z_q = Z_{p+q} \text{ for every } p, q \in C.$$

From (1.9) and (1.11) it follows, in particular, that

$$(1.12) \quad DZ_p = -Z_{p-1}.$$

From (1.4) it follows immediately that $xY_p = pY_{p+1}$, hence

$$(1.13) \quad xZ_p = -pZ_{p+1}.$$

Our last preliminary remark will be: If $S \in D'_-$, $S = 0$ for $x > 0$, and $T \in D'_-(\Omega)$, one can define $S * T \in D'_-(\Omega)$ by

$$(1.14) \quad \langle S * T, \varphi \rangle = \langle T(x), \langle S(y), \varphi(x+y) \rangle \rangle, \quad \varphi \in D_+(\Omega);$$

indeed, if $\theta(x) = \langle S(y), \varphi(x+y) \rangle$, then θ is infinitely differentiable. If $\varphi(x) = 0$ for $x \leq \alpha_\varphi$ ($\alpha_\varphi > 0$), then $\theta(x) = 0$ for $x \leq \alpha_\varphi$, so that one can consider θ as an element of $D_+(\Omega)$ and (1.14) is meaningful. If $\varphi \rightarrow 0$ in $D_+(\Omega)$, then $\theta \rightarrow 0$ in $D_+(\Omega)$, and consequently $S * T \in D'_-(\Omega)$. Moreover, $T \rightarrow S * T$ is a continuous linear mapping of $D'_-(\Omega)$ into itself. We have (SCHWARTZ [2], Chap. VI)

$$(1.15) \quad S * T(x) = \int_x^\infty S(x-y)T(y) dy.$$

In particular, we can take $S = Z_p$. Hence

$$(1.16) \quad T \rightarrow Z_p * T$$

is a continuous linear mapping of $D'(\Omega)$ to $D'(\Omega)$. We denote by Z_p^* the mapping (1.16), $Z_p^* \in L(D'(\Omega); D'(\Omega))$ (space of continuous linear mappings of $D'(\Omega)$ into itself, with the topology of uniform convergence on bounded sets of $D'(\Omega)$). The function $p \rightarrow Z_p^*$ is an entire analytic function with values in $L(D'(\Omega); D'(\Omega))$.

2. Operators $\mathcal{H}_k$, H_k .

We denote by L_k the operator

$$(2.1) \quad L_k = D^2 + \frac{k}{x} D, \quad D = \frac{d}{dx}, \quad k \in C.$$

Then $T \rightarrow L_k T$ is a continuous linear mapping of $D'(\Omega)$ into itself.

We want to prove in this section the following

Theorem 2.1. *For every $k \in C$, there exists an isomorphism $\mathcal{H}_k$ of $D'(\Omega)$ onto itself such that*

$$(2.2) \quad \mathcal{H}_k D^2 = L_k \mathcal{H}_k;^2$$

moreover, we can choose $\mathcal{H}_k$ in such a way that $k \rightarrow \mathcal{H}_k$ is an entire analytic function with values in $L(D'(\Omega); D'(\Omega))$.

We shall prove this theorem by explicitly constructing $\mathcal{H}_k$. We give in this section an *a priori* solution; another way to find $\mathcal{H}_k$ is given in section 3.

The operator $\mathcal{H}_k$ is a *transmutation operator* of D^2 and L_k . See J. DELSARTE [1] and J. L. LIONS [1], [2] for other cases.

Proof of Theorem 2.1. a. We consider *a priori* the operator

$$(2.3) \quad \mathcal{H}_k = M^{-1} \cdot Z_{-k}^* \cdot M \quad \text{on} \quad D'(\Omega_x).$$

This is meaningful: M maps $D'(\Omega_x)$ into $D'(\Omega_\xi)$, Z_{-k}^* maps $D'(\Omega_\xi)$ into $D'(\Omega_\xi)$ and M^{-1} maps $D'(\Omega_\xi)$ into $D'(\Omega_x)$. All of these mappings are continuous and, by the remarks of section 1, $k \rightarrow \mathcal{H}_k$ is an entire analytic function with values in $L(D'(\Omega); D'(\Omega))$.

b. Since $Z_{\frac{1}{2}k} * Z_{-\frac{1}{2}k} = \delta$ (by (1.11) and (1.9)), $\mathcal{H}_k$ is an isomorphism of $D'(\Omega_x)$ onto itself, with inverse H_k given by

$$(2.4) \quad \mathcal{H}_k^{-1} = H_k = M^{-1} \cdot Z_{\frac{1}{2}k}^* \cdot M = \mathcal{H}_{-k}.$$

c. In order to prove the theorem, it remains only to check that (2.2) holds, *i.e.*,

$$M^{-1} \cdot Z_{\frac{1}{2}k}^* \cdot M \cdot D_x^2 T = L_k \cdot M^{-1} \cdot Z_{-\frac{1}{2}k}^* \cdot M T, \quad T \in D'(\Omega_x),$$

² *I.e.* $\mathcal{H}_k D^2 T = L_k \mathcal{H}_k T$ for every $T \in D'(\Omega)$.

or

$$(2.5) \quad Z_{-\frac{1}{2}k} * (MD_x^2 T) = (ML_k M^{-1})(Z_{-\frac{1}{2}k} * MT).$$

But it is easily verified that

$$(2.6) \quad ML_k M^{-1} = 2(k+1)D_\xi + 4\xi D_\xi^2,$$

so that (2.5) reduces to

$$(2.7) \quad Z_{-\frac{1}{2}k} * (MD_x^2 T) = 2(k+1)D_\xi(Z_{-\frac{1}{2}k} * MT) + 4\xi D_\xi^2(Z_{-\frac{1}{2}k} * MT).$$

Using now (1.3) and setting $MT = S$, (2.7) gives

$$Z_{-\frac{1}{2}k} * (4\xi D_\xi^2 S + 2D_\xi S) = 2(k+1)D_\xi(Z_{-\frac{1}{2}k} * S) + 4\xi D_\xi^2(Z_{-\frac{1}{2}k} * S),$$

i.e.,

$$(2.8) \quad Z_{-\frac{1}{2}k} * (2\xi D_\xi^2 S) = kD_\xi(Z_{-\frac{1}{2}k} * S) + 2\xi D_\xi^2(Z_{-\frac{1}{2}k} * S).$$

But

$$Z_{-\frac{1}{2}k} * (\xi D_\xi^2 S) = \xi(Z_{-\frac{1}{2}k} * D_\xi^2 S) - (\xi Z_{-\frac{1}{2}k}) * D_\xi^2 S,$$

so that (2.8) reduces to

$$-2(\xi Z_{-\frac{1}{2}k}) * D_\xi^2 S = kD_\xi(Z_{-\frac{1}{2}k} * S),$$

i. e.,

$$D(-2\xi Z_{-\frac{1}{2}k}) = kZ_{-\frac{1}{2}k}.$$

By (1.13), $-2\xi Z_{-\frac{1}{2}k} = -kZ_{-\frac{1}{2}k+1}$ and the result follows from (1.12). Theorem 2.1 is proved.

Remark 2.1. Of course $\mathcal{H}_k^{-1} = H_k$ satisfies

$$(2.9) \quad D^2 H_k = H_k L_k \quad \text{on} \quad D'_-(\Omega).$$

Remark 2.2. If $T \in D'_-(\Omega)$ is zero for $x > a_T$, then $\mathcal{H}_k T$ and $H_k T$ are zero for $x > a_T$. Moreover, if a_T is the best limit for T , it is also the best limit for $\mathcal{H}_k T$ and $H_k T$.

Remark 2.3. It is easy to write down explicitly the kernel of $\mathcal{H}_k$ and $H_k = \mathcal{H}_{-k}$:

$$(2.10) \quad \mathcal{H}_k T(x) = \frac{2}{\Gamma(-\frac{1}{2}k)} \int_x^\infty y(y^2 - x^2)^{-\frac{1}{2}k-1} T(y) dy, \quad \text{Re } k < 0,$$

$$(2.11) \quad H_k T(x) = \frac{2}{\Gamma(\frac{1}{2}k)} \int_x^\infty y(y^2 - x^2)^{\frac{1}{2}k-1} T(y) dy, \quad \text{Re } k > 0.$$

For $\text{Re } k \geq 0$ (resp. ≤ 0), the kernel of $\mathcal{H}_k$ (resp. H_k) is a genuine distribution, obtained by analytic extension.

Remark 2.4. The operator Z_k^* applied to a function f with support bounded to the right is expressed by

$$(2.12) \quad Z_k * f(x) = \frac{1}{\Gamma(k)} \int_x^\infty (y-x)^{k-1} f(y) dy \quad (\operatorname{Re} k > 0).$$

This is the *Weyl fractional integral* (cf. H. WEYL [1]); we can say the Weyl fractional integral plays the same role with respect to the operators H_k and $\mathcal{H}_k$ as the Riemann-Liouville integral does with respect to the operators B_k and $\mathcal{B}_k$ (we recall the definition of these operators in the next section).

3. Another way to derive the operators $\mathcal{H}_k$ and H_k . We have constructed in LIONS [1], Chapter 2,³ a family of operators $\mathcal{B}_p$ such that

$$(3.1) \quad \mathcal{B}_p D^2 = L_{2p+1} \mathcal{B}_p, \quad p \neq -1, -2, \dots,$$

$\mathcal{B}_p$ being an *isomorphism* of $\mathcal{E}_*$ onto itself, $\mathcal{E}_*$ = space of infinitely differentiable even functions $\mathcal{B}_p$ is obtained by analytic extension in p of the formula

$$(3.2) \quad \mathcal{B}_p f(x) = \beta_p x^{-2p} \int_0^x (x^2 - y^2)^{p-\frac{1}{2}} f(y) dy, \quad f \in \mathcal{E}_*,$$

valid for $\operatorname{Re} p > -\frac{1}{2}$, where

$$(3.3) \quad \beta_p = \frac{2\Gamma(p+1)}{\sqrt{\pi}\Gamma(p+\frac{1}{2})}.$$

The idea is to “transpose” (3.1). To do that, we notice in the first place that

$$(3.4) \quad x^\lambda L_{2p+1} x^{-\lambda} = M_p, \quad \lambda = p + \frac{1}{2},$$

where

$$(3.5) \quad M_p = D^2 - (p^2 - \frac{1}{4})x^{-2}.$$

Now M_p is self adjoint. (3.1) can be written

$$(x^\lambda \mathcal{B}_p) D^2 = M_p (x^\lambda \mathcal{B}_p)$$

and by transposition

$$(3.6) \quad (x^\lambda \mathcal{B}_p)^* M_p = D^2 (x^\lambda \mathcal{B}_p)^*$$

where

$$\langle (x^\lambda \mathcal{B}_p)^* f, \varphi \rangle = \langle f, x^\lambda \mathcal{B}_p \varphi \rangle, \quad f \in \mathcal{E}'_*, \quad \varphi \in \mathcal{E}_*.$$

Replacing M_p by $x^\lambda L_{2p+1} x^{-\lambda}$ in (3.6) we obtain

$$(3.7) \quad (x^\lambda \mathcal{B}_p)^* x^\lambda L_{2p+1} = D^2 (x^\lambda \mathcal{B}_p)^* x^\lambda.$$

Now an easy calculation leads to

$$(x^\lambda \mathcal{B}_p)^* f(x) = \beta_p \int_x^\infty y^{\lambda-2p} (y^2 - x^2)^{p-\frac{1}{2}} f(y) dy,$$

³ Similar operators have been considered independently and simultaneously by A. ERDÉLYI [2] and A. WEINSTEIN [2].

for $\operatorname{Re} p > -\frac{1}{2}$, and consequently

$$(x^\lambda \mathcal{B}_p)^*(x^\lambda f)(x) = \beta_p \int_x^\infty y(y^2 - x^2)^{p-\frac{1}{2}} f(y) dy.$$

If we set $2p + 1 = k$, this leads to the operator

$$\beta_{\frac{1}{2}(k-1)} \int_x^\infty y(y^2 - x^2)^{\frac{1}{2}k-1} f(y) dy, \quad \operatorname{Re} k > 0,$$

and, up to a multiplicative factor, this is the operator H_k .

4. Distributions $S_k^{(a)}$. The expression (2.10) suggests introducing the distribution $S_k^{(a)}$ defined by

$$(4.1) \quad S_k^{(a)}(x) = \begin{cases} 0 & \text{for } x < a, \\ \frac{2}{\Gamma(-\frac{1}{2}k)} x(x^2 - a^2)^{-\frac{1}{2}k-1} & \text{for } x > a > 0, \end{cases} \quad \operatorname{Re} k < 0.$$

Since $\operatorname{Re} k < 0$, this defines a function locally in L^1 , which is a distribution $S_k^{(a)} \in D'_+$; $a > 0$ is a parameter. For the next §, it is fundamental to study the behavior of $S_k^{(a)}$ as $a \rightarrow 0$.

First we will prove the

Proposition 4.1. *For $a > 0$ fixed, the function $k \rightarrow S_k^{(a)}$, holomorphic for $\operatorname{Re} k < 0$ with values in D'_+ , can be analytically extended to an entire analytical function $k \rightarrow S_k^{(a)}$ with values in D'_+ .*

Proof. Let φ be in D_- . Then

$$\langle S_k^{(a)}, \varphi \rangle = \frac{2}{\Gamma(-\frac{1}{2}k)} \int_a^\infty x(x^2 - a^2)^{-\frac{1}{2}k-1} \varphi(x) dx, \quad \operatorname{Re} k < 0.$$

By an integration by parts we have

$$(4.2) \quad \langle S_k^{(a)}, \varphi \rangle = \frac{2}{k\Gamma(-\frac{1}{2}k)} \int_a^\infty (x^2 - a^2)^{-\frac{1}{2}k} \varphi'(x) dx,$$

and (4.2) is now valid for $\operatorname{Re} k < 2$. If we assume that $\operatorname{Re} k < 2$, and if we set

$$(4.3) \quad \partial g(x) = \frac{d}{dx} (x^{-1} g(x)),$$

it is easily checked that

$$(4.4) \quad \langle S_k^{(a)}, \varphi \rangle = \frac{2}{k(k-2) \cdots (k-2n)\Gamma(-\frac{1}{2}k)} \int_a^\infty (x^2 - a^2)^{-\frac{1}{2}k+n} \partial^n \varphi' dx,$$

and this gives the analytic extension for

$$(4.5) \quad \operatorname{Re} k < 2(n+1),$$

hence the proposition follows.

Proposition 4.2. *For every $k \in \mathbb{C}$, the function $a \rightarrow S_k^{(a)}$ is infinitely differentiable for $a > 0$ with values in D'_+ .*

Proof. Let φ be in D_- ; we choose n such that (4.5) holds and then $\langle S_k^{(a)}, \varphi \rangle$ is given by (4.4). Setting $\partial^n \varphi' = \psi$ ($\psi \in D_-$), we can write

$$(4.6) \quad \langle S_k^{(a)}, \varphi \rangle = \frac{2}{k(k-2) \cdots (k-2n)\Gamma(-\frac{1}{2}k)} a^{-k+2n+1} \int_1^\infty (t^2-1)^{-\frac{1}{2}k+n} \psi(at) dt,$$

and this function is infinitely differentiable in a for $a > 0$, hence the proposition follows.

Proposition 4.3. *If $\operatorname{Re} k < 1$ then*

$$S_k^{(a)} \rightarrow \frac{2\Gamma(-k)}{\Gamma(-\frac{1}{2}k)} Y_{-k}$$

in D'_+ as $a \rightarrow 0$.

Proof. By (4.2),

$$\langle S_k^{(a)}, \varphi \rangle \rightarrow \frac{2}{k\Gamma(-\frac{1}{2}k)} \int_0^\infty x^{-k} \varphi'(x) dx$$

if $\operatorname{Re} k < 1$, when $a \rightarrow 0$, uniformly for φ in bounded set of D_- . Then

$$S_k^{(a)} \rightarrow \frac{-2}{k\Gamma(-\frac{1}{2}k)} \frac{d}{dx} (x^{-k}) = \frac{2}{\Gamma(-\frac{1}{2}k)} \operatorname{Pf} (x^{-k-1}) = \frac{2\Gamma(-k)}{\Gamma(-\frac{1}{2}k)} \frac{1}{\Gamma(-k)} \operatorname{Pf} (x^{-k-1}),$$

hence the proposition follows.

Proposition 4.4. *For $k = 1$,*

$$(4.7) \quad S_1^{(a)} - \frac{2}{\Gamma(-\frac{1}{2})} (\log a) \delta' \rightarrow \frac{2}{\Gamma(-\frac{1}{2})} [\operatorname{Pf} (x^{-2})|_{x>0} - c\delta'] \quad \text{in } D'_+$$

as $a \rightarrow 0$, where

$$(4.8) \quad c = \int_1^\infty [(t^2-1)^{-\frac{1}{2}} - t^{-1}] dt.$$

Proof. By (4.2),

$$\begin{aligned} \langle S_1^{(a)}, \varphi \rangle &= \frac{2}{\Gamma(-\frac{1}{2})} \int_a^\infty (x^2 - a^2)^{-\frac{1}{2}} \varphi'(x) dx \\ &= \frac{2}{\Gamma(-\frac{1}{2})} \left[\int_a^\infty (x^2 - a^2)^{-\frac{1}{2}} \varphi'(x) dx + (\log a) \varphi'(0) \right] - \frac{2}{\Gamma(-\frac{1}{2})} (\log a) \varphi'(0). \end{aligned}$$

The bracket can be written

$$\left[\right] = \int_a^\infty ((x^2 - a^2)^{-\frac{1}{2}} - x^{-1}) \varphi'(x) dx + \int_a^\infty x^{-1} \varphi'(x) dx + (\log a) \varphi'(0).$$

According to SCHWARTZ [1], p. 42, formula (II, 2, 26), the last two terms converge, as $a \rightarrow 0$, to $\langle \text{Pf. } (x^{-1})|_{x>0}, \varphi' \rangle$. The integral can be written

$$\int_1^\infty [(t^2 - 1)^{-\frac{1}{2}} - t^{-1}] \varphi'(at) dt$$

which converges to $c\varphi'(0)$ as $a \rightarrow 0$. Consequently

$$\langle S_1^{(a)}, \varphi \rangle + \frac{2}{\Gamma(-\frac{1}{2})} (\log a) \varphi'(0) \rightarrow \frac{2}{\Gamma(-\frac{1}{2})} [c\varphi'(0) + \langle \text{Pf } x^{-1}|_{x>0}, \varphi' \rangle];$$

hence (4.7) is true.

Proposition 4.5. *We assume that $\text{Re } k = 1$, $k \neq 1$. Then we have*

$$(4.9) \quad S_k^{(a)} = U_k^{(a)} + a^{1-k} V_k^{(a)},$$

where

$$(4.10) \quad U_k^{(a)} \rightarrow \frac{2}{\Gamma(-\frac{1}{2}k)} \text{Pf } (x^{-k-1})|_{x>0} \quad \text{in } D'_+$$

and

$$(4.11) \quad V_k^{(a)} \rightarrow \frac{1}{2\sqrt{\pi}} \Gamma\left(\frac{k-1}{2}\right) \delta' \quad \text{in } D'_+.$$

Proof. Starting always with (4.2), we write

$$\begin{aligned} \langle S_k^{(a)}, \varphi \rangle &= \frac{2}{k\Gamma(-\frac{1}{2}k)} \left[\int_a^\infty (x^2 - a^2)^{-\frac{1}{2}k} \varphi'(x) dx + \frac{a^{1-k}}{1-k} \varphi'(0) \right] \\ &\quad - \frac{2}{k\Gamma(-\frac{1}{2}k)} \frac{a^{1-k}}{1-k} \varphi'(0). \end{aligned}$$

The bracket can be written

$$\left[\right] = \int_a^\infty ((x^2 - a^2)^{-\frac{1}{2}k} - x^{-k}) \varphi'(x) dx + \int_a^\infty x^{-k} \varphi'(x) dx + \frac{a^{1-k}}{1-k} \varphi'(0).$$

According to SCHWARTZ [1], p. 42, formula (II, 2, 26), the last two terms converge, as $a \rightarrow 0$, to $\langle \text{Pf. } (x^{-k})|_{x>0}, \varphi' \rangle$. If we define $U_k^{(a)}$ and $V_k^{(a)}$ by

$$(4.12) \quad \langle U_k^{(a)}, \varphi \rangle = \frac{2}{k\Gamma(-\frac{1}{2}k)} \left(\int_a^\infty x^{-k} \varphi'(x) dx + \frac{a^{1-k}}{1-k} \varphi'(0) \right)$$

then (4.10) is proved. Furthermore,

$$\begin{aligned} (4.13) \quad \langle V_k^{(a)}, \varphi \rangle &= -\frac{2}{k(1-k)\Gamma(-\frac{1}{2}k)} \varphi'(0) \\ &\quad + \frac{2}{k\Gamma(-\frac{1}{2}k)} a^{k-1} \int_a^\infty ((x^2 - a^2)^{-\frac{1}{2}k} - x^{-k}) \varphi'(x) dx, \end{aligned}$$

so that (4.9) holds. It remains only to check (4.11). We can write

$$\langle V_k^{(a)}, \varphi \rangle = -\frac{2}{k(1-k)\Gamma(-\frac{1}{2}k)} \varphi'(0) + \frac{2}{k\Gamma(-\frac{1}{2}k)} \int_1^\infty ((t^2-1)^{-\frac{1}{2}k} - t^{-k}) \varphi'(at) dt$$

and, when $a \rightarrow 0$, this converges to

$$\left[\frac{2}{k\Gamma(-\frac{1}{2}k)} \int_1^\infty ((t^2-1)^{-\frac{1}{2}k} - t^{-k}) dt - \frac{2}{k(1-k)\Gamma(-\frac{1}{2}k)} \right] \varphi'(0),$$

and an easy calculation gives (4.11). Proposition 4.5 is proved.

§2. THE RADIATION PROBLEM

5. Statement of the results. We consider in the plane (x, y) the open half plane $P = [y > 0]$; $D'(P)$ denotes the space of distributions in P ; if $u \in D'(P)$, we shall write also $(x, y) \rightarrow u(x, y)$. In P we consider $Q = [(x, y) \mid x \geq y]$. We shall prove

Theorem 5.1. *Let us consider a distribution $u(x, y) \in D'(P)$ which satisfies*

$$(5.1) \quad \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{k}{y} \frac{\partial}{\partial y} \right) u(x, y) = 0, \quad k \in \mathbb{C},$$

$$(5.2) \quad u \text{ has its support in } Q.$$

Then

(i) $u(x, y)$ equals a function $y \rightarrow u(\cdot, y)$ infinitely differentiable in $y > 0$ with values in $D'(R_+)$;

(ii) if $\operatorname{Re} k < 1$, $u(\cdot, y)$ converges in $D'_+(R_+)$ when $y \rightarrow 0$, $u(x, 0) = 0$ for $x < 0$;

(iii) if $\operatorname{Re} k > 1$ or $k = 1$, $u(\cdot, y)$ is not bounded in $D'_+(R_+)$ as $y \rightarrow 0$, unless $u = 0$;

(iv) if $\operatorname{Re} k = 1$, $k \neq 1$, $u(\cdot, y)$ is bounded in D'_+ as $y \rightarrow 0$, but has no limit when $y \rightarrow 0$, unless $u = 0$.

From this theorem, (iii), (iv), it results that the following radiation problem is meaningful only when $\operatorname{Re} k < 1$:⁴

Radiation problem. *We look for $u \in D'(P)$, satisfying (5.1), (5.2) and*

$$(5.3) \quad u(\cdot, y) \rightarrow F \text{ in } D'_+ \text{ as } y \rightarrow 0,$$

where F is given in D'_+ , $F(x) = 0$ for $x < 0$.

We shall prove

Theorem 5.2. *For $\operatorname{Re} k < 1$, the radiation problem admits a unique solution u , which depends continuously on F .⁵*

⁴ The statement of M. KAROL, *Referativni Journal* (1958), Ref. 2076, saying that HADAMARD solved the problem for $0 < k < 2$ is incorrect.

⁵ This means if $F_\alpha \rightarrow 0$ in D'_+ and keeps its support in $x \geq 0$, then $u_\alpha = u(F_\alpha) \rightarrow 0$ in $D'(P)$.

6. **Proof of Theorem 5.1.** We introduce

$$(6.1) \quad (H_k)_y u(x, y) = v(x, y),$$

where⁶ $u \in D'(R_x) \widehat{\otimes} D'(\Omega_y)$ (cf. L. SCHWARTZ [3], A. GROTHENDIECK [1] and, for remarks analogous to the present ones, J. L. LIONS [1]). Also one can consider $1 \otimes H_k$, where 1 = identity operator in $D'(R_x)$ and H_k = operator H_k of $D'(\Omega_y) \rightarrow D'(\Omega_y)$. Then $v = (1 \otimes H_k)u$ and *formally*

$$v(x, y) = \frac{2}{\Gamma(\frac{1}{2}k)} \int_y^\infty z(z^2 - y^2)^{\frac{1}{2}k-1} u(x, z) dz,$$

at least when $\operatorname{Re} k > 0$ (and this is correct when u is locally L^1 in P). By §1, (2.9), applying $(H_k)_y = 1 \otimes H_k$ to both members of (5.1), we get

$$(6.2) \quad \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) v = 0.$$

According to Remark 2.2, §1, H_k does not change the limit to the right of the supports; consequently

$$(6.3) \quad v \in D'(P) \quad \text{has its support in } Q.$$

But (6.2) and (6.3) are equivalent to

$$(6.4) \quad v(x, y) = f(x - y), \quad f \in D'(R), \quad f(t) = 0 \quad \text{for } t < 0.$$

Since (cf. §1) H_k is an isomorphism of $D'(\Omega_y)$ onto itself, with inverse $\mathcal{H}_k$, (6.1) gives

$$u(x, y) = (\mathcal{H}_k)_y v(x, y),$$

and, by (6.4), we obtain

$$(6.5) \quad u(x, y) = (\mathcal{H}_k)_y f(x - y).$$

Assume now that f is a smooth function (say $f \in D$, $f(t) = 0$ for $t < 0$) and that $\operatorname{Re} k < 0$. According to (2.10), §1, we can write

$$u(x, y) = \frac{2}{\Gamma(-\frac{1}{2}k)} \int_y^\infty z(z^2 - y^2)^{-\frac{1}{2}k-1} f(x - z) dz.$$

Using $S_k^{(a)}$ defined in §1, No. 4, this formula may be written

$$(6.6) \quad u(x, y) = S_k^{(y)} * f(x).$$

This formula is proved for $f \in D$, $f = 0$ for $t < 0$, and $\operatorname{Re} k < 0$. Extending by continuity, we see that (6.6) holds for every $f \in D'$, $f = 0$ for $t < 0$, and from extension by analyticity and Proposition 4.1, §1, we obtain

⁶ Of course, there is no difficulty at all if $u(x, y)$ is a function in x . We can always assume that, by regularization in x . But we think that the general method is more natural.

Proposition 6.1. *If u is a distribution in P which satisfies (5.1) and (5.2), then u is given by (6.6) where $f \in D'_+$, $f = 0$ for $t < 0$.*

Theorem 1.1 will follow now from Proposition 6.1 and the properties of $S_k^{(\nu)}$ given in §1, No. 4. Indeed, (i) follows immediately from Proposition 4.2, §1. By Proposition 4.3, §1, we have, if $\operatorname{Re} k < 1$,

$$(6.7) \quad u(\cdot, y) \rightarrow \frac{2\Gamma(-k)}{\Gamma(-\frac{1}{2}k)} Y_{-k} * f$$

as $y \rightarrow 0$. This proves (ii). For $k = 1$, we write

$$(6.8) \quad u(\cdot, y) = \left(S_1^{(\nu)} - \frac{2}{\Gamma(-\frac{1}{2})} (\log y) \delta' \right) * f + \frac{2}{\Gamma(-\frac{1}{2})} (\log y) \delta' * f.$$

By proposition 4.4, §1, the first term in the right hand side converges to

$$\frac{2}{\Gamma(-\frac{1}{2})} [\operatorname{Pf} (x^{-2})|_{x>0} - c\delta'] * f$$

as $y \rightarrow 0$. Since $(\log y) f'$ does not converge when $y \rightarrow 0$ unless $f = 0$, (iii) is proved for $k = 1$. We have proved more: If $k = 1$, there exists a distribution such that

$$(6.9) \quad u(\cdot, y) - \frac{2}{\Gamma(-\frac{1}{2})} (\log y) f' \rightarrow \frac{2}{\Gamma(-\frac{1}{2})} [\operatorname{Pf} (x^{-2})|_{x>0} - c\delta'] * f$$

as $y \rightarrow 0$.

If $\operatorname{Re} k = 1$, $k \neq 1$, we write

$$(6.10) \quad u(\cdot, y) = U_k^{(\nu)} * f + y^{1-k} (V_k^{(\nu)} * f),$$

with the notations of §1, Proposition 4.5. Since $U_k^{(\nu)} * f$ and $V_k^{(\nu)} * f$ converge in D'_+ as $y \rightarrow 0$, and y^{1-k} does not converge as $y \rightarrow 0$, $u(\cdot, y)$ does not converge in D'_+ as $y \rightarrow 0$ (unless $f = 0$). Since $|y^{1-k}| = 1$, $u(\cdot, y)$ remains in a bounded set of D'_+ ; this proves (iv). It remains only to prove (iii) for $\operatorname{Re} k > 1$. To do this we use relation (3.2), WEINSTEIN [1], p. 127.

Assume that u satisfies (5.1) and (5.2) with $\operatorname{Re} k > 1$. Then we introduce $U = y^{k-1}u$. By relation (3.2), *loc. cit.*, U satisfies

$$(6.11) \quad \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{2-k}{y} \frac{\partial}{\partial y} \right) U = 0,$$

and obviously

$$(6.12) \quad U \text{ has its support in } Q.$$

Since in (6.11), $\operatorname{Re} (2 - k) < 1$, we can apply (ii). Then $U(\cdot, y)$ converges in D'_+ as $y \rightarrow 0$. $u(\cdot, y) = y^{1-k} U(\cdot, y)$, $\operatorname{Re} k > 1$, does not remain in a bounded set of D'_+ as $y \rightarrow 0$.

Theorem 1.1 is completely proved.

5. Proof of Theorem 7.2. We look for u with (5.1), (5.2) and (5.3). We know that u is of the form (6.6). When $y \rightarrow 0$, by (6.7),

$$u(\cdot, y) \rightarrow \frac{2\Gamma(-k)}{\Gamma(-\frac{1}{2}k)} Y_{-k} * f,$$

and this limit must equal F ; hence

$$f = \frac{\Gamma(-\frac{1}{2}k)}{2\Gamma(-k)} Y_k * F,$$

and

$$(7.1) \quad u(x, y) = \frac{\Gamma(-\frac{1}{2}k)}{2\Gamma(-k)} S_k^{(\nu)} * Y_k * F.$$

This proves Theorem 5.2, and *gives explicitly the solution*.

We can slightly simplify formula (7.1). We note that

$$\frac{\Gamma(-\frac{1}{2}k)}{\Gamma(2(-k))} S_k^{(\nu)} = \frac{1}{\Gamma(1-k)} \frac{d}{dx} (x^2 - y^2)^{-\frac{1}{2}k} |_{x>y}, \quad \operatorname{Re} k < 2.$$

If we introduce

$$(7.2) \quad T_k^{(\nu)} = \begin{cases} 0 & \text{for } x < y, \quad \operatorname{Re} k < 2, \\ \frac{1}{\Gamma(1-k)} (x^2 - y^2)^{-\frac{1}{2}k} & \text{for } x > y, \end{cases}$$

then (7.1) can be written

$$(7.3) \quad u(x, y) = T_k^{(\nu)} * Y_{k-1} * F, \quad \operatorname{Re} k < 1.$$

This is identical with formula (8.1) of WEINSTEIN [1].⁷

Remark 7.1. By §1, we know that the distribution $k \rightarrow T_k^{(\nu)} * Y_{k-1}$ is an entire analytic function with values in D'_+ . We can give a simple expression valid for $\operatorname{Re} k > 0$. To do that, we consider the Laplace transform of $T_k^{(\nu)} * Y_{k-1}$:

$$\mathcal{L}(T_k^{(\nu)} * Y_{k-1}) = p^{1-k} \mathcal{L}(T_k^{(\nu)}).$$

By ERDÉLYI [1], p. 138, formula (11), we have

$$(7.4) \quad \mathcal{L}((x^2 - y^2)^{-\frac{1}{2}} |_{x>y}) = \pi^{-\frac{1}{2}} \Gamma(\nu + \tfrac{1}{2}) \left(\frac{2y}{p}\right)^\nu K_\nu(y p), \quad \operatorname{Re} \nu > -\tfrac{1}{2}.$$

Applying this with $\nu - \frac{1}{2} = -\frac{1}{2}k$, we get

$$\mathcal{L}(T_k^{(\nu)}) = \frac{\pi^{-\frac{1}{2}}}{\Gamma(1-k)} \Gamma(1 - \tfrac{1}{2}k) \left(\frac{2y}{p}\right)^{\frac{1}{2}(1-k)} K_{\frac{1}{2}(1-k)}(p y),$$

⁷ WEINSTEIN writes (7.3) for $F =$ "smooth" function; the formulas are *apparently* different because WEINSTEIN takes derivatives of F in the usual sense for $x \geq 0$ and in (7.3) the derivatives of F are calculated in the distribution sense, over the whole line.

and consequently

$$\begin{aligned}\mathfrak{L}(T_k^{(y)} * Y_{k-1}) &= \frac{\pi^{-\frac{1}{2}}}{\Gamma(1-k)} \Gamma(1 - \tfrac{1}{2}k) (2yp)^{\frac{1}{2}(1-k)} K_{\frac{1}{2}(1-k)}(py) \\ &= \frac{\pi^{-\frac{1}{2}}}{\Gamma(1-k)} \Gamma(1 - \tfrac{1}{2}k) 2^{1-k} y^{1-k} \left(\frac{2y}{p}\right)^{\frac{1}{2}(k-1)} K_{\frac{1}{2}(k-1)}(py),\end{aligned}$$

and applying again (7.4) with $\nu = \frac{1}{2}(k-1)$ we get

$$\mathfrak{L}(T_k^{(y)} * Y_{k-1}) = c_k y^{1-k} \mathfrak{L}((x^2 - y^2)^{\frac{1}{2}k-1}|_{x>y}),$$

where

$$(7.5) \quad c_k = \frac{2^{1-k} \Gamma(1 - \frac{1}{2}k)}{\Gamma(1-k) \Gamma(\frac{1}{2}k)}.$$

Consequently

$$(7.6) \quad T_k^{(y)} * Y_{k-1} = c_k y^{1-k} (x^2 - y^2)^{\frac{1}{2}k-1}|_{x>y}.$$

This formula is valid for $0 < \operatorname{Re} k < 2$. The right hand side is a locally L^1 function (of x) for $\operatorname{Re} k > 0$, and consequently *gives the analytic extension*. We remark that using (7.6) in (7.3), for $0 < \operatorname{Re} k < 1$, we get the GERMAIN & BADER form of the solution (cf. GERMAIN & BADER [1], WEINSTEIN [1]).⁸

Another proof. We have

$$T_k^{(y)} * Y_{k-1} = \frac{1}{\Gamma(k-1)\Gamma(1-k)} \int_y^x (x-\xi)^{k-2} (\xi^2 - y^2)^{-\frac{1}{2}k} d\xi,$$

for $1 < \operatorname{Re} k < 2$. Setting $\xi = y + (x-y)s$, we obtain

$$\begin{aligned}T_k^{(y)} * Y_{k-1} &= \frac{1}{\Gamma(k-1)\Gamma(1-k)} (x-y)^{\frac{1}{2}k-1} \int_0^1 s^{-\frac{1}{2}k} (1-s)^{k-2} (2y + (x-y)s)^{-\frac{1}{2}k} ds \\ &= \frac{1}{\Gamma(k-1)\Gamma(1-k)} (x-y)^{-1} \int_0^1 s^{-\frac{1}{2}k} (1-s)^{k-2} (s+c)^{-\frac{1}{2}k} ds,\end{aligned}$$

where $c = 2y/x - y$. Using the well known formula

$$\int_0^1 \frac{s^{p-1} (1-s)^{q-1}}{(s+c)^{p+q}} ds = c^{-q} (1+c)^{-p} B(p, q),$$

with $p = -\frac{1}{2}k + 1$, $q = k - 1$, we obtain

$$T_k^{(y)} * Y_{k-1} = \frac{(2y)^{1-k} \Gamma(1 - \frac{1}{2}k)}{\Gamma(\frac{1}{2}k) \Gamma(1-k)} (x^2 - y^2)^{\frac{1}{2}k-1}$$

which proves again the GERMAIN & BADER formula.

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⁸ The above calculation is essentially equivalent to the calculations done in WEINSTEIN [1], pp. 141-144.

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